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ON SET-VALUED STOCHASTIC EQUATIONS AND STOCHASTIC INCLUSIONS DRIVEN BY A BROWNIAN SHEET

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ABSTRACT. In this paper we developed studies on set-valued stochastic integral equations in the plane. We establish their connections with the theory of stochastic inclusions. We show that every solution to set-valued stochastic equation possesses a continuous selection belonging to the set of solutions of associated stochastic inclusion. We also present some applications to the study of reachable sets of solutions to stochastic integral inclusions as well as their viability properties.

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1. Introduction

The study of stochastic differential inclusions and set-valued stochastic differential (or integral) equations is inspired by the theory of stochastic controlled dynamic systems (see [22], [23], [39] and references therein). Similarly as in the case of deterministic differential inclusions, stochastic inclusions appear as generalizations of a family of stochastic equations

$$dx_t = f(t, x_t, u_t)dt + g(t, x_t, u_t)dB_t \tag{1.1}$$

dependent on a control parameter u belonging to the some set of controls U. Indeed, taking set-valued mappings $F(t, x) = \{f(t, x(t))\}_{u\in U}$ and $G(t, x) = \{g(t, x(t))\}_{u\in U}$ the equation (1.1) can be understood as a stochastic inclusion

$$dx_t \in F(t, x_t)dt + G(t, x_t)dB_t$$

or

$$x_t - x_s \in \int_s^t F(\tau, x_\tau)d\tau + \int_s^t G(\tau, x_\tau)dB_\tau. \tag{1.2}$$

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Then solutions of (1.1) are those of (1.2). Thus any control problem (1.1) can be transformed, by means of multivalued maps into problem (1.2).

Another different generalizations of stochastic systems can be developed in the spirit of set-valued stochastic equations that are understood as a set-valued relations of the type:

$$X_t = X_0 + \int_0^t F(\tau, X_\tau)d\tau + \int_0^t G(\tau, X_\tau)dB_\tau. \tag{1.3}$$

In a single-valued case both stochastic inclusions and stochastic set-valued equations reduce to single-valued stochastic equations. But in general, these two approaches are distinct. Indeed, every solution of (1.2) is always a single-valued stochastic process while solutions of (1.3) are set-valued mappings.

Although there exist a wide literature on stochastic inclusions and their applications (see e.g. [1], [2], [3], [4], [5], [15], [19], [20], [21], [22], [23], [26], [28], [33], [34], [35], [36], [37], [38], [39], [41], [42], [43], [44], and references therein), and intensive studies on set-valued stochastic equations ([29], [30], [31], [32], [40], [47], [50]), it seems that these two theories exist and are developed separately.

On the other hand in the deterministic case set-valued differential equations exhibit a useful tool for investigations of the dynamic of solutions to both differential inclusions and fuzzy differential equations (see e.g. [46], [27] and references therein). More precisely, in a deterministic case one can show that under appropriate assumptions solutions to set-valued differential equations admit continuous selections belonging to the set of solutions of associated differential inclusions.

The motivation of this paper is to establish similar connections between the well developed theory of stochastic inclusions and investigations focused on set-valued stochastic integral equations. In this paper we continue our study on set-valued stochastic equations in the plane initiated in [25] and earlier in considerations given in [45]. We consider stochastic inclusions and set-valued stochastic equations driven by a Brownian sheet (two-parameter Wiener process). We show that every solution to set-valued stochastic equation possesses a continuous selection belonging to the set of solutions of associated stochastic inclusion. We also present some comments on reachable sets of solutions to stochastic integral inclusions as well as their viability properties with connections to solutions of set-valued stochastic equations. Such stochastic inclusions and set-valued stochastic equations can be trated as a generalizations of stochastic differential equations in the plane which have a wide range of financial applications. set-valued stochastic equations can be treated as a generalizations of stochastic differential equations in the plane which have a wide range of financial applications. rates (see e.g. [12], [16], [17]).

The paper is organized as follows. In Section 2 we recall some basic notions and facts from the theory of stochastic and set-valued analysis, as well as the main results on set-valued stochastic integral equations driven by a Brownian sheet needed in the sequel. It will be done on the basis of our main reference [25]. In Section 3 we present main interrelations between solutions to set-valued stochastic integral equations and solutions to stochastic inclusions. Finally in Section 4 we present some concluding remarks on our results.

2. **Preliminaries**

Let $I \times J = [0,S] \times [0,T]$ denote the parameter set together with the partial ordering:

$$(s,t) \preceq (s',t') \text{ if and only if } s \leqslant s' \text{ and } t \leqslant t'.$$

We will also write

$$(s,t) \prec (s',t') \text{ if and only if } s < s' \text{ and } t < t'.$$

Throughout the paper we shall deal with a complete filtered probability space $(\Omega, \mathbb{F}, \{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J}, P)$, where $\{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J}$ is a family of sub-σ-fields of $\mathbb{F}$ such that $\mathbb{F}_{s,t} \subset \mathbb{F}_{s',t'}$, if $(s,t) \preceq (s',t')$. We will assume that $\{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J}$ satisfies the following additional conditions:

(i) $\mathbb{F}_{0,0}$ contains all P-null sets,
(ii) $\mathbb{F}_{s,t} = \bigcap_{(s,t)\prec(u,v)} \mathbb{F}_{u,v}$ for every $(s,t) \in [0,S) \times [0,T)$,
(iii) for every $(s,t) \in I\times J$, the σ-algebras $\mathbb{F}_{s,T}$ and $\mathbb{F}_{S,t}$ are conditionally independent relative to $\mathbb{F}_{s,t}$.

A stochastic process (or random field) $x : I \times J \times \Omega \to \mathbb{R}^d$ is said to be $\{\mathbb{F}_{s,t}\}$-adapted, if $x_{s,t} : \Omega \to \mathbb{R}^d$ is an $\mathbb{F}_{s,t}$-measurable random vector for every fixed $(s,t) \in I \times J$. Let $\{B_{s,t}\}_{(s,t)\in I\times J}$ be a two-parameter real valued $\{\mathbb{F}_{s,t}\}$-Wiener process (Brownian sheet). It is a two-parameter, continuous Gaussian process such that $\mathbb{E}B_{s,t} = 0$ and $\mathbb{E}(B_{s,t}B_{s',t'}) = \min\{s,s'\} \cdot \min\{t,t'\}$ for every $s,s' \in I$ and $t,t' \in J$ (c.f. [8]). Let $\mathcal{N}$ denote the σ-algebra of nonanticipating sets in $I \times J \times \Omega$, i.e.,

$$\mathcal{N} := \{A \in \mathcal{B}(I \times J) \otimes \mathbb{F} : A^{s,t} \in \mathbb{F}_{s,t} \ \text{ for every } (s,t) \in I \times J\},$$

where $A^{s,t} = \{\omega \in \Omega : (s,t,\omega) \in A\}$. A stochastic process $x : I \times J \times \Omega \to \mathbb{R}^d$ is nonanticipating if it is an $\mathcal{N}$-measurable mapping. It is easy to see that a stochastic process x is nonanticipating if and only if it is $\mathcal{B}(I \times J) \otimes \mathbb{F}$-measurable and $\{\mathbb{F}_{s,t}\}$-adapted. By the $\Delta_{s'\,t'}^{s\,t}(x)$ we denote the increment of x over the rectangle $[s',s]\times[t',t]$ i.e.

$$\Delta_{s'\,t'}^{s\,t}(x) = x_{s,t} - x_{s',t} - x_{s,t'} + x_{s',t'}$$

for $(s',t') \preceq (s,t)$ and (s',t'), $(s,t) \in I \times J$. Denote by λ the Lebesgue measure on the σ-algebra $\mathcal{B}(I \times J)$ of Borel sets in $I \times J$. For the sake of convenience we shall use the notations: $L^2_{\mathcal{N}}(\lambda \times P) := L^2(I \times J \times \Omega, \mathcal{N}, \lambda \times P; \mathbb{R}^d)$, $L^2_{s,t} := L^2(\Omega, \mathbb{F}_{s,t}, P; \mathbb{R}^d)$ and $L^2 := L^2(\Omega, \mathbb{F}, P; \mathbb{R}^d)$.

Let $\mathbb{X}$ be a separable Banach space. By a $K^b(\mathbb{X})$ we denote the family of all nonempty closed and bounded subsets of $\mathbb{X}$ while by $K^b_c(\mathbb{X})$ we mean those of elements from $K^b(\mathbb{X})$ that are also convex subsets of $\mathbb{X}$.

The Hausdorff metric $H_{\mathbb{X}}$ in $K^b(\mathbb{X})$ is defined by:

$$H_{\mathbb{X}}(A,B) := \max\{\overline{H}_{\mathbb{X}}(A,B), \overline{H}_{\mathbb{X}}(B,A)\},$$

where $\overline{H}_{\mathbb{X}}(A,B) = \sup_{a\in A} \operatorname{dist}_{\mathbb{X}}(a,B)$, $\operatorname{dist}_{\mathbb{X}}(a,B) := \inf_{b\in B}\|a-b\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{X}}$ is a norm in $\mathbb{X}$ (see e.g. [14]). Let $r>0$ and $\mathbb{V}(A,r) := \{x\in X : \operatorname{dist}_{\mathbb{X}}(x,A) < r\}$. Then from the above definitions it holds:

$$\overline{H}_{\mathbb{X}}(A,B) = \inf\{r>0 : A\subseteq \mathbb{V}(B,r)\} \tag{2.1}$$

and

$$H_{\mathbb{X}}(A,B) = \inf\{r>0 : A\subseteq \mathbb{V}(B,r) \text{ and } B\subseteq \mathbb{V}(A,r)\}. \tag{2.2}$$

Moreover we have:

$$H_{\mathbb{X}}(A+B, C+D) \leqslant H_{\mathbb{X}}(A,C) + H_{\mathbb{X}}(B,D)$$

and

$$H_{\mathbb{X}}(A+B, C+B) = H_{\mathbb{X}}(A,C)$$

for A, B, C, $D\in K^b_c(\mathbb{X})$, where $A+B$ denotes the Minkowski sum of A and B. By Theorem 1.5 and Corollary 1.9 in Chapter I in [14] $(K^b(\mathbb{X}), H_{\mathbb{X}})$ is a complete metric space and $K^b_c(\mathbb{X})$ is its closed subspace.

For $A\in K^b(\mathbb{X})$ we set $|||A||| := H_{\mathbb{X}}(A,\{0\}) = \sup_{a\in A}\|a\|_{\mathbb{X}}$. If $A,B\in K^b_c(\mathbb{X})$ then $A\ominus B$ denotes the Hukuhara difference (if it exists) between the sets A and B, i.e., the set $C\in K^b_c(\mathbb{X})$ such that $A = B + C$.

Let $F: I\times J\times\Omega \to K^b_c(\mathbb{R}^d)$ be a given set-valued mapping. It is called a two-parameter nonanticipating set-valued process if it is $\mathcal{N}$-measurable in the sense of set-valued analysis (c.f. [14]). It is called $L^2_{\mathcal{N}}(\lambda\times P)$-integrally bounded set-valued stochastic process if

$$|||F|||\in L^2(I\times J\times\Omega, \mathcal{N}, \lambda\times P; \mathbb{R}_+).$$

For such a mapping, by Kuratowski and Ryll-Nardzewski Measurable Selection Theorem (c.f. [14]) the set of its nonanticipating and square integrable selections

$$\mathcal{S}^2_{\mathcal{N}}(F,\lambda\times P) := \{f\in L^2_{\mathcal{N}}(\lambda\times P) : f\in F, \lambda\times P\text{-a.e.}\}$$

is nonempty. Then for every $f\in\mathcal{S}^2_{\mathcal{N}}(F,\lambda\times P)$ the Itô stochastic integral $\int_0^S\int_0^T f_{u,v}dB_{u,v}$ is well defined (see [8]). Moreover, the integral process $\left(\int_0^s\int_0^t f_{u,v}dB_{u,v}\right)_{(s,t)\in I\times J}$ is a continuous square integrable two-parameter martingale with respect to the filtration $\{\mathbb{F}_{s,t}\}_{(s,t)\in I\times J}$ and it satisfies Itô's isometry:

$$\mathbb{E}\left\|\int_{s'}^{s}\int_{t'}^{t} f_{u,v}dB_{u,v}\right\|_{\mathbb{R}^d}^2 = \mathbb{E}\int_{s'}^{s}\int_{t'}^{t} \|f_{u,v}\|_{\mathbb{R}^d}^2\lambda(du,dv) \tag{2.3}$$

for all $(s,t),(s',t')\in I\times J$ with $(s',t')\preceq(s,t)$. In view of Doob's maximal inequality for two-parameter martingales we have (c.f. [8]):

$$\mathbb{E}\left(\sup_{(s,t)\in I\times J}\left\|\int_{0}^{s}\int_{0}^{t} f_{u,v}dB_{u,v}\right\|_{\mathbb{R}^d}^2\right) \leqslant 16\mathbb{E}\int_{0}^{S}\int_{0}^{T} \|f_{u,v}\|_{\mathbb{R}^d}^2\lambda(du,dv). \tag{2.4}$$

For $F,G : I\times J\times\Omega\to K_c^b(\mathbb{R}^d)$ being set-valued and $L^2_{\mathcal{N}}(\lambda\times P)$-integrally bounded nonanticipating processes we define the following set-valued stochastic integrals in the plane.

Definition 2.1 ([25]). By a two-parameter set-valued stochastic Lebesgue integral of F and by a two-parameter set-valued Itô's integral of G, we mean the following sets contained in $L^2_{S,T}$:

$$\int_0^S\int_0^T F_{u,v}\lambda(du,dv) := \left\{\int_0^S\int_0^T f_{u,v}\lambda(du,dv) : f\in\mathcal{S}^2_{\mathcal{N}}(F,\lambda\times P)\right\}$$

and

$$\int_0^S\int_0^T G_{u,v}dB_{u,v} := \left\{\int_0^S\int_0^T g_{u,v}dB_{u,v} : g\in\mathcal{S}^2_{\mathcal{N}}(G,\lambda\times P)\right\}$$

respectively. Similarly, we define:

$$\int_{s'}^{s}\int_{t'}^{t} F_{u,v}\lambda(du,dv) := \int_0^S\int_0^T \mathbf{1}_{[s',s]\times[t',t]}(u,v)\, F_{u,v}\lambda(du,dv)$$

and

$$\int_{s'}^{s}\int_{t'}^{t} G_{u,v}dB_{u,v} := \int_0^S\int_0^T \mathbf{1}_{[s',s]\times[t',t]}(u,v)\, G_{u,v}dB_{u,v}$$

for every $(s,t),(s',t')\in I\times J$ with $(s',t')\preceq(s,t)$.

In [25] the following properties of set-valued stochastic integrals have been proved.

Theorem 2.2 ([25]). *Let $F : I\times J\times\Omega\to K_c^b(\mathbb{R}^d)$ be a set-valued and $L^2_{\mathcal{N}}(\lambda\times P)$-integrally bounded nonanticipating process. Then*

a) *$\mathcal{S}^2_{\mathcal{N}}(F,\lambda\times P)$ is a nonempty, closed, bounded, convex, decomposable and weakly compact subset of $L^2_{\mathcal{N}}(\lambda\times P)$.*
b) *The integrals $\int_{s'}^{s}\int_{t'}^{t} F_{u,v}\lambda(du,dv)$ and $\int_{s'}^{s}\int_{t'}^{t} F_{u,v}dB_{u,v}$ are nonempty, closed, bounded, convex and weakly compact subsets of $L^2_{s,t}$ for every $(s,t),(s',t')\in I\times J$ with $(s',t')\preceq(s,t)$.*

Theorem 2.3 ([25]). *Let $F, G : I \times J \times \Omega \to K_c^b(\mathbb{R}^d)$ be the set-valued, nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrally bounded stochastic processes. Then*

$$H^2_{L^2}\left(\int_{s'}^{s}\int_{t'}^{t} F_{u,v}\lambda(du,dv), \int_{s'}^{s}\int_{t'}^{t} G_{u,v}\lambda(du,dv)\right) \leqslant$$
$$\leqslant (s-s')(t-t')\int_{[s',s]\times[t',t]\times\Omega} H^2_{\mathbb{R}^d}(F,G)\, d\lambda \times dP$$

and

$$H^2_{L^2}\left(\int_{s'}^{s}\int_{t'}^{t} F_{u,v}dB_{u,v}, \int_{s'}^{s}\int_{t'}^{t} G_{u,v}dB_{u,v}\right) \leqslant$$
$$\leqslant \int_{[s',s]\times[t',t]\times\Omega} H^2_{\mathbb{R}^d}(F,G)\, d\lambda \times dP.$$

for every $(s,t), (s',t') \in I \times J$, $(s',t') \preceq (s,t)$.

In view of these we have the following result ([25]).

Corollary 2.4. *Let $F : I \times J \times \Omega \to K_c^b(\mathbb{R}^d)$ be a set valued, nonanticipating and $L^2_{\mathcal{N}}(\lambda \times P)$-integrally bounded stochastic processes. Then the correspondences*

$$I \times J \ni (s,t) \mapsto \int_0^s\int_0^t F_{u,v}\lambda(du,dv) \in K_c^b(L^2)$$

and

$$I \times J \ni (s,t) \mapsto \int_0^s\int_0^t F_{u,v}dB_{u,v} \in K_c^b(L^2)$$

are continuous set-valued mappings with respect to the metric H_{L^2}.

Using the notions of above defined set-valued stochastic integrals one can consider a multivalued stochastic integral equation driven by two-parameter Wiener process. Namely, let $\mathbb{F}$ be a separable σ-field with respect to probability P. Hence L^2 is a separable Banach space. Let $F,\ G : I \times J \times \Omega \times K_c^b(L^2) \to K_c^b(\mathbb{R}^d)$ be given set-valued mappings. Let $A : I \times J \to K_c^b(L^2)$ be a continuous mapping. By a set-valued stochastic integral equation generated by a triple (F, G, A) we mean the following equation considered in the metric space $\left(K_c^b(L^2), H_{L^2}\right)$

$$X_{s,t} + A_{0,0} = A_{s,0} + A_{0,t} + \int_0^s\int_0^t F(u,v,X_{u,v})\lambda(du,dv) + \int_0^s\int_0^t G(u,v,X_{u,v})dB_{u,v}, \tag{2.5}$$

for every $(s,t) \in I \times J$.

Thus the equation (2.5) is thought as an abstract relation in the hyperspace of nonempty, bounded, closed and convex subsets of the space L^2. Note also that for F, G and A being single-valued maps, the multivalued equation (2.5) reduces to single-valued one considered in [49], [48], [51].

Definition 2.5. By a solution to equation (2.5) we mean an H_{L^2}-continuous mapping $X : I \times J \to K_c^b(L^2)$ such that (2.5) is satisfied.

Below we formulate main assumptions imposed on set-valued mappings F, G and A:

(A1) for every $U \in K_c^b(L^2)$ the mappings

$$F(\cdot,\cdot,\cdot,U),\ G(\cdot,\cdot,\cdot,U) : I \times J \times \Omega \to K_c^b(\mathbb{R}^d)$$

are nonanticipating set-valued two-parameter stochastic processes.

(A2) there exists a constant $L > 0$ such that

$$\max\{H^2_{\mathbb{R}^d}(F(s,t,\omega,C_1), F(s,t,\omega,C_2)), H^2_{\mathbb{R}^d}(G(s,t,\omega,C_1), G(s,t,\omega,C_2))\} \leqslant LH^2_{L^2}(C_1, C_2),$$

for every $(s,t) \in I \times J$, every $C_1, C_2 \in K_c^b(L^2)$, and P-a.e.

(A3) there exists a constant $K > 0$ such that

$$\max\{H_{\mathbb{R}^d}(F(s,t,\omega,C),\{\theta\}), H_{\mathbb{R}^d}(G(s,t,\omega,C),\{\theta\})\} \leqslant K(1 + H_{L^2}(C,\{\Theta\})),$$

for every $(s,t) \in I \times J$, every $C \in K_c^b(L^2)$, and P-a.e.

(A4) the mapping $A : I \times J \to K_c^b(L^2)$ is assumed to be continuous with respect to the Hausdorff metric H_{L^2} and such that the Hukuhara difference $(A_{s,0} + A_{0,t}) \ominus A_{0,0}$ exists for every $(s,t) \in I \times J$, and $\sup_{(s,t)\in I\times J} H_{L^2}((A_{s,0} + A_{0,t}) \ominus A_{0,0}, \{\Theta\}) < \infty$.

The symbols θ and Θ denote the zero elements in $\mathbb{R}^d$ and L^2, respectively. We recall the following results from [25], needed in the sequel.

Theorem 2.6. *Let F, $G : I \times J \times \Omega \times K_c^b(L^2) \to K_c^b(\mathbb{R}^d)$ and $A : I \times J \to K_c^b(L^2)$ satisfy conditions* (A1)–(A4). *Then equation* (2.5) *has a unique solution.*

Theorem 2.7. *Under assumptions of Theorem* 2.6 *the solution X to equation* (2.5) *satisfies:*

$$H^2_{L^2}(X(s,t),\{\Theta\}) \leqslant [3 \sup_{(s,t)\in I\times J} H^2_{L^2}(A_{s,0} + A_{0,t}, A_{0,0}) + 6K^2 st(st+1)] \exp\{6K^2 st(st+1)\},$$

for every $(s,t) \in I \times J$.

3. Stochastic integral inclusions and set-valued stochastic integral equations

Let us assume that set-valued mappings

$$F_1, G_1 : I \times J \times \Omega \times L^2 \to K_c^b(\mathbb{R}^d)$$

satisfy the following conditions:

(B1) $F_1(\cdot,\cdot,\cdot,\eta)$, $G_1(\cdot,\cdot,\cdot,\eta) : I \times J \times \Omega \to K_c^b(\mathbb{R}^d)$ are nonanticipating set-valued stochastic processes for every $\eta \in L^2$, (B2) there exist $L > 0$, such that P-a.e. for every $(s,t) \in I \times J$ and every $\eta_1, \eta_2 \in L^2$ it holds:

$$\max\left\{H^2_{\mathbb{R}^d}\left(F_1(s,t,\omega,\eta_1), F_1(s,t,\omega,\eta_2)\right), H^2_{\mathbb{R}^d}\left(G_1(s,t,\omega,\eta_1), G_1(s,t,\omega,\eta_2)\right)\right\} \leqslant L\left\|\eta_1-\eta_2\right\|^2_{L^2},$$

(B3) there exists a constant $K > 0$ such that

$$\max\{H_{\mathbb{R}^d}(F_1(s,t,\omega,\eta),\{\theta\}), H_{\mathbb{R}^d}(G_1(s,t,\omega,\eta),\{\theta\})\} \leqslant K(1+\|\eta\|_{L^2})$$

for every $(s,t) \in I \times J$, every $\eta \in L^2$, and P-a.e.

Let $\xi : I \times J \times \Omega \to \mathbb{R}^d$ be an $\{\mathbb{F}_{s,t}\}$−adapted and square integrable stochastic process. We assume that $\xi : I \times J \to L^2$ is a continuous mapping such that $\xi_{s,t} \in L^2_{s,t}$, for every $(s,t) \in I \times J$. By a stochastic differential inclusion generated by (F_1, G_1, ξ) in a plane driven by a two-parameter Wiener process B we mean the following relation:

$$\begin{cases} \Delta^{s\ t}_{s'\ t'}(x) \in \int_{s'}^{s}\int_{t'}^{t} F_1(u,v,x_{u,v})\ \lambda(du,dv) + \int_{s'}^{s}\int_{t'}^{t} G_1(u,v,x_{u,v})\ dB_{u,v}, \\ x_{0,t} = \xi_{0,t} \\ x_{s,0} = \xi_{s,0} \end{cases} \tag{3.1}$$

for every (s',t'), $(s,t) \in I \times J$, $(s',t') \preceq (s,t)$.

Definition 3.1. By a solution to stochastic integral inclusion (3.1) generated by (F_1, G_1, ξ) we mean a continuous stochastic process $x : I \times J \times \Omega \to \mathbb{R}^d$, which has the following representation:

$$x_{s,t} - \xi_{0,t} - \xi_{s,0} + \xi_{0,0} = \int_0^s\int_0^t f_1(u,v)\lambda(du,dv) + \int_0^s\int_0^t g_1(u,v)dB_{u,v},$$

for some $f_1 \in S^2_{\mathcal{N}}(F_1 \circ x)$, $g_1 \in S^2_{\mathcal{N}}(G_1 \circ x)$ where $(F_1 \circ x)(s,t,\omega) = F_1(s,t,\omega,x_{s,t})$, $(G_1 \circ x)(s,t,\omega) = G_1(s,t,\omega,x_{s,t})$ and $(s,t) \in I \times J$.

Denote by $SI(F_1, G_1, \xi)$ the set of all solutions to stochastic inclusion (3.1). Then every element $x \in SI(F_1, G_1, \xi)$ can be understood as a continuous mapping:

$$x : I \times J \to L^2,$$

Moreover it holds $x_{s,t} \in L^2_{s,t}$ for every $(s,t) \in I \times J$.

Now, our aim is to introduce a notion of a set-valued stochastic integral equation generated by stochastic integral inclusion (3.1). First let $\overline{co}(U)$ denote the closed convex hull of the set $U \subset \mathbb{R}^d$, i.e. it is an intersection of all closed convex subsets of $\mathbb{R}^d$ containing U. Similarly by $co(U)$ we denote the convex hull of the set U. One can show the following relation (see Lemma 2.1 in [18]):

$$\overline{co}(U) = clco(U), \tag{3.2}$$

where cl denotes the closure operation. We will also recall the well-known Carathéodory's theorem.

Theorem 3.2 ([18])**.** *Let A be a subset of $\mathbb{R}^d$. Then every point $x \in co(A)$ is the convex combination of at most $d+1$ points in A.*

Let us consider now the mappings:

$$F_2, G_2 : I \times J \times \Omega \times K_c^b(L^2) \to K_c^b(\mathbb{R}^d)$$

defined as follows:

$$F_2(s,t,\omega,C) := \overline{co}\left(\bigcup_{c\in C} F_1(s,t,\omega,c)\right), \tag{3.3}$$

$$G_2(s,t,\omega,C) := \overline{co}\left(\bigcup_{c\in C} G_1(s,t,\omega,c)\right) \tag{3.4}$$

for $(s,t,\omega,C) \in I \times J \times \Omega \times K_c^b(L^2)$.

By a set-valued stochastic integral equation associated with a stochastic integral inclusion (3.1) generated by (F_1, G_1, ξ) we mean the following relation in L^2:

$$\begin{aligned} X_{s,t} + A_{0,0} = A_{s,0} + A_{0,t} + \int_0^s \int_0^t F_2(u,v,X_{u,v})\lambda(du,dv) \\ + \int_0^s \int_0^t G_2(u,v,X_{u,v})dB_{u,v}, \end{aligned} \tag{3.5}$$

for every $(s,t) \in I \times J$, where $A : I \times J \to K_c^b(L^2)$ is a continuous mapping satisfying (A4).

Moreover we assume that the following condition on ξ and A is imposed:

(B4) $\xi_{0,t} + \xi_{s,0} - \xi_{0,0} \in (A_{s,0} + A_{0,t}) \ominus A_{0,0}$, for every $(s,t) \in I \times J$.

Proposition 1. *Assume that $F_1, G_1 : I \times J \times \Omega \times L^2 \to K_c^b(\mathbb{R}^d)$ satisfy the conditions* (B1)–(B3)*. Then $F_2, G_2 : I \times J \times \Omega \times K_c^b(L^2) \to K_c^b(\mathbb{R}^d)$ defined as in* (3.3) *and* (3.4) *satisfy* (A1) *and* (A3)*.*

Proof. Firstly, we show that $F_2(\cdot,\cdot,\cdot,C)$ is a nonanticipating set-valued stochastic process for every $C \in K_c^b(L^2)$. By Theorem 1.0 in [13] it is enough to show that for every fixed $C \in K_c^b(L^2)$ the mapping $F_2(\cdot,\cdot,\cdot,C)$ possesses a nonanticipating Castaigne representation. To ensure this, we recall that L^2 is separable. Therefore every set $C \in K_c^b(L^2)$ is separable too. Then there exist a sequence $\{x_n\}_{n\geqslant 1} \subset C$ such that $cl_{L^2}\{x_n : n \geqslant 1\} = C$.

By (B1) the mapping $(s,t,\omega) \to F_1(s,t,\omega,x_n)$ is nonanticipating for every $n \geqslant 1$. Hence by Theorem 1.0 in [13] there exist a sequence $\{v_n^k\}_{k\geqslant 1}$ of nonanticipating selections for $F_1(\cdot,\cdot,\cdot,x_n)$ (i.e. Castaigne representation for $F_1 \circ x_n$) such that $(F_1 \circ x_n)(s,t,\omega) = F_1(s,t,\omega,x_n) = cl_{\mathbb{R}^d}\{v_n^k(s,t,\omega) : k \geqslant 1\}$. Then we have:

$$F_1(s,t,\omega,C) = F_1(s,t,\omega,cl_{L^2}\{x_n : n \geqslant 1\}) \subseteq cl_{\mathbb{R}^d} F_1(s,t,\omega,\{x_n : n \geqslant 1\})$$
$$= cl_{\mathbb{R}^d}\left(\bigcup_{n\geqslant 1} F_1(s,t,\omega,x_n)\right).$$

Thus:

$$cl_{\mathbb{R}^d}\left(\bigcup_{n\geqslant 1} F_1(s,t,\omega,x_n)\right) = cl_{\mathbb{R}^d}\left(\bigcup_{n\geqslant 1} cl_{\mathbb{R}^d}\{v_n^k(s,t,\omega) : k \geqslant 1\}\right).$$

Moreover we also have that

$$cl_{\mathbb{R}^d}(\bigcup_{n\geqslant 1} cl_{\mathbb{R}^d}\{v_n^k(s,t,\omega) : k \geqslant 1\}) = cl_{\mathbb{R}^d}(\bigcup_{n\geqslant 1}\{v_n^k(s,t,\omega) : k \geqslant 1\}).$$

Hence it follows that

$$cl_{\mathbb{R}^d} F_1(s,t,\omega,C) = cl_{\mathbb{R}^d}\{v_n^k(s,t,\omega) : k \geqslant 1\}.$$

Thus using Theorem 1.0 in [13] again we ensure that the mapping $(s,t,\omega) \to cl_{\mathbb{R}^d} F_1(s,t,\omega,C)$ is nonanticipating.

Now we will show that

$$\overline{co}(cl_{\mathbb{R}^d} F_1(s,t,\omega,C)) = \overline{co}(F_1(s,t,\omega,C)). \tag{3.6}$$

It is obvious that

$$\overline{co}(cl_{\mathbb{R}^d} F_1(s,t,\omega,C)) \supset cl_{\mathbb{R}^d} co(F_1(s,t,\omega,C),$$

for every $(s,t,\omega,C) \in I \times J \times \Omega \times K_c^b(L^2)$.

It remains to show that

$$\overline{co}(cl_{\mathbb{R}^d} F_1(s,t,\omega,C)) \subset cl_{\mathbb{R}^d} co(F_1(s,t,\omega,C),$$

for every $(s,t,\omega,C) \in I \times J \times \Omega \times K^b(L^2)$.

For this, it suffices to prove that $co(cl_{\mathbb{R}^d} F_1(s,t,\omega,C)) \subset cl_{\mathbb{R}^d} co(F_1(s,t,\omega,C))$, because by (3.2) we have:

$$cl_{\mathbb{R}^d} co(cl_{\mathbb{R}^d} F_1(s,t,\omega,C)) = \overline{co}(cl_{\mathbb{R}^d} F_1(s,t,\omega,C))$$

and

$$cl_{\mathbb{R}^d} co(F_1(s,t,\omega,C) = \overline{co}(F_1(s,t,\omega,C).$$

Let $x \in co(cl_{\mathbb{R}^d} F_1(s,t,\omega,C))$. Then due to Theorem 3.2 there exist $w_1,\ldots,w_{d+1} \in cl_{\mathbb{R}^d} F_1(s,t,\omega,C)$ and $\lambda_1,\ldots,\lambda_{d+1} \geqslant 0$, $\sum_{i=1}^{d+1}\lambda_i = 1$ such that $x = \sum_{i=1}^{d+1}\lambda_i w_i$.

Since $w_1, \dots, w_{d+1} \in cl_{\mathbb{R}^d} F_1(s,t,\omega,C)$, it follows that for every $i = 1, \dots, d+1$ there exists $\{u_n^i\}_{n \geqslant 1} \subset F_1(s,t,\omega,C)$ such that $u_n^i \to w_i$, for $n \to \infty$. Let us take a sequence $y^n := \lambda_1 u_n^1 + \lambda_2 u_n^2 + \cdots + \lambda_{d+1} u_n^{d+1}$. Then we have that

$$y^n = \lambda_1 u_n^1 + \lambda_2 u_n^2 + \cdots + \lambda_{d+1} u_n^{d+1} \to x = \lambda_1 w_n^1 + \lambda_2 w_n^2 + \cdots + \lambda_{d+1} w_n^{d+1}.$$

Since for every $n \in N$, $y^n \in co(F_1(s,t,\omega,C))$ it follows that $x \in cl_{\mathbb{R}^d} co(F_1(s,t,\omega,C))$. Thus equality (3.6) follows.

Finally by Proposition 2.26, Chapter II in [14] we conclude that a mapping $(s,t,\omega) \to \overline{co}(F_1(s,t,\omega,C) = F_2(s,t,\omega,C)$ is nonanticipating, for every $C \in K_c^b(L^2)$.

Now we show that if F_1 satisfies the condition (B2) then F_2 satisfies (A2) with some positive constant. Let C_1, $C_2 \in K_c^b(L^2)$. Then for every $\eta \in C_1$ and every $\varepsilon > 0$ there exist $\gamma \in C_2$ such that

$$\begin{aligned}\|\eta - \gamma\|_{L^2} \leqslant dist(\eta, C_2) + \varepsilon \leqslant \sup_{\eta \in C_1} \, dist(\eta, C_2) + \varepsilon \\ = \overline{H}_{L^2}(C_1, C_2) + \varepsilon \leqslant H_{L^2}(C_1, C_2) + \varepsilon.\end{aligned}$$

Then by (B2) we get

$$\overline{H}_{\mathbb{R}^d}\left(F_1(s,t,\omega,\eta), F_1(s,t,\omega,\gamma)\right) \leqslant L^{1/2}\,\|\eta - \gamma\|_{L^2} \leqslant L^{1/2} H_{L^2}(C_1, C_2) + L^{1/2}\varepsilon.$$

Let us put $r(\varepsilon) := L^{1/2} H_{L^2}(C_1, C_2) + L^{1/2}\varepsilon$. Then by the last inequality above we have

$$F_1(s,t,\omega,\eta) \subset \mathbb{V}(F_1(s,t,\omega,\gamma), r(\varepsilon)) \subset \mathbb{V}(F_1(s,t,\omega,C_2), r(\varepsilon)),$$

for every $\eta \in C_1$. Hence

$$F_1(s,t,\omega,C_1) \subset \mathbb{V}(F_1(s,t,\omega,C_2), r(\varepsilon)).$$

Thus by (2.1) it follows that

$$\overline{H}_{\mathbb{R}^d}\left(F_1(s,t,\omega,C_1), F_1(s,t,\omega,C_2)\right) \leqslant r(\varepsilon).$$

Since $\varepsilon > 0$ was arbitrary it follows that

$$\overline{H}_{\mathbb{R}^d}\left(F_1(s,t,\omega,C_1), F_1(s,t,\omega,C_2)\right) \leqslant r(0).$$

In a similar way one can prove that

$$\overline{H}_{\mathbb{R}^d}\left(F_1(s,t,\omega,C_2), F_1(s,t,\omega,C_1)\right) \leqslant r(0).$$

Hence we have

$$H_{\mathbb{R}^d}\left(F_1(s,t,\omega,C_1), F_1(s,t,\omega,C_2)\right) \leqslant r(0).$$

Therefore by Remark 1.19 in [14] we conclude that

$$H_{\mathbb{R}^d}\left(F_2(s,t,\omega,C_1), F_2(s,t,\omega,C_2)\right) \leqslant L^{1/2} H_{L^2}(C_1, C_2).$$

In a similar way as above one can show that F_2 satisfies also condition (A3), provided F_1 attends $(B3)$. □

Before we formulate the main result of this section we establish a slightly more general version of Carathéodory/Lipschitz Selection Theorem needed in the sequel. Although its proof goes by using simillar argumentations as in the proof of Theorem 9.5.3 in [6], we shall present it below for the readers convinience. For this aim we recall first some notions (see [6] for details).

For $A \in K_c^b(\mathbb{R}^d)$ by $\sigma(A, \cdot) : \mathbb{R}^d \to \mathbb{R}$,

$$\sigma(A, p) := \sup \{ < a, p >: a \in A \}$$

we denote a support function of the set A. Then it is easy to see that for every $p_1, p_2 \in \mathbb{R}^d$ one has

$$|\sigma(A, p_1) - \sigma(A, p_2)| \leqslant H_{\mathbb{R}^d}(A, \{\theta\}) \|p_1 - p_2\|_{\mathbb{R}^d}. \tag{3.7}$$

Let $s_d : K_c^b(\mathbb{R}^d) \to \mathbb{R}^d$ be a Steiner Point, i.e.

$$s_d(A) = \begin{cases} \sigma(A, 1)/2 - \sigma(A, -1)/2, & \text{for } d = 1 \\ d \int_{\Sigma^{d-1}} p\sigma(A, p)\mu(dp), & \text{for } d \geqslant 2 \end{cases} \tag{3.8}$$

where Σ^{d-1} denotes the unit sphere in $\mathbb{R}^d$ and μ is a measure on Σ^{d-1} proportional to the Lebesgue measure and $\mu(\Sigma^{d-1}) = 1$. Then by Theorem 9.4.1 in [6], it follows that s_d is a Lipschitz selection map, i.e. $s_d(A) \in A$ for every $A \in K_c^b(\mathbb{R}^d)$ and $\|s_d(A_1) - s_d(A_2)\|_{\mathbb{R}^d} \leqslant dH_{\mathbb{R}^d}(A_1, A_2)$ for $A_1, A_2 \in K_c^b(\mathbb{R}^d)$. Let $(\Gamma, \mathbb{M}, \nu)$ be a measure space and $\mathbb{X}$ be a linear normed space. Assume that $F : \Gamma \times \mathbb{X} \to K_c^b(\mathbb{R}^d)$ satisfies:

(i) $F(\cdot, x)$ is $\mathbb{M}$-measurable for every $x \in \mathbb{X}$,
(ii) $F(\gamma, \cdot)$ is Lipschitz continuous, i.e. $H_{\mathbb{R}^d}(F(\gamma, x), F(\gamma, y)) \leqslant L \|x - y\|_{\mathbb{X}}$, for some constatnt $L > 0$ and for every $x, y \in \mathbb{X}$,
(iii) $H_{\mathbb{R}^d}(F(\gamma, x), \{\theta\}) \leqslant K(1 + \|x\|_{\mathbb{X}})$ for $K > 0$ and for every $x \in \mathbb{X}$.

Then we have the following version of Carathéodory/Lipschitz selection property for F.

Proposition 2. *Let $F : \Gamma \times \mathbb{X} \to K_c^b(\mathbb{R}^d)$ be a set-valued mapping satisfying conditions* (i)–(iii). *Then there exist a function $f : \Gamma \times \mathbb{X} \to \mathbb{R}^d$ such that:*

(a) *$f(\gamma, x) \in F(\gamma, x)$ for all $(\gamma, x) \in \Gamma \times \mathbb{X}$,*
(b) *$f(\cdot, x)$ is $\mathbb{M}$-measurable for each $x \in \mathbb{X}$,*
(c) *$\|f(\gamma, x) - f(\gamma, y)\|_{\mathbb{R}^d} \leqslant Ld \|x - y\|_{\mathbb{X}}$, for all $\gamma \in \Gamma$ and $x, y \in \mathbb{X}$,*
(d) *$\|f(\gamma, x)\|_{\mathbb{R}^d} \leqslant K(1 + \|x\|_{\mathbb{X}})$ for every $\gamma \in \Gamma$ and $x \in \mathbb{X}$.*

Proof. Firstly, let us note that by Proposition 2.32 in [14] a mapping $\gamma \to \sigma(F(\gamma, x), p)$ is $\mathbb{M}$-measurable for every fixed $x \in \mathbb{X}$ and $p \in \mathbb{R}^d$. On the other hand by (3.7) a function $p \to p\sigma(F(\gamma, x), p)$ is continuous and hence integrable on Σ^{d-1} with respect to the measure μ. Moreover the mapping $(\gamma, p) \to p\sigma(F(\gamma, x), p)$ is $\mathbb{M} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and by the assuption (iii) it is also integrable on $\Gamma \otimes \Sigma^{d-1}$ with respect to the measure $\nu \times \mu$ for every fixed $x \in \mathbb{X}$. Hence by the definition of Steiner Point and by the Fubini Theorem the mapping $\gamma \to s_d(F(\gamma, x))$ is $\mathbb{M}$-measurable for each $x \in \mathbb{X}$, in the case when $d \geqslant 2$. The measurability of the mapping $\gamma \to s_1(F(\gamma, x))$ follows directly by the definition of Steiner Point for $d = 1$ and again by Proposition 2.32 in [14]. Then by properties of Steiner Point and the assmumptions (i)–(iii) imposed on set-valued mapping F it follows that a function $f(\gamma, x) := s_d(F(\gamma, x))$ is a desired selection of F. □

Now we formulate the main result of this section.

Theorem 3.3. *Assume that* F_1 *and* G_1 *satisfy* (B1)–(B3), *and* A *satisfies* (A4). *Moreover, let* ξ *and* A *satisfy condition* (B4). *Then there exists a solution* $X : I \times J \to K_c^b(L^2)$ *to the set-valued stochastic integral equation* (3.5) *and a solution* $x : I \times J \times \Omega \to \mathbb{R}^d$ *to the stochastic integral inclusion* (3.1) *such that:*

$$dist_{L^2}(x_{s,t}, X_{s,t}) = 0,$$

for every $(s,t) \in I \times J$.

Proof. By Proposition 1 and Theorem 2.6 there exist a unique solution X to the set-valued stochastic integral equation (3.5). Now, let as consider the set $K(\xi, X)$ defined by

$$\begin{aligned} K(\xi, X) := \Bigg\{ & x \in C(I \times J, L^2) : x_{s,t} - \xi_{s,0} - \xi_{0,t} + \xi_{0,0} \\ & = \int_0^s \int_0^t f(u,v)\lambda(du,dv) + \int_0^s \int_0^t g(u,v) dB_{u,v} \ P\text{-p.w.} \\ & \text{for every } (s,t) \in I \times J, \text{ and some } f \in S^2_{\mathcal{N}}(F_2 \circ X),\ g \in S^2_{\mathcal{N}}(G_2 \circ X) \Bigg\}. \end{aligned}$$

Notice that $K(\xi, X)$ is a nonempty subset of $C(I \times J, L^2)$. Indeed, by Proposition 1 and by the continuity of X it follows that the set-valued mappings $F_2 \circ X : I \times J \times \Omega \to K_c^b(\mathbb{R}^d)$ and $G_2 \circ X : I \times J \times \Omega \to K_c^b(\mathbb{R}^d)$ are nonanticipating. Moreover by (B3) and again by Proposition 1 we have:

$$|||(F_2 \circ X)(s,t,\omega)|||^2_{\mathbb{R}^d} = |||F_2(s,t,\omega, X_{s,t})|||^2_{\mathbb{R}^d} \leqslant 2K(1 + H^2_{L^2}(X_{s,t}, \{\Theta\})).$$

Consequently, due to Theorem 2.7 we get

$$\sup_{(s,t) \in I \times J} |||(F_2 \circ X)(s,t,\omega)|||^2_{\mathbb{R}^d} < \infty.$$

Thus the set $S^2_{\mathcal{N}}(F_2 \circ X)$ is nonempty. In a similar way one can show the nonemptiness of the set $S^2_{\mathcal{N}}(G_2 \circ X)$. It shows the nonemptiness of the set $K(\xi, X)$. Next, let us note that if $x \in K(\xi, X)$, then for every $(s,t) \in I \times J$ it holds $dist_{L^2}(x_{s,t}, X_{s,t}) = 0$. Indeed, by the definition of $K(\xi, X)$ and assumption (B4) we have:

$$\begin{aligned} x_{s,t} &= \xi_{s,0} + \xi_{0,t} - \xi_{0,0} + \int_0^s \int_0^t f(u,v)\lambda(du,dv) + \int_0^s \int_0^t g(u,v)dB_{u,v} \\ &\in A_{s,0} + A_{0,t} \ominus A_{0,0} + \int_0^s \int_0^t F_2(u,v,X_{u,v})\lambda(du,dv) \\ &\quad + \int_0^s \int_0^t G_2(u,v,X_{u,v})dB_{u,v} = X_{s,t}, \end{aligned}$$

for $(s,t,\omega) \in I \times J \times \Omega$.

Next we will show that the set $K(\xi, X)$ is bounded.

Let $x \in K(\xi, X)$. Then there exist $f \in S^2_{\mathcal{N}}(F_2 \circ X)$ and $g \in S^2_{\mathcal{N}}(G_2 \circ X)$ such that:

$$x_{s,t} - \xi_{s,0} - \xi_{0,t} + \xi_{0,0} = \int_0^s \int_0^t f(u,v)\lambda(du,dv) + \int_0^s \int_0^t g(u,v)dB_{u,v},$$

for $(s,t) \in I \times J$. Hence by (A4) and two-parameter Itô's isometry (2.3) we have:

$$\begin{aligned} \sup_{(s,t)\in I\times J} E\|x_{s,t}\|^2_{\mathbb{R}^d} &\leqslant 3 \sup_{(s,t)\in I\times J} E\|\xi_{s,0} + \xi_{0,t} - \xi_{0,0}\|^2_{\mathbb{R}^d} \\ &\quad + 3 \sup_{(s,t)\in I\times J} E\left\|\int_0^s \int_0^t f(u,v)\lambda(du,dv)\right\|^2_{\mathbb{R}^d} \\ &\quad + 3 \sup_{(s,t)\in I\times J} E\left\|\int_0^s \int_0^t g(u,v)dB_{u,v}\right\|^2_{\mathbb{R}^d} \\ &\leqslant 3 \sup_{(s,t)\in I\times J} H^2_{L^2}((A_{s,0} + A_{0,t}) \ominus A_{0,0}, \{\Theta\}) \\ &\quad + 3 \sup_{(s,t)\in I\times J} stE\int_0^s \int_0^t \|f(u,v)\|^2_{\mathbb{R}^d}\lambda(du,dv) \\ &\quad + 3 \sup_{(s,t)\in I\times J} E\left\|\int_0^s \int_0^t g(u,v)dB_{u,v}\right\|^2_{\mathbb{R}^d} \\ &\leqslant 3 \sup_{(s,t)\in I\times J} H^2_{L^2}((A_{s,0} + A_{0,t}) \ominus A_{0,0}, \{\Theta\}) \\ &\quad + 3STE\int_0^S \int_0^T \|f(u,v)\|^2_{\mathbb{R}^d}\lambda(du,dv) \\ &\quad + 3E\int_0^S \int_0^T \|g(u,v)\|^2_{\mathbb{R}^d}\lambda(du,dv). \end{aligned}$$

On the other hand, again by (A3) and Theorem 2.7 for every $(s,t) \in I \times J$ we infer:

$$\begin{aligned} \|f(s,t,\omega)\|^2_{\mathbb{R}^d} &\leqslant \|\|F_2(s,t,\omega,X_{s,t})\|\|^2_{\mathbb{R}^d} \\ &\leqslant [3 \sup_{(s,t)\in I\times J} H^2_{L^2}(A_{s,0} + A_{0,t}, A_{0,0}) \\ &\quad + 6K^2st(st+1)]\exp\{6K^2st(st+1)\} < \infty \end{aligned}$$

and

$$\|g(s,t,\omega)\|^2_{\mathbb{R}^d} \leqslant \|\|G_2(s,t,\omega,X_{s,t})\|\|^2_{\mathbb{R}^d}$$

$$\leqslant [3 \sup_{(s,t)\in I\times J} H^2_{L^2}(A_{s,0}+A_{0,t},A_{0,0}) \\ + 6K^2 st(st+1)] \exp\{6K^2 st(st+1)\} < \infty.$$

Therefore

$$\sup_{(s,t)\in I\times J} E\|x_{s,t}\|^2_{\mathbb{R}^d} < M$$

where M is a positive constant which does not depend on x.

In the next step we will show that $K(\xi,X)$ is a closed subset of $C(I\times J,L^2)$. Let us take a sequence $(x^n)\subset K(\xi,X)$ such that $x^n\to x$ for $n\to\infty$ in the space $C(I\times J,L^2)$, where $x\in C(I\times J,L^2)$. For every $n\in N$, x^n belongs to the set $K(\xi,X)$. Therefore:

$$x^n_{s,t} = \xi_{s,0}+\xi_{0,t}-\xi_{0,0}+\int_0^s\int_0^t f^n(u,v)\lambda(du,dv)+\int_0^s\int_0^t g^n(u,v)dB_{u,v},$$

where $f^n\in S^2_{\mathcal{N}}(F_2\circ X)$, $g^n\in S^2_{\mathcal{N}}(G_2\circ X)$ for every $n\in N$ and $(s,t)\in I\times J$. By Theorem 2.2a), due to weak compactness of the sets $S^2_{\mathcal{N}}(F_2\circ X)$ and $S^2_{\mathcal{N}}(G_2\circ X)$ we infer that there exist subsequences (f^{n_k}) and (g^{n_k}) and also $f\in S^2_{\mathcal{N}}(F_2\circ X)$ and $g\in S^2_{\mathcal{N}}(G_2\circ X)$ such that $f^{n_k}\rightharpoonup f$ and $g^{n_k}\rightharpoonup g$ weakly in $L^2_{\mathcal{N}}(\lambda\times P)$. Therefore we get:

$$\xi_{s,0}+\xi_{0,t}-\xi_{0,0}+\int_0^s\int_0^t f^{n_k}(u,v)\lambda(du,dv)+\int_0^s\int_0^t g^{n_k}(u,v)dB_{u,v}\to x_{s,t} \tag{3.9}$$

in L^2, for $k\to\infty$, and $(s,t)\in I\times J$.

Let us define the linear operators $I_{s,t}: L^2_{\mathcal{N}}(\lambda\times P)\to L^2$, and $J_{s,t}: L^2_{\mathcal{N}}(\lambda\times P)\to L^2$ as follows:

$$I_{s,t}(f) := \int_0^s\int_0^t f(u,v)\lambda(du,dv)$$

and

$$J_{s,t}(g) := \int_0^s\int_0^t g(u,v)dB_{u,v}.$$

By Itô's isometry (2.3) and Doob's maximal inequality (2.4) we infer that $I_{s,t}$ and $J_{s,t}$ are norm-to-norm continuous. Now by Theorem 3.4.12 in [9] we obtain that they are also continuous with respect to weak topologies in $L^2_{\mathcal{N}}(\lambda\times P)$ and L^2, respectively. Hence

$$I_{s,t}(f^{n_k}) := \int_0^s\int_0^t f^{n_k}(u,v)\lambda(du,dv)\rightharpoonup I_{s,t}(f) := \int_0^s\int_0^t f(u,v)\lambda(du,dv)$$

and

$$J_{s,t}(g^{n_k}) := \int_0^s\int_0^t g^{n_k}(u,v)dB_{u,v}\rightharpoonup J_{s,t}(g) := \int_0^s\int_0^t g(u,v)dB_{u,v},$$

when $k\to\infty$. Thus

$$\xi_{s,0}+\xi_{0,t}-\xi_{0,0}+\int_0^s\int_0^t f^{n_k}(u,v)\lambda(du,dv)+\int_0^s\int_0^t g^{n_k}(u,v)dB_{u,v} \\ \rightharpoonup \xi_{s,0}+\xi_{0,t}-\xi_{0,0}+\int_0^s\int_0^t f(u,v)\lambda(du,dv)+\int_0^s\int_0^t g(u,v)dB_{u,v}$$

in L^2, for every $(s,t) \in I \times J$, for $k \to \infty$. This convergence and (3.9) allow us to claim that:

$$x_{s,t} = \xi_{s,0} + \xi_{0,t} - \xi_{0,0} + \int_0^s \int_0^t f(u,v)\lambda(du,dv) + \int_0^s \int_0^t g(u,v)dB_{u,v} \ P\text{-p.w.}$$

Thus $x \in K(\xi, X)$ which proves the closedness of $K(\xi, X)$ in $C(I \times J, L^2)$.

In order to finish the proof, we will show that there exist $\widehat{x} \in SI(F_1, G_1, \xi)$ such that $\widehat{x} \in K(\xi, X)$.

Because F_1 and G_1 satisfy (B1)–(B3), then by Proposition 2 there exist mappings $\overline{f}, \overline{g} : I \times J \times \Omega \times L^2 \to \mathbb{R}^d$ such that:

(i) $\overline{f}(s,t,\omega,\eta) \in F_1(s,t,\omega,\eta)$, $\overline{g}(s,t,\omega,\eta) \in G_1(s,t,\omega,\eta)$, for every $(s,t,\omega,\eta) \in I \times J \times \Omega \times L^2$,

(ii) for every $\eta \in L^2$ the mappings $\overline{f}(\cdot,\cdot,\cdot,\eta)$, $\overline{g}(\cdot,\cdot,\cdot,\eta) : I \times J \times \Omega \to \mathbb{R}^d$ are nonanticipating,

(iii) there exists $\overline{L} > 0$ such that for every $(s,t) \in I \times J$, $\omega \in \Omega$, η_1, $\eta_2 \in L^2$ it holds

$$\max\left\{\left\|\overline{f}(s,t,\omega,\eta_1) - \overline{f}(s,t,\omega,\eta_2)\right\|^2_{\mathbb{R}^d}, \left\|\overline{g}(s,t,\omega,\eta_1) - \overline{g}(s,t,\omega,\eta_2)\right\|^2_{\mathbb{R}^d}\right\} \leqslant \overline{L}\left\|\eta_1 - \eta_2\right\|^2_{L^2}.$$

(iv)

$$\max\left\{\left\|\overline{f}(s,t,\omega,\eta)\right\|_{\mathbb{R}^d}, \left\|\overline{g}(s,t,\omega,\eta)\right\|_{\mathbb{R}^d}\right\} \leqslant K(1 + \|\eta\|_{L^2})$$

for every $(s,t) \in I \times J$, every $\eta \in L^2$, and P-a.e., with a positive constant K the same as in (B3).

Then for every $x \in K(\xi, X)$ the mappings

$$I \times J \times \Omega \ni (s,t,\omega) \to \overline{f}(s,t,\omega,x_{s,t}) \in \mathbb{R}^d,$$

$$I \times J \times \Omega \ni (s,t,\omega) \to \overline{g}(s,t,\omega,x_{s,t}) \in \mathbb{R}^d$$

are elements of $L^2_{\mathcal{N}}(\lambda \times P)$. Moreover $\overline{f}(s,t,\omega,x_{s,t}) \in F_1(s,t,\omega,x_{s,t})$ and $\overline{g}(s,t,\omega,x_{s,t}) \in G_1(s,t,\omega,x_{s,t})$.

Let us define the operator $V : K(\xi, X) \to K(\xi, X)$ as follows:

$$V(x)(s,t) = \xi_{s,0} + \xi_{0,t} - \xi_{0,0} + \int_0^s \int_0^t \overline{f}(u,v,x_{u,v})\lambda(du,dv) + \int_0^s \int_0^t \overline{g}(u,v,x_{u,v})dB_{u,v}$$

for every $x \in K(\xi, X)$ and $(s,t) \in I \times J$. Then $V(x) \in C(I \times J, L^2)$. For every $x \in K(\xi, X)$ we have

$$\overline{f}(s,t,\omega,x_{s,t}) \in F_1(s,t,\omega,x_{s,t}) \subset \bigcup_{\eta \in X_{s,t}} F_1(s,t,\omega,\eta) \subset F_2(s,t,\omega,X_{s,t}).$$

In a similar way we conclude the same relation for $\overline{g}$, G_1 and G_2. Thus we get $V(x) \in K(\xi, X)$ for every $x \in K(\xi, X)$.

Hence it is sufficient to show that the mapping V has a fixed point. Obviously, such a fixed point will be also a solution to stochastic integral inclusion (3.1) generated by a triple (F_1, G_1, ξ). We will show that V is a contraction under the metric

$$\rho(x,y) := \sup_{(s,t)\in I\times J} e^{-\overline{L}(ST+1)st}\left[E\left\|x_{s,t}-y_{s,t}\right\|^2_{\mathbb{R}^d}\right]^{\frac{1}{2}}$$

in $C(I\times J, L^2)$. Indeed, for $x, y\in K(\xi, X)$ by properties of $\overline{f}$, $\overline{g}$ and (2.3) we have:

$$\begin{aligned}
&\rho^2(V(x),V(y))\\
&\quad\leqslant 2\sup_{(s,t)\in I\times J} e^{-2\overline{L}(ST+1)st}\left[E\left\|\int_0^s\int_0^t(\overline{f}(u,v,x_{u,v})-\overline{f}(u,v,y_{u,v}))\lambda(du,dv)\right\|^2_{\mathbb{R}^d}\right.\\
&\quad\left.+\,E\left\|\int_0^s\int_0^t(\overline{g}(u,v,x_{u,v})-\overline{g}(u,v,y_{u,v}))dB_{u,v}\right\|^2_{\mathbb{R}^d}\right]\\
&\leqslant 2\sup_{(s,t)\in I\times J} e^{-2\overline{L}(ST+1)st}\left[ST\cdot E\int_0^s\int_0^t\left\|\overline{f}(u,v,x_{u,v})-\overline{f}(u,v,y_{u,v})\right\|^2_{\mathbb{R}^d}\lambda(du,dv)\right.\\
&\quad\left.+E\int_0^s\int_0^t\left\|(\overline{g}(u,v,x_{u,v})-\overline{g}(u,v,y_{u,v}))\right\|^2_{\mathbb{R}^d}\lambda(du,dv)\right]\\
&\leqslant 2\overline{L}(ST+1)\sup_{(s,t)\in I\times J} e^{-2\overline{L}(ST+1)st}E\int_0^s\int_0^t E\left\|x_{u,v}-y_{u,v}\right\|^2_{\mathbb{R}^d}\lambda(du,dv)\\
&\leqslant 2\overline{L}(ST+1)\sup_{(s,t)\in I\times J} e^{-2\overline{L}(ST+1)st}\int_0^s\int_0^t e^{-2\overline{L}(ST+1)uv}e^{2\overline{L}(ST+1)uv}E\left\|x_{u,v}-y_{u,v}\right\|^2_{\mathbb{R}^d}\lambda(du,dv)\\
&\leqslant 2\overline{L}(ST+1)\sup_{(s,t)\in I\times J} e^{-2\overline{L}(ST+1)st}\int_0^s\int_0^t\left[\sup_{(u,v)\in[0,s]\times[0,t]} e^{-2\overline{L}(ST+1)uv}E\left\|x_{u,v}-y_{u,v}\right\|^2_{\mathbb{R}^d}\times\right.\\
&\quad\left.e^{2\overline{L}(ST+1)uv}\right]\lambda(du,dv)\\
&\leqslant 2\overline{L}(ST+1)\sup_{(s,t)\in I\times J} e^{-2\overline{L}(ST+1)st}\rho^2(x,y)\int_0^s\int_0^t e^{2\overline{L}(ST+1)uv}\lambda(du,dv).
\end{aligned}$$

We put:

$$W := 2\overline{L}(ST+1)\sup_{(s,t)\in I\times J} e^{-2\overline{L}(ST+1)st}\rho^2(x,y)\int_0^s\int_0^t e^{2\overline{L}(ST+1)uv}\lambda(du,dv).$$

Let us note that

$$\begin{aligned}
\int_0^s\int_0^t e^{2\overline{L}(ST+1)uv}du\,dv &\leqslant \int_0^s\left(\int_0^t e^{2\overline{L}(ST+1)tv}du\right)dv\\
&\leqslant \int_0^s t\cdot e^{2\overline{L}(ST+1)tv}dv = \frac{\left(e^{2\overline{L}(ST+1)ts}-1\right)}{2\overline{L}(ST+1)}.
\end{aligned}$$

Thus the expression W is less or equal than:

$$\rho^2(x,y)\sup_{(s,t)\in I\times J}(1-e^{-2\overline{L}(ST+1)st}).$$

Therefore we get:

$$\rho^2(V(x),V(y)) \leqslant (1-e^{-2\overline{L}(ST+1)ST})\rho^2(x,y).$$

Then applying Banach's Contraction Principle we infer that there exists a unique $\widehat{x} \in K(\xi, X)$ such that:

$$\widehat{x}_{s,t} = \xi_{s,0} + \xi_{0,t} - \xi_{0,0} + \int_0^s \int_0^t \overline{f}(u, v, \widehat{x}_{u,v})\lambda(du, dv) + \int_0^s \int_0^t \overline{g}(u, v, \widehat{x}_{u,v})dB_{u,v}.$$

Thus the proof is completed. □

4. Concluding remarks

Let X be a set-valued solution to equation (3.5) generated by (F_2, G_2, A) with F_2 and G_2 given by (3.3) and (3.4), respectively. Since $X : I \times J \to K_c^b(L^2)$ is continuous and closed convex valued multifunction it follows by Michael's Continuous Selection Theorem (see e.g. [6], [9]) that X admits a continuous selection i.e., there exist a continuous mapping $x : I \times J \to L^2$ such that $x(s,t) \in X(s,t)$ for every $(s,t) \in I \times J$. By Theorem 3.3 it follows that the set-valued solution to equation (3.5) possess also a continuous selection belonging to the set of solutions of an associated stochastic inclusion (3.1). It reflects the situation known in the case of deterministic one-parameter set-valued differential equations and inclusions (c.f. [46]). Let $CS(X)$ denote the set of all continuous selections for X. Then we get:

Corollary 4.1. *Under assumptions of Theorem* 3.3 *it holds*

$$CS(X) \cap SI(F_1, G_1, \xi) \neq \emptyset.$$

On the other hand Theorem 3.3 can be helpfull in analysis of reachable sets of solutions to stochastic inclusion (3.1) generated by (F_1, G_1, ξ). Indeed, for $(s,t) \in I \times J$ let $\mathcal{A}((s,t), \xi, F_1, G_1)$ be such the set, i.e.

$$\mathcal{A}((s,t), \xi, F_1, G_1) := \{x_{s,t} \in L^2 : x \in SI(F_1, G_1, \xi)\}.$$

It means that it is the set of all possible values that are attained by trajectories from $SI(F_1, G_1, \xi)$ at the point (s,t). Let $A : I \times J \to K_c^b(L^2)$ be a given continuous mapping. By $\mathcal{C}(A)$ we denote the set of all continuous functions $\xi : I \times J \to L^2$ such that (B4) is satisfied. Let

$$\mathcal{A}((s,t), \mathcal{C}(A), F_1, G_1) = \bigcup_{\xi \in \mathcal{C}(A)} \mathcal{A}((s,t), \xi, F_1, G_1).$$

Then we have:

Corollary 4.2. *Let assumptions of Theorem* 3.3 *be satisfied and* $X : I \times J \to K_c^b(L^2)$ *be a unique solution to the equation* (3.5). *Then*

$$\mathcal{A}((s,t), \mathcal{C}(A), F_1, G_1) \cap X_{s,t} \neq \emptyset$$

for every $(s,t) \in I \times J$.

Let us note that the statement given in Theorem 3.3 can be also interpreted in the spirit of viability property for solutions to stochastic inclusion (3.1). Here, the viability property means that for a given family of sets $\mathcal{K} = \{X(s,t) : (s,t) \in I \times J\} \subset K_c^b(L^2)$ there exists a solution $\widehat{x} \in SI(F_1, G_1, \xi)$ such that $\widehat{x}(s,t) \in X(s,t)$ for every $(s,t) \in I \times J$. Such a solution $\widehat{x}$ is said to be viable in $\mathcal{K}$. Thus by Theorem 3.3 we have:

Corollary 4.3. *Suppose that assumptions of Theorem* 3.3 *are satisfied. Let the family* $\mathcal{K} = \{X(s,t) : (s,t) \in I \times J\} \subset K_c^b(L^2)$ *satisfy equation* (3.5)*. Then there exist a solution* $\widehat{x} \in SI(F_1, G_1, \xi)$ *viable in* $\mathcal{K}$*.*

Finally we apply Theorem 3.3 to stochastic inclusions with expectations in the coefficients. Let us consider set-valued random functions $F, G : I \times J \times \Omega \times \mathbb{R}^d \times \mathbb{R}^1 \to K_c^b(\mathbb{R}^d)$ satisfying the following conditions:

(C1) $F(\cdot,\cdot,\cdot,x,u)$, $G(\cdot,\cdot,\cdot,x,u) : I \times J \times \Omega \to K_c^b(\mathbb{R}^d)$ are nonanticipating set-valued processes for every $x \in \mathbb{R}^d$, and $u \in \mathbb{R}^1$,

(C2) there exist $L_1 > 0$ such that for every $(s,t) \in I \times J$, $x, y \in \mathbb{R}^d$, and $u, v \in \mathbb{R}^1$ it holds:

$$\max\{H^2_{\mathbb{R}^d}(F(s,t,\omega,x,u), F(s,t,\omega,y,v)), H^2_{\mathbb{R}^d}(G(s,t,\omega,x,u), G(s,t,\omega,y,v))\}$$
$$\leqslant L_1(\|x-y\|^2_{\mathbb{R}^d} + |u-v|) \text{ P-a.e.},$$

(C3) there exist $K_1 > 0$ such that

$$\max\{H_{\mathbb{R}^d}(F(s,t,\omega,x,u), \{\Theta\}), H_{\mathbb{R}^d}(G(s,t,\omega,x,u), \{\Theta\})\}$$
$$\leqslant K_1\left(1 + \frac{\|x\|_{\mathbb{R}^d} + |u|}{2}\right) \text{ P-a.e.}$$

for every $(s,t) \in I \times J$, $x \in \mathbb{R}^d$ and $u \in \mathbb{R}^1$.

Let $\xi : I \times J \times \Omega \to \mathbb{R}^d$ be an $\mathbb{F}_{s,t}-$adapted and square integrable stochastic process. As previously, we assume that $\xi : I \times J \to L^2$ is a continuous function. Let us consider the following stochastic inclusion:

$$\text{(4.1)} \qquad \begin{cases} \Delta^{s\ t}_{s'\ t'}(x) \in \int_{s'}^{s}\int_{t'}^{t} F(u,v,E(x_{u,v}), \|x_{u,v}\|_{L^2})\ \lambda(du,dv) \\ + \int_{s'}^{s}\int_{t'}^{t} G(u,v,E(x_{u,v}), \|x_{u,v}\|_{L^2})\ dB_{u,v}, \\ x_{0,t} = \xi_{0,t} \\ x_{s,0} = \xi_{s,0} \end{cases}$$

for every (s',t'), $(s,t) \in I \times J$, $(s',t') \preceq (s,t)$. It is easy to check that taking:

$$F_1, G_1 : I \times J \times \Omega \times L^2 \to K_c^b(\mathbb{R}^d)$$

with

$$F_1(s,t,\omega,\eta) := F(s,t,\omega,E\eta, \|\eta\|_{L^2})$$

and

$$G_1(s,t,\omega,\eta) := G(s,t,\omega,E\eta, \|\eta\|_{L^2})$$

it follows that these set-valued mappings satisfy conditions (B1), (B2) and (B3). Let $C \in K_c^b(L^2)$ be given. Let us define:

$$F_2(s,t,\omega,C) := \overline{co}(\bigcup_{\eta\in C} F(s,t,\omega,E\eta,\|\eta\|_{L^2})),$$

and

$$G_2(s,t,\omega,C) := \overline{co}(\bigcup_{\eta\in C} G(s,t,\omega,E\eta,\|\eta\|_{L^2})).$$

Thus by Proposition 1, Theorem 3.3 can be applied for stochastic inclusion (4.1) generated by (F,G,ξ) and for set-valued stochastic equation (3.5) generated by (F_2,G_2,A) with F_2,G_2 as above and $A : I\times J \to K_c^b(L^2)$ being a continuous mapping satisfying (A4), and ξ and A satisfying condition (B4). Also Corollaries 4.1, 4.2 and 4.3 are valid in this case.

It is also worth to note that in a single-valued case stochastic inclusion (4.1) reduces to the stochastic integral equation with expectations in the coefficients

$$x_{s,t} - \xi_{s,0} - \xi_{0,t} + \xi_{0,0} = \int_0^s\int_0^t f(u,v,E(x_{u,v}),\|x_{u,v}\|_{L^2})\lambda(du,dv) + \int_0^s\int_0^t g(u,v,E(x_{u,v}),\|x_{u,v}\|_{L^2})dB_{u,v}. \tag{4.2}$$

Stochastic differential equations described by special forms of (4.2) can be found as models in finance. For example they were used in the theory of term structure of interest rates (see e.g. [16], [17], [12] and references therein).

Finally, let us note that the same methods we have presented in this paper can by applied to the study of interrelations between solutions of stochastic inclusions and solutions of set-valued stochastic equations in one parameter case. Therefore, one-parameter counterparts of Theorem 3.3 and Corollaries 4.1, 4.2 and 4.3 hold true. Moreover in this case the one-parameter counterpart of equation (4.2) has the form:

$$\begin{cases} dx_t = f(t,E(x_t),\|x_t\|_{L^2})dt + g(t,E(x_t),\|x_t\|_{L^2})dB_t \\ x_0 = \xi. \end{cases}$$

Such equations have a wide range of applications. For example in [7] they were used in wildlife models. In [29] similar equations were applied for modeling of dynamics of stock prices (see also [24]).

References

[1] N. U. Ahmed, Nonlinear stochastic differential inclusion on Banach space, *Stoch. Anal. Appl.,* 12:1–10, 1994.

[2] N. U. Ahmed, Impulsive perturbation of C_0 semigroups and stochastic evolution inclusions, *Discuss. Math. Differ. Incl. Control Optim.*, 22:125–149, 2002.

[3] N. U. Ahmed, Optimal relaxed controls for nonlinear infinite dimensional stochastic differential inclusions, *Optimal Control of Differential Equations*, M. Dekker Lect. Notes, 160:1–19, 1994.

[4] N. U. Ahmed, Differential inclusions operator valued measures and optimal control, *Dynam. Systems Appl.*, 16:13–36, 2007.

[5] J.-P. Aubin and G. Da Prato, The viability theorem for stochastic differential inclusions, *Stoch. Anal. Appl.,* 16(1):1–15, 1998.

[6] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkh äuser, Basel, 1990.

[7] D. R. Brillinger, H. K. Preisler, A. A. Ager, J. G. Kie and B. S. Stewart, Employing stochastic differential equations to model wildlife motion, *Bull. Braz. Math. Soc.,* 33:385–408, 2002.

[8] R. Cairoli and J. B. Walsh, Stochastic integrals in the plane. *Acta Math.*, 134:111–183, 1975.

[9] Z. Denkowski, S. Migórski and N.S. Papageorgiou, *An Introduction to Nonlinear Analysis and Its Applications, Part I*, Kluwer Acad. Publ., Boston, 2003.

[10] G. Di Nunno and I. B. Eide, Minimal-variance hedging in large financial markets: random fields approach, *Stoch. Anal. Appl.* 28, no. 1, 54–85, 2010.

[11] N. Dunford and J. T. Schwartz, *Linear Operators, Part I: General Theory*, Int. Publ., INC, New York, 1958.

[12] R. S. Goldstein, The term structure of interest rates as a random field, *The Review of Financial Studies,* 13(2):365–384, 2000.

[13] F. Hiai and H. Umegaki, Integrals, conditional expectations, and martingales of multivalued functions, *J. Multivariate Anal.*, 7:149–182, 1977.

[14] S. Hu, N. Papageorgiou, *Handbook of Multivalued Analysis, Vol. 1, Theory*, Kluwer Acad. Publ., Boston, 1997.

[15] A. Jakubowski, M. Kamenskii and P. R. de Fitte, Existence of weak solutions to stochastic evolution inclusions, *Stoch. Anal. Appl.,* 23:723–749, 2005.

[16] D. P. Kennedy, The term structure of interest rates as a Gaussian random field, *Mathematical Finance,* 4(3):247–258, 1994.

[17] D. P. Kennedy, Characterizing Gaussian models of the term structure of interest rates, *Mathematical Finance,* 7(2):107–118, 1997.

[18] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer Acad. Publ., Dordrecht, 1991.

[19] M. Kisielewicz, Backward stochastic differential inclusions, *Dynam. Systems Appl.*, 16:121–140, 2007.

[20] M. Kisielewicz, Weak compactness of weak solutions to backward stochastic differential inclusions, *Dynam. Systems Appl.*, 17:351–370, 2008.

[21] M. Kisielewicz, Set-valued stochastic integrals and stochastic inclusions, *Stoch. Anal. Appl.,* 15(5):783–800, 1997.

[22] M. Kisielewicz, M. Michta and J. Motyl, Set-valued approach to stochastic control. Part I. Existence and regularity properties, *Dynam. Systems Appl.*, 12:405–431, 2003.

[23] M. Kisielewicz, M. Michta and J. Motyl, Set-valued approach to stochastic control. Part II. Viability and semimartingale issue, *Dynam. Systems Appl.*, 12:433–466, 2003.

[24] P. E. Kloeden and T. Lorenz, Stochastic differential equation with nonlocal sample dependence, *Stoch. Anal. Appl.,* 28:1–9, 2010.

[25] M. Kozaryn, M. Malinowski, M. Michta, K. Świątek, On multivalued stochastic integral equations driven by a Wiener process in the plane, *Dynam. Systems Appl.*, 21:293318, 2012.

[26] P. Kree, Diffusion equation for multivalued stochastic differential equation, *J. Func. Anal.,* 49:73–90, 1982.

[27] V. Lakshmikantham and R. N. Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, Taylor and Francis Publishers, London, 2003.

[28] A. A. Levakov, Stochastic differential inclusions, *J. Diff. Eq.,* 71(2):212–221, 2003.

[29] T. Lorenz, Mutational analysis: a joint framework for dynamical systems in and beyond vector spaces, Habilitationsschrift, Heidelberg University, 2009.

[30] M. T. Malinowski and M. Michta, Fuzzy integral equation, *Dynam. Systems Appl.,* 19:473–494, 2010.

[31] M. T. Malinowski and M. Michta, Set-valued stochastic integral equation driven by martingales, *J. Math. Anal. Appl.,* 394:30–47, 2012.

[32] M. T. Malinowski, M. Michta and J. Sobolewska, Set-valued stochastic differential equations driven by semimartingales, *Nonlin. Anal.*, 79:204–220, 2013.

[33] M. Michta, Optimal solutions to stochastic differential inclusions, *Applicationes Math.*, 29:387–398, 2002.

[34] M. Michta, On weak solutions to stochastic differential inclusions driven by semimartingales, *Stoch. Anal. Appl.*, 22:1341–1361, 2004

[35] M. Michta, On set-valued stochastic integrals and fuzzy stochastic equations, *Fuzzy Sets and Systems*, 177:1–19, 2011.

[36] M. Michta and J. Motyl, Weak solutions to a stochastic inclusion with multivalued integrator, *Dynam. Systems Appl.*, 14:323–334, 2005.

[37] M. Michta and J. Motyl, Second order stochastic inclusions and equations, *Dynam. Systems Appl.*, 15:301–315, 2006.

[38] M. Michta and J. Motyl, Set valued Stratonovich integral and Stratonovich type stochastic inclusion, *Dynam. Systems Appl.*, 16:141–154, 2007.

[39] M. Michta and J. Motyl, Stochastic inclusion with a non-lipschitz right hand side, In: Stochastic Differential Equations, Ed. N. Halidas, Nova Science Publ. Inc., 189–232, 2011.

[40] I. Mitoma, Y. Okazaki and J. Zhang, Set-valued stochastic differential equation in M-type 2 Banach space, *Commun. Stoch. Anal.,* 4(2):215–237, 2010.

[41] J. Motyl, Stochastic functional inclusion driven by semimartingale, *Stoch. Anal. Appl.,* 16(3):517–532, 1998.

[42] J. Motyl, Existence of solutions of set-valued Itô equation, *Bull PAN*, 46(4):419–430, 1998.

[43] J. Motyl, Viable solution of set-valued stochastic equations, *Optimization,* 48:157–176, 2000.

[44] J. Motyl, Stochastic Itô inclusion with upper separated multifunctions, *J. Math. Anal. Appl.*, 400:505–509, 2013.

[45] W. Sosulski, Set-valued stochastic integrals and stochastic inclusions in a plane, *Discuss. Math. Differ. Incl. Control Optim.,* 21(2):249–259, 2001.

[46] A. A. Tolstonogov, *Differential Inclusions in Banach Space*, Kluwer Acad. Publ., Dordrecht, 2000.

[47] H. Vu, L. S. Dong, N. N. Phung, N. V. Hoa and N. D. Phu, Stability criteria of solution for stochastic set differential equations, *Appl. Math.*, 3:354–359, 2012.

[48] E. Wong and M. Zakai, Martingales and stochastic integrals for processes with a multidimensional parameter, *Z. Wahrsch. Verw. Gebiete*, 29:109–122, 1974.

[49] J. Yeh, Existence of strong solutions for stochastic differential equations in the plane. *Pacific J. Math.*, 97:217–247, 1981.

[50] J. Zhang and S. Li, Approximate solutions of set-valued stochastic differential equations, *Journal of Uncertain Systems,* 7(1):3–12, 2013.

[51] X. Zhang and J. Zhu, Non-Lipschitz stochastic differential equations driven by multi-parameter Brownian motions, *Stoch. Dyn.*, 6:329–340, 2006.