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# A COMPUTATIONAL METHOD FOR THE QUENCHING TIME FOR A PARABOLIC PROBLEM DUE TO A CONCENTRATED NONLINEAR SOURCE IN AN INFINITE STRIP

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**ABSTRACT.** An upper bound of the quenching time $t_q$ for a semilinear parabolic initial-boundary value problem with a concentrated nonlinear source in an $N$-dimensional infinite strip is given. A computational method is devised to compute $t_q$ under different conditions.

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## 1. INTRODUCTION

Let a point $(x_1, x_2, \ldots, x_{N-1}, x_N)$ in the $N$-dimensional Euclidean space $\mathbb{R}^N$ be denoted by $(x, \tilde{x})$ where $x$ stands for $x_1$, $L$ and $b$ be positive numbers such that $b < L$, $S = (-L, L)\times\mathbb{R}^{N-1}$, $s = (-b, b)\times\mathbb{R}^{N-1}$, $\partial S = \left\{(x, \tilde{x}) : x \in \{-L, L\}\,, \text{ and } \tilde{x} \in \mathbb{R}^{N-1}\right\}$, and $\partial s = \left\{(x, \tilde{x}) : x \in \{-b, b\}\,, \text{ and } \tilde{x} \in \mathbb{R}^{N-1}\right\}$. Let $\nu(x, \tilde{x})$ denote the unit outward normal at $(x, \tilde{x}) \in \partial s$, and $\chi_s(x, \tilde{x})$ denote a function which is 1 for $|x| > b$, and 0 for $|x| < b$. Since the Dirac delta function is the derivative of the Heaviside function, it follows that $\partial\chi_s(x, \tilde{x})/\partial\nu$ yields a Dirac delta function at each point on $x = |b|$, and is zero everywhere else (cf. Chan and Tragoonsirisak [1]), and hence we have a concentrated source on $\partial s$. Recently, Chan and Tragoonsirisak [2], [3] studied the following problem with a concentrated nonlinear source on $\partial s$:

$$(1.1)\qquad \begin{cases} u_t - \triangle u = \alpha \dfrac{\partial\chi_s(x, \tilde{x})}{\partial\nu} f(u) \text{ in } S \times (0, T], \\ u(x, 0) = 0 \text{ on } \bar{S},\ u(x, t) = 0 \text{ on } \partial S \times (0, T], \end{cases}$$

where $\alpha$ and $T$ are positive real numbers, $\bar{S}$ is the closure of $S$, $f$ is a given function such that $\lim_{u\to c^-} f(u) = \infty$ for some positive constant $c$, and $f(u)$ and its derivatives $f'(u)$ and $f''(u)$ are positive for $0 \le u < c$. Let $H = \partial/\partial t - \partial^2/\partial x^2$, $D = (0, L)$,

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$\bar{D} = [0, L]$, and $\Omega = D \times (0, T]$. Due to symmetry, the problem (1.1) is equivalent to the following one-dimensional problem:

$$\begin{cases} Hu = \alpha\delta\left(x-b\right)f(u) \text{ in } \Omega, \\ u(x,0) = 0 \text{ on } \bar{D},\ u_x\left(0,t\right) = u\left(L,t\right) = 0 \text{ for } 0 < t \leq T, \end{cases} \tag{1.2}$$

where $\delta\left(x-b\right)$ is the Dirac delta function. A solution $u$ is said to quench if there exists an extended real number $t_q \in (0, \infty]$ such that

$$\sup\left\{u(x,t) : x \in \bar{D}\right\} \to c^- \text{ as } t \to t_q.$$

If $t_q < \infty$, then $u$ is said to quench in a finite time. If $t_q = \infty$, then $u$ quenches in infinite time. Green's function $g\left(x,t;\xi,\tau\right)$ corresponding to the problem (1.2) is given by

$$g\left(x,t;\xi,\tau\right) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\cos\frac{\left(2n-1\right)\pi x}{2L}\right)\left(\cos\frac{\left(2n-1\right)\pi\xi}{2L}\right)\exp\left(-\frac{\left(2n-1\right)^2\pi^2\left(t-\tau\right)}{4L^2}\right)$$

(cf. Chan and Tragoonsirisak [2]). For ease of reference, let us summarize the main results of Chan and Tragoonsirisak [2] in the following theorem.

**Theorem 1.1.** (a) *For* $(x,t;\xi,\tau) \in \left(\bar{D} \times (\tau, T]\right) \times \left(\bar{D} \times [0, T)\right)$, $g\left(x,t;\xi,\tau\right)$ *is continuous.*

(b) *For* $x, \xi \in D$ *and* $0 \leq \tau < t \leq T$, $g\left(x,t;\xi,\tau\right)$ *is positive.*

(c) *If* $r\left(t\right) \in C\left([0,T]\right)$, *then* $\int_0^t g(x,t;b,\tau)r(\tau)d\tau$ *is continuous for* $x \in \bar{D}$ *and* $t \in [0, T]$.

(d) *There exists some* $t_q$ *such that for* $0 \leq t < t_q$, *the nonlinear integral equation*

$$u\left(x,t\right) = \alpha\int_0^t g\left(x,t;b,\tau\right)f\left(u\left(b,\tau\right)\right)d\tau \tag{1.3}$$

*has a unique continuous nonnegative solution* $u$. *This solution* $u$ *is the unique solution of the problem (1.2), and is a strictly increasing function of* $t$ *in* $D$. *For any* $t \in (0, t_q)$, $u\left(x,t\right)$ *attains its absolute maximum at* $(b,t)$ *on the region* $\bar{D} \times [0,t]$. *If* $t_q$ *is finite, then at* $t_q$, $u$ *quenches at* $x = b$ *only.*

In this paper, we give a method to compute the quenching time $t_q$. In Section 2, we show how to find an upper bound for $t_q$. In Section 3, a computational method is devised to find $t_q$ by making use of the several criteria given by Chan and Tragoonsirisak [3] for quenching to occur.

## 2. UPPER BOUND

To find the upper bound of the quenching time $t_q$, let

$$w\left(t\right) = \int_0^L \phi\left(x\right)u\left(x,t\right)dx, \tag{2.1}$$

where $\phi(x)$ denotes the eigenfunction corresponding to the fundamental eigenvalue $\lambda_1$ of the eigenvalue problem:

$$\left\{ \begin{array}{l} \phi'' = -\lambda_1 \phi \text{ in } D, \\ \phi'(0) = \phi(L) = 0, \end{array} \right. \tag{2.2}$$

such that

$$\int_0^L \phi(x)\, dx = 1. \tag{2.3}$$

By a direct computation, we obtain

$$\lambda_1 = \frac{\pi^2}{4L^2}, \tag{2.4}$$

$$\phi(x) = \frac{\pi}{2L} \cos\left(\frac{\pi x}{2L}\right). \tag{2.5}$$

Our next result gives an upper bound for the quenching time.

**Theorem 2.1.** *If*

$$\alpha\ \phi(b) - \lambda_1 \sup_{0<w<c} \left(\frac{w}{f(w)}\right) > 0, \tag{2.6}$$

*then*

$$t_q \leq \frac{1}{\alpha\ \phi(b) - \lambda_1 \sup\limits_{0<w<c} \left(\frac{w}{f(w)}\right)} \int_0^c \frac{dw}{f(w)}. \tag{2.7}$$

*Proof.* Let us multiply the differential equation in (1.2) by $\phi(x)$ and integrate over $x$ from 0 to L. We have

$$\int_0^L u_t(x,t)\, \phi(x)\, dx - \int_0^L u_{xx}(x,t)\, \phi(x)\, dx = \int_0^L \alpha\delta(x-b)\, f(u)\phi(x)\, dx.$$

We have

$$\int_0^L u_t(x,t)\, \phi(x)\, dx = \frac{d}{dt} \int_0^L u(x,t)\, \phi(x)\, dx = \frac{dw}{dt},$$

$$\int_0^L \alpha\delta(x-b)\, f(u)\phi(x)\, dx = \alpha\ \phi(b)\, f(u(b,t)).$$

By using integration by parts, it follows from (1.2) and (2.2) that

$$\begin{aligned} \int_0^L u_{xx}(x,t)\, \phi(x)\, dx &= \phi(L)\, u_x(L,t) - \phi(0)\, u_x(0,t) - \int_0^L \phi'(x)\, u_x(x,t)\, dx \\ &= -\left[\phi'(L)\, u(L,t) - \phi'(0)\, u(0,t)\right] + \int_0^L \phi''(x)\, u(x,t)\ dx \\ &= -\lambda_1 \int_0^L \phi(x)\, u(x,t)\ dx \\ &= -\lambda_1 w(t). \end{aligned}$$

Thus,

$$\frac{dw}{dt} + \lambda_1 w = \alpha\, \phi(b)\, f(u(b,t)). \tag{2.8}$$

From Theorem 1.1(d), (2.1), and (2.3), we obtain

$$w(t) \le \int_0^L \phi(x)\, u(b,t)\, dx = u(b,t). \tag{2.9}$$

It follows from (2.8), (2.9), and $f$ being an increasing function that

$$\begin{aligned} \frac{dw}{dt} &= \alpha\, \phi(b)\, f(u(b,t)) - \lambda_1 w(t) \\ &\ge \alpha\, \phi(b)\, f(w(t)) - \lambda_1 w(t) \\ &= f(w(t)) \left[\alpha\, \phi(b) - \lambda_1 \frac{w(t)}{f(w(t))}\right] \\ &\ge f(w(t)) \left[\alpha\, \phi(b) - \lambda_1 \sup_{0<w<c} \left(\frac{w(t)}{f(w(t))}\right)\right]. \end{aligned}$$

Using (2.6) and $f(w(t))$ being positive, we have

$$\frac{1}{\alpha\, \phi(b) - \lambda_1 \sup\limits_{0<w<c} \left(\frac{w(t)}{f(w(t))}\right)} \cdot \frac{dw}{f(w(t))} \ge dt. \tag{2.10}$$

Integrating (2.10), we obtain

$$\frac{1}{\alpha\, \phi(b) - \lambda_1 \sup\limits_{0<w<c} \left(\frac{w(t)}{f(w(t))}\right)} \int_{w(0)}^{w(t_q)} \frac{dw}{f(w(t))} \ge \int_0^{t_q} dt = t_q.$$

Since

$$w(0) = \int_0^L \phi(x)\, u(x,0)\, dx = 0, \text{ and } \lim_{t \to t_q^-} w(t) \le c,$$

we have (2.7). □

## 3. QUENCHING TIME

To compute the quenching time $t_q$, we use Mathematica version 9. From Theorem 1.1(d), it is sufficient to consider (1.3) at $x = b$ in order to determine the quenching time $t_q$. We consider

$$\begin{aligned} M(t) &= u(b,t) \\ &= \frac{2\alpha}{L} \int_0^t \sum_{n=1}^{\infty} \left(\cos \frac{(2n-1)\pi b}{2L}\right)^2 \exp\left(-\frac{(2n-1)^2 \pi^2 (t-\tau)}{4L^2}\right) f(M(\tau))\, d\tau. \end{aligned}$$

We give below the steps to compute $t_q$:

Step 1. We input the function $f(u)$ and $c$.

Step 2. We input the tolerances $\delta$ and $\epsilon$ to be defined later, and the upper limit $Q$ (instead of $\infty$) in the summation.

Step 3. We compute $\lambda_1$ from (2.4), and $\phi(b)$ from (2.5).

Step 4. Let $t_l^{(n)}$ and $t_u^{(n)}$ denote the $(n+1)$th estimates of the lower and upper bounds of $t_q$ respectively, and $t_q^{(n)} = \left(t_l^{(n)} + t_u^{(n)}\right)/2$ be the $(n+1)$th approximation of $t_q$. Initially, we let its lower bound $t_l^{(0)}$ be zero, and we compute its upper bound $t_u^{(0)}$ by using the right-hand side of the inequality (2.7). We remark that (2.7) is satisfied if $t_u^{(0)}$ can be computed.

Step 5. Let $h = t_q^{(n)}/m$, where $m$ denotes the number of subdivisions. For $k = 1, 2, 3, \ldots$, we use the finite sum,

$$
\begin{aligned}
&M^{(k)}(rh) \\
&= \frac{2\alpha}{L}\int_0^{rh}\sum_{n=1}^{Q}\left(\cos\frac{(2n-1)\pi b}{2L}\right)^2 \exp\left(-\frac{(2n-1)^2\pi^2(rh-\tau)}{4L^2}\right) f\left(M^{(k-1)}(\tau)\right)d\tau,
\end{aligned}
$$

where $r = 1, 2, 3, \ldots, m$ in the iterative process with $M^{(0)}(\tau) = 0$, and $M^{(k)}(0) = 0$. To compute $M^{(k+1)}(rh)$, we use the interpolation to approximate $M^{(k)}(\tau)$.

Step 6. At the $k$th iteration, if for some $r$, $M^{(k)}(rh) \geq c$, then $t_l^{(n+1)} = t_l^{(n)}$, and $t_u^{(n+1)} = t_q^{(n)}$. We go to Steps 4 and 5. If $\max_{r=0,1,2,\ldots,m}\left|M^{(k)}(rh) - M^{(k-1)}(rh)\right| < \delta$ which is a given tolerance, then the sequence $\left\{M^{(k)}(t)\right\}$ converges; in this case, let $t_l^{(n+1)} = t_q^{(n)}, t_u^{(n+1)} = t_u^{(n)}$, and go to Steps 4 and 5. We use the interpolation to approximate $M^{(k)}(\tau)$ if $\max_{r=0,1,2,\ldots,m}\left|M^{(k)}(rh) - M^{(k-1)}(rh)\right| \geq \delta$, and continue the iterative process for the $(k+1)$th iteration to obtain an upper bound and a lower bound before going to Steps 4 and 5.

Step 7. If $\left|t_u^{(n)} - t_l^{(n)}\right| < \epsilon$ for a given tolerance $\epsilon$, then $t_q^{(n)} = \left(t_l^{(n)} + t_u^{(n)}\right)/2$ is accepted as the final estimate of $t_q$.

For illustrations of the above computational scheme, let $f(u) = 1/(1-u)$. Then, $c = 1$. By a direct computation, we have $\sup_{0<w<c}(w/f(w)) = 1/4$. From (2.4) and (2.5),

$$\lambda_1 = \frac{\pi^2}{4L^2}, \text{ and } \phi(b) = \frac{\pi}{2L}\cos\left(\frac{\pi b}{2L}\right).$$

Since $t_l^{(0)} = 0$, it follows from the right-hand side of the inequality (2.7) that

$$t_u^{(0)} = \frac{1}{\alpha\,\phi(b) - \frac{1}{4}\lambda_1}\int_0^1 (1-w)\,dw = \frac{2}{4\,\alpha\,\phi(b) - \lambda_1}.$$

Using Steps 1 to 7 with $\epsilon = 10^{-4}$, $\delta = 10^{-6}$, $Q = 10$, and $m = 40$, and the quenching criteria given by Chan and Tragoonsirisak [3], we obtain the following tables for $t_q$ (to four significant figures).

First, we fix $L$ and $b$, and would like to study the effect of $\alpha$ on the quenching time. Let $L = 1$, and $b = 1/2$. From Theorems 3.1 and 2.5 of Chan and Tragoonsirisak

[3], $u$ quenches in a finite time for

$$\alpha > \frac{1}{L-b} \sup_{0<w<c} \left( \frac{w}{f(w)} \right),$$

that is $\alpha > 1/2$. We obtain the following table:

| $\alpha$ | $t_u^{(0)}$ | $t_q$ |
|---|---|---|
| .8000 | 1.840 | .7245 |
| .9000 | 1.306 | .5532 |
| 1.000 | 1.012 | .4427 |

We note that the quenching time $t_q$ is a decreasing function of $\alpha$. Physically, this means that the larger the source, the smaller the quenching time.

Next, we fix $\alpha$ and $L$, and study the effect of $b$ on the quenching time. From Corollaries 3.2 and 3.3 of Chan and Tragoonsirisak [3], if

$$L > \frac{1}{\alpha} \sup_{0<w<c} \left( \frac{w}{f(w)} \right)$$

and $b \in (0, b^*)$, where

$$b^* = L - \frac{1}{\alpha} \sup_{0<w<c} \left( \frac{w}{f(w)} \right),$$

then $u$ quenches in a finite time. Let $\alpha = 1$, and $L = 1$. Then for $u$ to quench in a finite time, the above conditions give $L > 1/4$, and $b^* = 3/4$. We obtain the following table:

| $b$ | $t_u^{(0)}$ | $t_q$ |
|---|---|---|
| .5000 | 1.012 | .4427 |
| .6000 | 1.632 | .6013 |
| .7000 | 5.193 | 1.335 |

We see that when $b$ is closer to $L$, the quenching time $t_q$ is larger. Physically, this means that the closer the nonlinear concentrated source to the boundary $x = L$ where $u = 0$, the larger the quenching time.

Lastly, we fix $\alpha$ and $b$, and study the effect of $L$ on the quenching time. From Theorem 3.1(ii) of Chan and Tragoonsirisak [3], if

$$L > b + \frac{1}{\alpha} \sup_{0<w<c} \left( \frac{w}{f(w)} \right),$$

then $u$ quenches in a finite time. Let $\alpha = 1$, and $b = 1/2$. Then for $u$ to quench in a finite time, the above condition requires $L > 3/4$. We obtain the following table:

| $L$ | $t_u^{(0)}$ | $t_q$ |
|---|---|---|
| 1.000 | 1.012 | .4427 |
| 1.100 | .8781 | .4093 |
| 1.200 | .8195 | .3929 |

We observe that when $L$ is further away from $b$, the quenching time $t_q$ is smaller. Physically, this means that the further away the nonlinear concentrated source to the boundary $x = L$ where $u = 0$, the smaller the quenching time.

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