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ASYMPTOTICALLY PRACTICAL STABILITY OF IMPULSIVE FUNCTIONAL DIFFERENTIAL SYSTEMS IN TERMS OF TWO MEASUREMENTS

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ABSTRACT. This paper deals with the practical stability problem for impulsive functional differential systems with finite delays in terms of two measurements. Some sufficient conditions which guarantee the uniformly asymptotically practical stability of the addressed systems are derived by using Lyapunov functions and the Razumikhin technique. Finally, two examples are given to show the effectiveness of the obtained results.

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1. INTRODUCTION

Recently, special interest was paid to the practical stability of differential systems arising in engineering, economics and neural networks, see [24, 25]. In fact, the desired state of a system may be mathematically unstable and yet the system may oscillate sufficiently near this state and its performance is acceptable. To deal with such situations, the notion of practical stability is useful. Based on the theory of impulsive differential systems, see [1–17], some results for practical stability of impulsive differential systems were obtained in the literature, see [18–27].

For asymptotical stability, in the sense of Lyapunov, the domain of attraction $h_0(t_0, x_0) < \delta$, where δ is related to ϵ, may not be large enough to allow the desired deviations to cancel out. However, asymptotically practical stability requires the given domain of attraction $h_0(t_0, x_0) < u$ to be independent of ϵ. Hence, in practice, asymptotically practical stability is more useful. In [21–26], the authors obtained some results for practical stability of ordinary differential systems or impulsive systems. Unfortunately, there are only a few results concerning uniformly asymptotically practical stability of impulsive functional differential systems. The purpose of this

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paper is to establish some criteria which guarantee uniformly asymptotically practical stability of impulsive functional differential systems by using Lyapunov functions and the Razumikhin technique. This work is organized as follows. In Section 2, we introduce some basic definitions and notations. In Section 3, the main results are presented. In Section 4, two examples are discussed to illustrate the results.

2. PRELIMINARIES

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, $\mathbb{R}^n$ the n-dimensional real space equipped with the Euclidean norm $\|\cdot\|$, and $\mathbb{Z}_+$ the set of positive integers. For any interval $I \subseteq \mathbb{R}$, set $C(I, \mathbb{R}^n) \triangleq \{\varphi : I \to \mathbb{R}^n \mid \varphi \text{ is continuous}\}$, and $PC(I, \mathbb{R}^n) \triangleq \{\varphi : I \to \mathbb{R}^n \mid \varphi(t^+) = \varphi(t) \text{ for } t \in I,\ \varphi(t^-) \text{ exists for } t \in I,\ \varphi(t^-) = \varphi(t) \text{ for all but the points } t_k \in I\}$, $\varphi(t^+)$ and $\varphi(t^-)$ denote the left limit and right limit of function $\varphi(t)$, respectively. For $\varphi \in PC([-\tau, 0], \mathbb{R}^n)$, the norm of φ is defined by $\|\varphi\| = \sup_{-\tau\le\theta\le 0} |\varphi(\theta)|$, where $-\infty < -\tau < 0$. The impulse times t_k satisfy $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = +\infty$. Let $\mathbb{R}_\tau^+ = [-\tau, \infty)$.

Consider the impulsive functional differential system:

$$
\begin{cases}
\dot{x}(t) = f(t, x_t), & t \ge t_0, t \ne t_k, k \in \mathbb{Z}_+, \\
\Delta x(t_k) = I_k(t_k^-, x(t_k^-)), & k \in \mathbb{Z}_+, \\
x(t_0 + s) = \varphi(s), & s \in [-\tau, 0],
\end{cases}
\tag{2.1}
$$

where $0 \le t_0 < t_1$, $\varphi \in PC([-\tau, 0], \mathbb{R}^n)$, $f \in C([t_k, t_{k+1}) \times \mathbb{D}, \mathbb{R}^n)$, $f(t, 0) = 0$, $\mathbb{D}$ is an open set in $PC([-\tau, 0], \mathbb{R}^n)$. For each $t \ge t_0$, $x_t \in \mathbb{D}$ is defined by $x_t(s) = x(t + s)$, $s \in [-\tau, 0]$. For each $k \in \mathbb{Z}_+$, $I_k \in C([-\tau, \infty) \times \mathbb{R}^n, \mathbb{R}^n)$, $I_k(t, 0) = 0$, and for any $\rho > 0$, there exists a $\rho_1 \in (0, \rho)$ such that $x \in S(\rho_1)$ implies that $x + I_k \in S(\rho)$, where $S(\rho) = \{x :| x |< \rho, x \in \mathbb{R}^n\}$.

In this paper, we assume that f and I_k satisfy certain conditions such that the solution of system (2.1) exists on $[t_0, +\infty)$ and is unique [22]. We denote by $x(t) = x(t, t_0, \varphi)$ the solution of system (2.1) with initial value (t_0, φ).

For convenience, we define the following classes of functions:
$K = \{w \in C(\mathbb{R}_+, \mathbb{R}_+) : w \text{ is strictly increasing and } w(0) = 0\}$;
$K_1 = \{w \in C(\mathbb{R}_+, \mathbb{R}_+) : w(0) = 0 \text{ and } w(s) > 0 \text{ for } s > 0\}$;
$K_2 = \{\psi \in C(\mathbb{R}_+, \mathbb{R}_+) : \psi \text{ is increasing and } \psi(s) < s \text{ for } s > 0\}$;
$\Gamma^n = \{h \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+) : \forall t \in \mathbb{R}_+, \inf_x h(t, x) = 0\}$;
$\Gamma_\tau^n = \{h \in C(\mathbb{R}_\tau^+ \times \mathbb{R}^n, \mathbb{R}_+) : \forall t \in \mathbb{R}_\tau^+, \inf_x h(t, x) = 0\}$.
$\tilde{h_0}(t, x_t) = \sup_{-\tau\le\theta\le 0} h_0(t + \theta, x_t(\theta))$, where $h_0 \in \Gamma_\tau^n$, $x_t \in PC([-\tau, 0], \mathbb{R}^n)$, $t \in \mathbb{R}_+$.

In addition, we introduce some definitions as follows:

Definition 2.1 ([16])**.** The function $V : [-\tau, \infty) \times \mathbb{D} \to \mathbb{R}_+$ belongs to class ν_0 if

(i) V is continuous on each of the sets $[t_{k-1}, t_k)\times\mathbb{D}$, $k \in \mathbb{Z}_+$, and $\lim_{(t,\varphi)\to(t_k^-,\psi)} V(t,\varphi) = V(t_k^-, \psi)$ exists;

(ii) $V(t,x)$ is locally Lipschitzian in x and for all $t \geq t_0$, $V(t,0) \equiv 0$.

Definition 2.2 ([16])**.** Given a function $V \in \nu_0$, for any $(t,\psi) \in [t_{k-1}, t_k) \times \mathbb{D}$, the upper right-hand Dini derivative of $V(t,x)$ along the solution of (2.1) is defined by

$$D^+V(t,\psi(0)) = \lim_{h\to 0^+} \sup\{V(t+h, \psi(0) + hf(t,\psi)) - V(t,\psi(0))\}/h.$$

Definition 2.3 ([24, 25])**.** Given two constants u, v, $0 < u < v$, and let $h_0 \in \Gamma_\tau^n$, $h \in \Gamma^n$. Then, the impulsive functional differential system (2.1) with respect to (u,v) is said to be

(S$_1$) $(\tilde{h}_0, h)$-practically stable, if given (u,v) with $0 < u < v$, we have $\tilde{h}_0(t_0, x_{t_0}) < u$ implies $h(t,x(t)) < v$, $t \geq t_0$ for some $t_0 \in \mathbb{R}_+$;

(S$_2$) $(\tilde{h}_0, h)$-uniformly practically stable if (S_1) holds for every $t_0 \in \mathbb{R}_+$;

(S$_3$) $(\tilde{h}_0, h)$-asymptotically practically stable, if (S_1) holds and for any $\epsilon > 0$ there exists $T = T(t_0, \epsilon) > 0$ such that $\tilde{h}_0(t_0, x_{t_0}) < u$ implies $h(t, x(t)) < \epsilon$, $t \geq t_0 + T$ for some $t_0 \in \mathbb{R}_+$;

(S$_4$) $(\tilde{h}_0, h)$-uniformly asymptotically practically stable if (S_2) holds and the latter part of (S_3) holds for a constant $T = T(\epsilon) > 0$ only dependent on ϵ.

3. MAIN RESULTS

Theorem 3.1. *Assume that there exist functions $\alpha, \beta, \phi, \omega \in K$, $g \in PC(\mathbb{R}_+, \mathbb{R}_+)$, $\psi \in K_2$, $V \in v_0$ such that*

(i) $0 < u < v$ *are given;*

(ii) $h_0 \in \Gamma_\tau^n$, $h \in \Gamma^n$, $h(t,x) \leq \phi(\tilde{h}_0(t,x_t))$ *whenever* $\tilde{h}_0(t,x_t) < u$;

(iii) $\beta(h(t,x)) \leq V(t,x) \leq \alpha(h_0(t,x))$ *for* $(t,x) \in [t_0 - \tau, \infty) \times S(\rho)$;

(iv) $V(t_k, x(t_k^-) + I_k(t_k^-, x(t_k^-))) \leq \psi(V(t_k^-, x(t_k^-)))$;

(v) $P(V(t,x(t))) \geq V(t+s, x(t+s))$, $s \in [-\tau, 0]$, $t \in [t_{k-1}, t_k)$, $k \in \mathbb{Z}_+$, *implies that*

$$D^+V(t,x(t)) \leq g(t)\omega(V(t,x(t))),$$

where $P(s) > \psi^{-1}(s)$, $s > 0$,

$$\sup_{s>0} \frac{\omega(s)}{s} \cdot \sup_{k\in\mathbb{Z}_+} \int_{t_{k-1}}^{t_k} g(s)ds + \sup_{s>0} \frac{\psi(s)}{s} < 1,$$

and $x(t)$ *is a solution of system* (2.1);

(vi) $\phi(u) < v, \alpha(u) < \psi(\beta(v))$.

Then the system (2.1) *with respect to* (u,v) *is* $(\tilde{h}_0, h)$*-uniformly asymptotically practically stable.*

Proof. Let

$$A \triangleq \sup_{k\in\mathbb{Z}_+}\int_{t_{k-1}}^{t_k} g(s)ds, \eta \triangleq \sup_{s>0}\frac{\omega(s)}{s}, q \triangleq \Big(\sup_{s>0}\frac{\psi(s)}{s}\Big)^{-1} > 1.$$

Since $\psi \in K_2$,

$$\inf_{s>0}\frac{\psi^{-1}(s)}{s} \geq q.$$

Then it follows from (v) that $\eta A + q^{-1} < 1$, which implies that

$$\frac{\omega(s)}{s}A + q^{-1} \leq \eta A + q^{-1} < 1, \quad s > 0.$$

Thus

$$\frac{1}{\omega(s)} > \frac{A}{s(1-q^{-1})}, \quad s > 0.$$

Now we show that $\ln q > 1 - q^{-1}$. Let $F(t) = \ln t - (1 - t^{-1})$, $t > 1$, and it can be deduced that $F^{'}(t) = \frac{t-1}{t^2} > 0$, $F(1) = 0$, and therefore, $F(t)$ is nondecreasing. Thus $\ln q > 1 - q^{-1}$, $q > 1$.

For any $t_0 \geq 0$, let $x(t) \doteq x(t, t_0, \varphi)$ be the solution of system (2.1) through (t_0, φ), where $(t_0, \varphi) \in \mathbb{R}_+ \times PC([-\tau, 0], \mathbb{R}^n)$, and $\tilde{h}_0(t_0, x_{t_0}) < u$. It suffices to show that

$$h(t, x(t)) < v, \quad t \geq t_0.$$

By conditions (ii) and (vi),

$$h(t_0, x(t_0)) \leq \phi(\tilde{h}_0(t_0, x_{t_0})) < \phi(u) < v.$$

Next we shall prove that

$$V(t, x(t)) \leq \psi^{-1}(\alpha(u)), \quad t \in [t_0 - \tau, +\infty). \tag{3.1}$$

For any $t \in [t_0 - \tau, t_0]$, there exists a $s \in [-\tau, 0]$, such that $t = t_0 + s$, and then from the definition of $\tilde{h}_0(t, x_t)$, we know that for $t \in [t_0 - \tau, t_0]$

$$h_0(t, x(t)) = h_0(t_0 + s, x(t_0 + s)) = h_0(t_0 + s, x_{t_0}(s)) \leq \tilde{h}_0(t_0, x_{t_0}) < u.$$

Thus for all $t \in [t_0 - \tau, t_0]$

$$V(t, x(t)) \leq \alpha(h_0(t, x(t))) \leq \alpha(\tilde{h}_0(t_0, x_{t_0})) < \alpha(u) < \psi^{-1}(\alpha(u)). \tag{3.2}$$

Now we show that

$$V(t, x(t)) \leq \psi^{-1}(\alpha(u)), t \in [t_0, t_1). \tag{3.3}$$

If it does not hold, then there exists a $r \in [t_0, t_1)$, such that $V(r, x(r)) > \psi^{-1}(\alpha(u))$. Let $r_2 = \inf\{t : V(t, x(t)) > \psi^{-1}(\alpha(u)), t \in [t_0, t_1)\}$. Since $V(t_0, x(t_0)) \leq \psi^{-1}(\alpha(u))$, it is clear that

$$r_2 > t_0, \quad V(r_2, x(r_2)) = \psi^{-1}(\alpha(u)).$$

Let $r_1 = \sup\{t : V(t, x(t)) \leq \alpha(u), t \in [t_0, r_2)\}$. Thus

$$V(r_1, x(r_1)) = \alpha(u), \alpha(u) \leq V(t, x(t)) \leq \psi^{-1}(\alpha(u)), \quad t \in [r_1, r_2].$$

By (3.2), we obtain that for any $t \in [r_1, r_2]$

$$P(V(t,x(t))) > \psi^{-1}(V(t,x(t))) \geq \psi^{-1}(\alpha(u)) \geq V(t+s, x(t+s)), \quad s \in [-\tau, 0].$$

Using condition (v), the inequality $D^+V(t,x(t)) \leq g(t)\omega(V(t,x(t)))$ holds for all $t \in [r_1, r_2]$. Hence we obtain

$$\int_{V(r_1,x(r_1))}^{V(r_2,x(r_2))} \frac{ds}{\omega(s)} \leq \int_{r_1}^{r_2} g(t)dt \leq \int_{t_0}^{t_1} g(t)dt \leq A. \tag{3.4}$$

On the other hand,

$$\begin{aligned}\int_{V(r_1,x(r_1))}^{V(r_2,x(r_2))} \frac{ds}{\omega(s)} &= \int_{\alpha(u)}^{\psi^{-1}(\alpha(u))} \frac{ds}{\omega(s)} \\ &> \frac{A}{1-q^{-1}} \int_{\alpha(u)}^{\psi^{-1}(\alpha(u))} \frac{ds}{s} \\ &= \frac{A}{1-q^{-1}} \ln \frac{\psi^{-1}(\alpha(u))}{\alpha(u)} \\ &\geq \frac{A}{1-q^{-1}} \ln \inf_{s>0} \frac{\psi^{-1}(s)}{s} \\ &= \frac{A}{1-q^{-1}} \ln q \\ &> A,\end{aligned}$$

which is a contradiction with the inequality (3.4) and thus (3.3) holds.

Then it follows from condition (iv) that

$$V(t_1, x(t_1^-) + I_1(t_1^-, x(t_1^-))) \leq \psi(V(t_1^-, x(t_1^-))) \leq \alpha(u).$$

Next, we claim that

$$V(t,x(t)) \leq \psi^{-1}(\alpha(u)), \quad t \in [t_1, t_2). \tag{3.5}$$

If this assertion is not true, then there exists a $r \in [t_1, t_2)$, such that $V(r, x(r)) > \psi^{-1}(\alpha(u))$. Let $r_4 = \inf\{t : V(t,x(t)) > \psi^{-1}(\alpha(u)), t \in [t_1, t_2)\}$. Since $V(t_1, x(t_1)) \leq \alpha(u) \leq \psi^{-1}(\alpha(u))$, we have

$$r_4 > t_1, V(r_4, x(r_4)) = \psi^{-1}(\alpha(u)).$$

Let $r_3 = \sup\{t : V(t,x(t)) \leq \alpha(u), t \in [t_1, r_4)\}$. Thus

$$V(r_3, x(r_3)) = \alpha(u), \alpha(u) \leq V(t,x(t)) \leq \psi^{-1}(\alpha(u)), \quad t \in [r_3, r_4].$$

Considering (3.2) and (3.4), we obtain for any $t \in [r_3, r_4]$

$$P(V(t,x(t))) > \psi^{-1}(V(t,x(t))) \geq \psi^{-1}(\alpha(u)) \geq V(t+s, x(t+s)), \quad s \in [-\tau, 0].$$

Using condition (v), the inequality $D^+V(t,x(t)) \le g(t)\omega(V(t,x(t)))$ holds for all $t \in [r_3, r_4]$. Hence

$$\begin{aligned} A < \frac{A}{1-q^{-1}} \ln q &\le \frac{A}{1-q^{-1}} \ln \frac{\psi^{-1}(\alpha(u))}{\alpha(u)} \\ &= \frac{A}{1-q^{-1}} \int_{\alpha(u)}^{\psi^{-1}(\alpha(u))} \frac{ds}{s} \\ &< \int_{V(r_3,x(r_3))}^{V(r_4,x(r_4))} \frac{ds}{\omega(s)} \\ &\le \int_{r_3}^{r_4} g(t)dt \\ &\le A, \end{aligned}$$

which is a contradiction and thus (3.5) holds.

Then from (iv), we get

$$V(t_2, x(t_2^-) + I_2(t_2^-, x(t_2^-))) \le \psi(V(t_2^-, x(t_2^-))) \le \alpha(u).$$

Similarly, it can be deduced that

$$V(t,x(t)) \le \psi^{-1}(\alpha(u)), \quad t \in [t_2, t_3).$$

By simple induction, we can prove that

$$V(t,x(t)) \le \psi^{-1}(\alpha(u)), t \in [t_k, t_{k+1}), \quad k \in \mathbb{Z}_+,$$

and

$$V(t_{k+1}, x(t_{k+1}^-) + I_{k+1}(t_{k+1}^-, x(t_{k+1}^-))) \le \psi(V(t_{k+1}^-, x(t_{k+1}^-))) \le \alpha(u) < \psi^{-1}(\alpha(u)).$$

It follows from condition (iii) and (vi) that

$$V(t,x(t)) \le \psi^{-1}(\alpha(u)) < \beta(v),$$

$$h(t,x(t)) \le \beta^{-1}(V(t,x(t))) < \beta^{-1}(\beta(v)) = v, \quad t \ge t_0.$$

This inequality implies that the system (2.1) with respect to (u,v) is $(\tilde{h}_0, h)$-uniformly practically stable.

Next, we show that the system (2.1) with respect to (u,v) is $(\tilde{h}_0, h)$-uniformly asymptotically practically stable. For any ϵ, $0 < \epsilon < v$, there exist numbers $a = a(\epsilon) > 0$, $0 < d < a$, such that

$$P(s) > \psi^{-1}(s) + a, \psi^{-1}(s) + a > \psi^{-1}(s+d), \quad s \in [\beta(\epsilon), \psi^{-1}(\alpha(u))].$$

Let $N = N(\epsilon)$ satisfy $\beta(\epsilon) + (N-1)d \le \psi^{-1}(\alpha(u)) \le \beta(\epsilon) + Nd$, and $T = (N-1)\lambda\tau$, where $\lambda \ge 1$. We shall prove that

$$V(t,x(t)) \le \psi^{-1}(\beta(\epsilon)), \quad t \ge t_0 + T.$$

In order to do this, we first prove that there exists a $T_1 \geq t_0$, such that

$$V(T_1, x(T_1)) \leq \beta(\epsilon) + (N-1)d. \tag{3.6}$$

If (3.6) does not hold, then for any $t \geq t_0$, $V(t, x(t)) > \beta(\epsilon) + (N-1)d$. Note that for $s \in [-\tau, 0]$

$$\begin{aligned} P(V(t,x(t))) &> \psi^{-1}(V(t,x(t))) + a \geq \psi^{-1}(\beta(\epsilon) + (N-1)d) + a \\ &> \psi^{-1}(\beta(\epsilon) + Nd) \geq \psi^{-1}(\alpha(u)) \geq V(t+s, x(t+s)). \end{aligned}$$

Thus

$$D^+V(t, x(t)) \leq g(t)\omega(V(t, x(t))), \quad t \geq t_0,$$

which implies that

$$\int_{t_0}^{t} D^+V(s, x(s)) \leq \int_{t_0}^{t} g(s)\omega(V(s, x(s)))ds. \tag{3.7}$$

Suppose $t \in [t_l, t_{l+1}), l \in \mathbb{Z}_+$, so from (3.7) and $\eta A + q^{-1} < 1$, it can be derived that

$$\begin{aligned} V(t,x(t)) &\leq V(t_0, x(t_0)) + \int_{t_0}^{t} g(s)\omega(V(s,x(s)))ds \\ &\quad + \sum_{t_0 < t_k \leq t} [V(t_k, x(t_k)) - V(t_k^-, x(t_k^-))] \\ &\leq \psi^{-1}(\alpha(u)) + \sum_{j=0}^{l} \int_{t_j}^{t_{j+1}} g(s)\omega(V(s,x(s)))ds \\ &\quad + \sum_{t_0 < t_k \leq t} [\psi(V(t_k^-, x(t_k^-))) - V(t_k^-, x(t_k^-))] \\ &\leq \psi^{-1}(\alpha(u)) + \omega(\psi^{-1}(\alpha(u))) \sum_{j=0}^{l} \int_{t_j}^{t_{j+1}} g(s)ds \\ &\quad + \sum_{t_0 < t_k \leq t} V(t_k^-, x(t_k^-))[\frac{\psi(V(t_k^-, x(t_k^-))}{V(t_k^-, x(t_k^-))} - 1] \\ &\leq \psi^{-1}(\alpha(u)) + \omega(\psi^{-1}(\alpha(u)))(l+1)A + l\psi^{-1}(\alpha(u))[\sup_{s>0} \frac{\psi(s)}{s} - 1] \\ &\leq \psi^{-1}(\alpha(u)) + \psi^{-1}(\alpha(u)) \frac{\omega(\psi^{-1}(\alpha(u)))}{\psi^{-1}(\alpha(u))}(l+1)A + l\psi^{-1}(\alpha(u))[q^{-1} - 1] \\ &\leq \psi^{-1}(\alpha(u)) + \psi^{-1}(\alpha(u))\eta(l+1)A + l\psi^{-1}(\alpha(u))[q^{-1} - 1] \\ &\leq \psi^{-1}(\alpha(u)) + \psi^{-1}(\alpha(u))l\Big(\eta A + q^{-1} - 1\Big) + \psi^{-1}(\alpha(u))\eta A \\ &\to -\infty, \text{ as } l \to +\infty, \end{aligned}$$

which is a contradiction. Thus, there exists a $T_1 \geq t_0$, such that (3.6) holds.

Next we prove that

$$V(t, x(t)) \leq \psi^{-1}(\beta(\epsilon) + (N-1)d), \quad t \geq T_1. \tag{3.8}$$

Let $m = \min\{m \in \mathbb{Z}_+ : t_m \geq T_1\}$, and we show that

$$V(t, x(t)) \leq \psi^{-1}(\beta(\epsilon) + (N-1)d), \quad t \in [T_1, t_m). \tag{3.9}$$

If (3.9) does not hold, then there is a $r \in [T_1, t_m)$ such that

$$V(r, x(r)) > \psi^{-1}(\beta(\epsilon) + (N-1)d).$$

Let $r^\star = \inf\{t : V(t, x(t)) > \psi^{-1}(\beta(\epsilon) + (N-1)d), t \in [T_1, t_m)\}$. Since

$$V(T_1, x(T_1)) \leq \beta(\epsilon) + (N-1)d \leq \psi^{-1}(\beta(\epsilon) + (N-1)d),$$

we have

$$r^\star > t_1, V(r^\star, x(r^\star)) = \psi^{-1}(\beta(\epsilon) + (N-1)d).$$

Let $\hat{r} = \sup\{t : V(t, x(t)) \leq \beta(\epsilon) + (N-1)d, t \in [T_1, r^\star)\}$. Note

$$V(r^\star, x(r^\star)) = \psi^{-1}(\beta(\epsilon) + (N-1)d) > \beta(\epsilon) + (N-1)d.$$

Thus

$$\hat{r} < r^\star, \quad V(\hat{r}, x(\hat{r})) = \beta(\epsilon) + (N-1)d,$$

$$\beta(\epsilon) + (N-1)d \leq V(t, x(t)) \leq \psi^{-1}(\beta(\epsilon) + (N-1)d), \quad t \in [\hat{r}, r^\star].$$

Thus for any $t \in [\hat{r}, r^\star]$

$$\begin{aligned} P(V(t, x(t))) &> \psi^{-1}(V(t, x(t))) + a \geq \psi^{-1}(\beta(\epsilon) + (N-1)d) + a \\ &> \psi^{-1}(\beta(\epsilon) + Nd) \geq \psi^{-1}(\alpha(u)) \geq V(t+s, x(t+s)), \quad s \in [-\tau, 0]. \end{aligned}$$

Using condition (v), the inequality $D^+V(t, x(t)) \leq g(t)\omega(V(t, x(t)))$ holds for all $t \in [\hat{r}, r^\star]$. Hence we obtain

$$\begin{aligned} A < \frac{A}{1-q^{-1}} \ln q &\leq \frac{A}{1-q^{-1}} \ln \frac{\psi^{-1}(\beta(\epsilon) + (N-1)d)}{\beta(\epsilon) + (N-1)d} \\ &= \frac{A}{1-q^{-1}} \int_{\beta(\epsilon)+(N-1)d}^{\psi^{-1}(\beta(\epsilon)+(N-1)d)} \frac{ds}{s} \\ &< \int_{V(\hat{r}, x(\hat{r}))}^{V(r^\star, x(r^\star))} \frac{ds}{\omega(s)} \\ &\leq \int_{\hat{r}}^{r^\star} g(t)dt \\ &\leq A, \end{aligned}$$

which is a contradiction and thus (3.9) holds.

Then from condition (iv), we get

$$V(t_m, x(t_m^-) + I_m(t_m^-, x(t_m^-))) \leq \psi(V(t_m^-, x(t_m^-))) \leq \beta(\epsilon) + (N-1)d.$$

Similarly, it can be deduced that

$$V(t, x(t)) \leq \psi^{-1}(\beta(\epsilon) + (N-1)d), \quad t \in [t_m, t_{m+1}).$$

By simple induction, one may derive that

$$V(t,x(t)) \le \psi^{-1}(\beta(\epsilon)+(N-1)d), \quad t\in[t_k,t_{k+1}), \quad k\ge m,$$

and

$$V(t_{k+1},x(t_{k+1}^-)+I_{k+1}(t_{k+1}^-,x(t_{k+1}^-))) \le \psi(V(t_{k+1}^-,x(t_{k+1}^-))) \le \beta(\epsilon)+(N-1)d.$$

Thus (3.8) holds.

Now we prove that there exists a $T_2 \ge T_1+\lambda\tau, \lambda\ge 1$, such that

$$V(T_2,x(T_2)) \le \beta(\epsilon)+(N-2)d. \tag{3.10}$$

If (3.10) does not hold, then for any $t\ge T_1+\lambda\tau, V(t,x(t)) > \beta(\epsilon)+(N-2)d$. Note that for $s\in[-\tau,0]$

$$\begin{aligned} P(V(t,x(t))) &> \psi^{-1}(V(t,x(t)))+a \ge \psi^{-1}(\beta(\epsilon)+(N-2)d)+a \\ &> \psi^{-1}(\beta(\epsilon)+(N-1)d) \ge V(t+s,x(t+s)). \end{aligned}$$

Hence, it follows from condition (v) that

$$D^+V(t,x(t)) \le g(t)\omega(V(t,x(t))), \quad t\ge T_1+\lambda\tau,$$

which implies that

$$\int_{T_1+\lambda\tau}^t D^+V(s,x(s)) \le \int_{T_1+\lambda\tau}^t g(s)\omega(V(s,x(s)))ds. \tag{3.11}$$

Suppose $T_1+\lambda\tau\in[t_{\hat n-1},t_{\hat n})$, $\hat n\in\mathbb{Z}_+$, $t\in[t_{\hat n+i-1},t_{\hat n+i})$, $i\in\mathbb{Z}_+$, so from (3.11) and $\eta A+q^{-1}<1$, it can be derived that

$$\begin{aligned} V(t,x(t)) &\le V(T_1+\lambda\tau,x(T_1+\lambda\tau)) + \int_{T_1+\lambda\tau}^t g(s)\omega(V(s,x(s)))ds \\ &\quad + \sum_{T_1+\lambda\tau<t_k\le t}[V(t_k,x(t_k)-V(t_k^-,x(t_k^-))] \\ &\le \psi^{-1}(\alpha(u)) + \sum_{j=0}^{i}\int_{t_{n+j-1}}^{t_{n+j}} g(s)\omega(V(s,x(s)))ds \\ &\quad + \sum_{T_1+\lambda\tau<t_k\le t}[\psi(V(t_k^-,x(t_k^-))-V(t_k^-,x(t_k^-))] \\ &\le \psi^{-1}(\alpha(u)) + \omega(\psi^{-1}(\alpha(u)))\sum_{j=0}^{i}\int_{t_{n-1+j}}^{t_{n+j}} g(s)ds \\ &\quad + \sum_{T_1+\lambda\tau<t_k\le t} V(t_k^-,x(t_k^-))[\frac{\psi(V(t_k^-,x(t_k^-))}{V(t_k^-,x(t_k^-)}-1] \\ &\le \psi^{-1}(\alpha(u)) + \omega(\psi^{-1}(\alpha(u)))(i+1)A + i\psi^{-1}(\alpha(u))[\sup_{s>0}\frac{\psi(s)}{s}-1] \\ &\le \psi^{-1}(\alpha(u)) + \psi^{-1}(\alpha(u))\frac{\omega(\psi^{-1}(\alpha(u)))}{\psi^{-1}(\alpha(u))}(i+1)A + i\psi^{-1}(\alpha(u))[q^{-1}-1] \end{aligned}$$

$$\begin{aligned}
&\leq \psi^{-1}(\alpha(u)) + \psi^{-1}(\alpha(u))\eta(i+1)A + i\psi^{-1}(\alpha(u))[q^{-1}-1] \\
&\leq \psi^{-1}(\alpha(u)) + \psi^{-1}(\alpha(u))i\Big(\eta A + q^{-1} - 1\Big) + \psi^{-1}(\alpha(u))\eta A, \\
&\to -\infty, \text{as } l \to +\infty,
\end{aligned}$$

which is a contradiction. Therefore, there exists a $T_2 \geq T_1 + \lambda\tau$ such that (3.10) holds. Next we shall show that

$$V(t, x(t)) \leq \psi^{-1}(\beta(\epsilon) + (N-2)d), \quad t \geq T_2. \tag{3.12}$$

Let $n = \min\{n \in \mathbb{Z}_+ : t_n \geq T_2\}$, and we claim that

$$V(t, x(t)) \leq \psi^{-1}(\beta(\epsilon) + (N-2)d), \quad t \in [T_2, t_n). \tag{3.13}$$

If (3.13) does not hold, then there is a $r \in [T_2, t_n)$ such that

$$V(t, x(t)) > \psi^{-1}(\beta(\epsilon) + (N-2)d).$$

Let $\bar{r} = \inf\{t : V(t, x(t)) > \psi^{-1}(\beta(\epsilon) + (N-2)d), t \in [T_2, t_n)\}$. Since $V(T_2, x(T_2)) \leq \epsilon) + (N-2)d$, we have

$$\bar{r} > T_2, V(\bar{r}, x(\bar{r})) = \psi^{-1}(\beta(\epsilon) + (N-2)d).$$

Let $\tilde{r} = \sup\{t : V(t, x(t)) \leq \beta(\epsilon) + (N-2)d, t \in [T_2, \bar{r})\}$. Note

$$V(\bar{r}, x(\bar{r})) = \psi^{-1}(\beta(\epsilon) + (N-2)d) > \beta(\epsilon) + (N-2)d.$$

Thus

$$\tilde{r} < \bar{r}, V(\tilde{r}, x(\tilde{r})) = \beta(\epsilon) + (N-2)d,$$
$$\beta(\epsilon) + (N-2)d \leq V(t, x(t)) \leq \psi^{-1}(\beta(\epsilon) + (N-2)d), \quad t \in [\tilde{r}, \bar{r}].$$

Thus for any $t \in [\tilde{r}, \bar{r}]$

$$\begin{aligned}
P(V(t, x(t))) &> \psi^{-1}(V(t, x(t))) + a \geq \psi^{-1}(\beta(\epsilon) + (N-2)d) + a \\
&> \psi^{-1}(\beta(\epsilon) + (N-1)d) \geq V(t+s, x(t+s)), \quad s \in [-\tau, 0].
\end{aligned}$$

Using condition (v), the inequality $D^+V(t, x(t)) \leq g(t)\omega(V(t, x(t)))$ holds for all $t \in [\tilde{r}, \bar{r}]$. Hence we obtain

$$\begin{aligned}
A < \frac{A}{1-q^{-1}} \ln q &< \frac{A}{1-q^{-1}} \ln \frac{\psi^{-1}(\alpha(u))}{\alpha(u)}) \\
&< \frac{A}{1-q^{-1}} \int_{\alpha(u))}^{\psi^{-1}(\alpha(u))} \frac{ds}{s} \\
&< \int_{V(\tilde{r}, x(\tilde{r}))}^{V(\bar{r}, x(\bar{r}))} \frac{du}{\omega(u)} \\
&\leq \int_{\tilde{r}}^{\bar{r}} g(t)dt \\
&\leq A,
\end{aligned}$$

which is a contradiction and thus (3.13) holds.

Then from condition (iv), we obtain

$$V(t_n, x(t_n^-) + I_n(t_n^-, x(t_n^-))) \le \psi(V(t_n^-, x(t_n^-))) \le \beta(\epsilon) + (N-2)d.$$

Similarly, we have

$$V(t, x(t)) \le \psi^{-1}(\beta(\epsilon) + (N-2)d), \quad t \in [t_n, t_{n+1}).$$

By simple induction, one may derive that

$$V(t, x(t)) \le \psi^{-1}(\beta(\epsilon) + (N-2)d), \quad t \in [t_k, t_{k+1}), \quad k \ge n,$$

and

$$V(t_{k+1}, x(t_{k+1}^-) + I_{k+1}(t_{k+1}^-, x(t_{k+1}^-))) \le \psi(V(t_{k+1}^-, x(t_{k+1}^-))) \le \beta(\epsilon) + (N-2)d).$$

Thus (3.12) holds. Similarly, we can prove that there exists a $T_3 \ge T_2 + \lambda\tau$, $\lambda \ge 1$, such that

$$V(T_3, x(T_3)) \le \beta(\epsilon) + (N-3)d,$$

and

$$V(t, x(t)) \le \psi^{-1}(\beta(\epsilon) + (N-3)d), \quad t \ge T_3.$$

By simple induction, we can prove, in general, that

$$V(T_j, x(T_j)) \le \beta(\epsilon) + (N-j)d,$$

and

$$V(t, x(t)) \le \psi^{-1}(\beta(\epsilon) + (N-j)d), \quad t \ge T_j, \quad j = 1, 2, \ldots, N.$$

Therefore, when choosing $j = N$, we obtain

$$V(t, x(t)) \le \psi^{-1}(\beta(\epsilon)), \quad t \ge T_N,$$

where $T_N \ge t_0 + (N-1)\lambda\tau$. Therefore

$$h(t, x(t)) \le \beta^{-1}(\psi^{-1}(\beta(\epsilon))), \quad t \ge t_0 + T,$$

where $T = (N-1)\lambda\tau$.

The proof is complete. □

Theorem 3.2. *Assume that there exist functions $\alpha, \beta, \phi \in K$, $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$, $P \in K_1, V \in v_0$ such that*

(i) $0 < u < v$ *are given;*
(ii) $h_0 \in \Gamma_\tau^n$, $h \in \Gamma^n$, $h(t,x) \le \phi(\tilde{h}_0(t, x_t))$ *whenever* $\tilde{h}_0(t, x_t) < u$*;*
(iii) $\beta(h(t,x)) \le V(t,x) \le \alpha(h_0(t,x))$ *for* $(t,x) \in [t_0 - \tau, \infty) \times S(\rho)$*;*
(iv) $V(t_k, x(t_k^-) + I_k(t_k^-, x(t_k^-))) \le (1+\beta_k)V(t_k^-, x(t_k^-))$*, where* $\beta_k \ge 0, \sum_{k=1}^{\infty} \beta_k < \infty$*;*

(v) *$P(V(t,x(t))) \geq V(t+s,x(t+s))$, $s \in [-\tau,0]$, $t \in [t_{k-1}, t_k)$, $k \in \mathbb{Z}_+$, implies that*

$$D^+V(t,x(t)) \leq -\omega(V(t,x(t))),$$

with $P(s) > Ms$, $s > 0$, $M = \prod_{k=1}^{\infty}(1+\beta_k) < \infty$, where $x(t)$ is a solution of system (2.1)*;*

(vi) *$\phi(u) < v$, $M\alpha(u) < \beta(v)$.*

Then the system (2.1) *with respect to (u,v) is $(\tilde{h}_0,h)$-uniformly asymptotically practically stable.*

Proof. For any $t_0 \geq 0$, let $x(t) \doteq x(t,t_0,\varphi)$ be the solution of system (2.1) through (t_0,φ), where $(t_0,\varphi) \in \mathbb{R}_+ \times PC([-\tau,0],\mathbb{R}^n)$, and $\tilde{h}_0(t_0,x_{t_0}) < u$. It suffices to show that

$$h(t,x(t)) < v, \quad t \geq t_0.$$

By conditions (ii) and (vi),

$$h(t_0,x(t_0)) \leq \phi(\tilde{h}_0(t_0,x_{t_0})) < \phi(u) < v.$$

Next we prove that

$$V(t,x(t)) \leq M\alpha(u), \quad t \in [t_0-\tau,+\infty).$$

noindent For any $t \in [t_0-\tau,t_0]$, there exists a $s \in [-\tau,0]$, such that $t = t_0+s$, and then from the definition of $\tilde{h}_0(t,x_t)$, we know that for $t \in [t_0-\tau,t_0]$

$$h_0(t,x(t)) = h_0(t_0+s,x(t_0+s)) = h_0(t_0+s,x_{t_0}(s)) \leq \tilde{h}_0(t_0,x_{t_0}) < u.$$

Thus for all $t \in [t_0-\tau,t_0]$

$$V(t,x(t)) \leq \alpha(h_0(t,x(t))) \leq \alpha(\tilde{h}_0(t_0,x_{t_0})) < \alpha(u) < M\alpha(u).$$

In particular,

$$V(t_0,x(t_0)) < \alpha(u) < M\alpha(u).$$

Now, we show that

$$V(t,x(t)) \leq \alpha(u), \quad t \in [t_0,t_1). \tag{3.14}$$

If it does not hold, then there exists a $r \in [t_0,t_1)$, such that

$$V(r,x(r)) > \alpha(u).$$

Let $r_1 = \inf\{t : V(t,x(t)) > \alpha(u), t \in [t_0,t_1)\}$. Since $V(t_0,x(t_0)) \leq \alpha(u)$, it is clear that

$$r_1 > t_0, V(r_1,x(r_1)) = \alpha(u), D^+V(r_1,x(r_1)) \geq 0.$$

Thus for $s \in [-\tau,0]$

$$P(V(r_1,x(r_1))) > MV(r_1,x(r_1)) \geq \alpha(u) \geq V(r_1+s,x(r_1+s)).$$

By condition (v), we have that

$$D^+(V(r_1, x(r_1))) \le -\omega(V(r_1, x(r_1))) < 0,$$

which is a contradiction. Thus (3.14) holds. From (3.14) and condition (iii), we obtain

$$V(t_1, x(t_1)) \le (1+\beta_1)V(t_1^-, x(t_1^-)) \le (1+\beta_1)\alpha(u).$$

Next, we show that

$$V(t, x(t)) \le (1+\beta_1)\alpha(u), \quad t \in [t_1, t_2). \tag{3.15}$$

If this assertion is not true, then there exists a $r \in [t_1, t_2)$, such that

$$V(r, x(r)) > (1+\beta_1)\alpha(u).$$

Let $r_2 = \inf\{t : V(t, x(t)) > (1+\beta_1)\alpha(u), t \in [t_1, t_2)\}$. Since $V(t_1, x(t_1)) \le (1+\beta_1)\alpha(u)$, we get

$$r_2 > t_1, V(r_2, x(r_2)) = (1+\beta_1)\alpha(u), D^+V(r_2, x(r_2)) \ge 0.$$

Thus for $s \in [-\tau, 0]$

$$P(V(r_2, x(r_2))) > MV(r_2, x(r_2)) \ge (1+\beta_1)\alpha(u) \ge V(r_2 + s, x(r_2 + s)).$$

By condition (v), we have that

$$D^+(V(r_2, x(r_2))) \le -\omega(V(r_2, x(r_2))) < 0,$$

which is a contradiction. Thus (3.15) holds.

Considering (3.15) and condition (iii), it can be deduced that

$$V(t_2, x(t_2)) \le (1+\beta_2)V(t_2^-, x(t_2^-)) \le (1+\beta_1)(1+\beta_2)\alpha(u).$$

By simple induction, we have that

$$V(t, x(t)) \le (1+\beta_1)(1+\beta_2)\cdots(1+\beta_k)\alpha(u), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{Z}_+,$$

which, together with (3.14) yields

$$V(t, x(t)) \le M\alpha(u) < \beta(v), \quad t \ge t_0.$$

Therefore from condition (iii), we have that

$$h(t, x(t)) \le v, \quad t \ge t_0.$$

Thus the system (2.1) with respect to (u, v) is $(\tilde{h}_0, h)$-uniformly practically stable.

Next, we show that the system (2.1) with respect to (u, v) is $(\tilde{h}_0, h)$-uniformly asymptotically practically stable. For any ϵ, $0 < \epsilon < v$, there exists number $d = d(\epsilon) > 0$ such that

$$P(s) > Ms + d, \quad s \in \left[\frac{\beta(\epsilon)}{M}, M\alpha(u)\right].$$

Let $N = N(\epsilon)$ be the smallest integer such that

$$\frac{\beta(\epsilon) + Nd}{M} \geq M\alpha(u),$$

and

$$\gamma = \inf_{\frac{\beta(\epsilon)}{M} \leq s \leq M\alpha(u)} \omega(s), \quad h = \max\left\{\frac{M\alpha(u)(1+\bar{M})}{\gamma}, \tau\right\},$$

where $\bar{M} = \sum_{k=1}^{\infty} \beta_k$. Let $T = T(\epsilon) = (2N-1)h$, and we shall prove that

$$V(t, x(t)) \leq \beta(\epsilon), \quad t \geq t_0 + T.$$

To this end, we first prove that

$$V(t, x(t)) \leq \beta(\epsilon) + (N-1)d, \quad t \geq t_0 + h. \tag{3.16}$$

In fact, when $t \in [t_0, t_0 + h]$, there exists a $r \in [t_m, t_{m+1}) \subset [t_0, t_0 + h]$, $m \in \mathbb{Z}_+$, such that

$$V(r, x(r)) \leq \frac{\beta(\epsilon) + (N-1)d}{M}. \tag{3.17}$$

If (3.17) does not hold, it is clear that for any $t \in [t_0, t_0 + h]$

$$V(t, x(t)) > \frac{\beta(\epsilon) + (N-1)d}{M},$$

$$\frac{\beta(\epsilon)}{M} \leq V(t, x(t)) \leq M\alpha(u).$$

Thus for $s \in [-\tau, 0]$

$$\begin{aligned} P(V(t, x(t))) &\geq MV(t, x(t)) + d \geq \beta(\epsilon) + (N-1)d + d = \beta(\epsilon) + Nd \\ &\geq M\alpha(u) \geq V(t+s, x(t+s)). \end{aligned}$$

It follows from condition (v) that

$$D^+V(t, x(t)) \leq -\omega(V(t, x(t))) \leq -\gamma, \quad t \in [t_0, t_0 + h],$$

which implies that

$$\int_{t_0}^{t} D^+V(s, x(s)) \leq \int_{t_0}^{t} -\gamma ds, \quad t \in [t_0, t_0 + h].$$

Consequently, for $t \in [t_0, t_0 + h]$

$$\begin{aligned} V(t, x(t)) &\leq V(t_0, x(t_0)) + \sum_{t_0 < t_k \leq t} [V(t_k, x(t_k)) - V(t_k^-, x(t_k^-))] - \gamma(t - t_0) \\ &\leq M\alpha(u) + \sum_{t_0 < t_k \leq t} \beta_k V(t_k^-, x(t_k^-)) - \gamma(t - t_0). \\ &\leq M\alpha(u) + \bar{M}M\alpha(u) - \gamma(t - t_0). \end{aligned}$$

In particular,

$$V(t_0 + h, x(t_0 + h)) \leq M\alpha(u)(1 + \bar{M}) - \gamma h \leq 0,$$

which is a contradiction. Hence when $t \in [t_0, t_0+h]$, there exists a $r \in [t_m, t_{m+1}) \subset [t_0, t_0+h], m \in \mathbb{Z}_+$, such that

$$V(r, x(r)) \le \frac{\beta(\epsilon) + (N-1)d}{M}.$$

Then, we claim that

$$V(t, x(t)) \le \frac{\beta(\epsilon) + (N-1)d}{M}, \quad t \in [r, t_{m+1}). \tag{3.18}$$

If (3.18) does not hold, there exists a $\hat{r} \in [r, t_{m+1})$, such that

$$V(\hat{r}, x(\hat{r})) > \frac{\beta(\epsilon) + (N-1)d}{M}.$$

Let $\tilde{r} = \inf\{t : V(t, x(t)) > \frac{\beta(\epsilon)+(N-1)d}{M}, t \in [r, t_{m+1})\}$. Since

$$V(r, x(r)) \le \frac{\beta(\epsilon) + (N-1)d}{M},$$

we have that

$$\tilde{r} > r, \quad V(\tilde{r}, x(\tilde{r})) = \frac{\beta(\epsilon) + (N-1)d}{M}, \quad D^+(V(\tilde{r}, x(\tilde{r}))) \ge 0.$$

Noting that $\frac{\beta(\epsilon)}{M} \le V(\tilde{r}, x(\tilde{r})) \le M\alpha(u)$, thus for $s \in [-\tau, 0]$

$$\begin{aligned} P(V(\tilde{r}, x(\tilde{r}))) &\ge MV(\tilde{r}, x(\tilde{r})) + d \ge \beta(\epsilon) + (N-1)d + d = \beta(\epsilon) + Nd \\ &\ge M\alpha(u) \ge V(\tilde{r}+s, x(\tilde{r}+s)). \end{aligned}$$

It follows from condition (v) that

$$D^+(V(\tilde{r}, x(\tilde{r}))) \le -\omega(V(\tilde{r}, x(\tilde{r}))) < 0,$$

which is a contradiction. Thus (3.18) holds.

Considering (3.18) and condition (iii), it can be deduced that

$$V(t_{m+1}, x(t_{m+1})) \le (1+\beta_{m+1})V(t_{m+1}^-, x(t_{m+1}^-)) \le (1+\beta_{m+1})\frac{\beta(\epsilon) + (N-1)d}{M}.$$

Similarly, we may show

$$V(t, x(t)) \le (1+\beta_{m+1})\frac{\beta(\epsilon) + (N-1)d}{M}, \quad t \in [t_{m+1}, t_{m+2}),$$

and

$$V(t_{m+2}, x(t_{m+2})) \le (1+\beta_{m+2})V(t_{m+2}^-, x(t_{m+2}^-)) \le (1+\beta_{m+2})(1+\beta_{m+1})\frac{\beta(\epsilon) + (N-1)d}{M}.$$

By simple induction, we can prove in general that

$$\begin{aligned} V(t, x(t)) &\le \prod_{j=1}^{i}(1+\beta_{m+j})\frac{\beta(\epsilon) + (N-1)d}{M} \\ &\le \beta(\epsilon) + (N-1)d, \quad t \in [t_{m+i}, t_{m+i+1}), \quad i \in \mathbb{Z}_+. \end{aligned}$$

Thus, (3.16) holds.

Next, we prove that

$$V(t,x(t)) \le \beta(\epsilon) + (N-2)d, \quad t \ge t_0 + 3h. \tag{3.19}$$

In fact, when $t \in [t_0+2h, t_0+3h]$, there exists a $\bar{r} \in [t_n, t_{n+1}) \subset [t_0+2h, t_0+3h]$, $n \in \mathbb{Z}_+$, such that

$$V(\bar{r}, x(\bar{r})) \le \frac{\beta(\epsilon) + (N-2)d}{M}. \tag{3.20}$$

If (3.20) does not hold, then for any $t \in [t_0+2h, t_0+3h]$

$$V(t,x(t)) > \frac{\beta(\epsilon) + (N-2)d}{M},$$

$$\frac{\beta(\epsilon)}{M} \le V(t,x(t)) \le M\alpha(u),$$

and thus for $s \in [-\tau, 0]$

$$\begin{aligned} P(V(t,x(t))) &\ge MV(t,x(t)) + d \ge \beta(\epsilon) + (N-2)d + d > \beta(\epsilon) + (N-1)d \\ &\ge V(t+s, x(t+s)). \end{aligned}$$

By condition (v), we have that

$$D^+V(t,x(t)) \le -\omega(V(t,x(t))) \le -\gamma, \quad t \in [t_0+2h, t_0+3h],$$

which implies that

$$\int_{t_0+2h}^{t} D^+V(s,x(s)) \le \int_{t_0+2h}^{t} -\gamma ds, \quad t \in [t_0+2h, t_0+3h].$$

Therefore for any $t \in [t_0+2h, t_0+3h]$

$$\begin{aligned} V(t,x(t)) &\le V(t_0, x(t_0)) + \sum_{t_0+2h<t_k\le t} [V(t_k, x(t_k)) - V(t_k^-, x(t_k^-))] - \gamma(t - t_0 - 2h) \\ &\le M\alpha(u) + \sum_{t_0+2h<t_k\le t} \beta_k V(t_k^-, x(t_k^-)) - \gamma(t - t_0 - 2h) \\ &\le M\alpha(u) + \bar{M}M\alpha(u) - \gamma(t - t_0 - 2h). \end{aligned}$$

In particular,

$$V(t_0+3h, x(t_0+3h)) \le M\alpha(u)(1+\bar{M}) - \gamma h \le 0,$$

which is a contradiction. Hence when $t \in [t_0+2h, t_0+3h]$, there exists a $\bar{r} \in [t_n, t_{n+1}) \subset [t_0+2h, t_0+3h], n \in \mathbb{Z}_+$, such that

$$V(\bar{r}, x(\bar{r})) \le \frac{\beta(\epsilon) + (N-2)d}{M}.$$

Similarly, we can prove that (3.19) holds. By simple induction we have that

$$V(t,x(t)) \le \beta(\epsilon) + (N-i)d, t \ge t_0 + (2i-1)h, \quad i = 1, 2, \cdots, N.$$

Therefore, when choosing $i = N$, we obtain

$$V(t,x(t)) \le \beta(\epsilon), \quad t \ge t_0 + T.$$

From condition (iii), we have that

$$h(t, x(t)) \le \epsilon, \quad t \ge t_0 + T.$$

The proof is complete. □

4. APPLICATIONS

The following illustrative examples will demonstrate the effectiveness of our results.

Example 4.1. Consider the following impulsive functional differential system:

$$\begin{cases} \dot{x}(t) = a(t)x(t) + \int_{t-\tau(t)}^{t} b(s)x(s)ds, & t \ge t_0, t \ne t_k, k \in \mathbb{Z}_+, \\ x(t_k) = qx(t_k^-), & k \in \mathbb{Z}_+, \\ x(t_0 + s) = \varphi(s), & s \in [-\tau, 0], \end{cases} \tag{4.1}$$

where $q < 1$, $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b \in C(\mathbb{R}, \mathbb{R})$, $\tau(t) \in C([t_0, +\infty), [0, \tau])$, τ is a nonnegative constant. For convenience, we consider $h_0(x) = h(x) = \|x\|$.

Property 4.1. Given constants u, v satisfying $u < qv$, and a constant $\lambda > \frac{1}{q}$. Then the system (4.1) with respect to (u, v) is $(\tilde{h}_0, h)$-uniformly asymptotically practically stable if

$$\sup_{k \in \mathbb{Z}_+} \int_{t_{k-1}}^{t_k} g(s)ds < 1 - q,$$

where $g(t) = a(t) + \lambda \int_{t-\tau}^{t} b(s)ds$.

Proof. Choose $V(t, x(t)) = \|x(t)\|$, where $x(t)$ is a solution of system (4.1). Let $\alpha(s) = \beta(s) = s$, $P(s) = \lambda s$, $\omega(s) = s$, $\psi(s) = qs$, $s > 0$, and then

$$\begin{aligned} D^+V(t, x(t)) &= \|a(t)x(t) + \int_{t-\tau(t)}^{t} b(s)x(s)ds\| \\ &\le a(t)\|x(t)\| + \int_{t-\tau}^{t} \|b(s)x(s)\|ds \\ &\le a(t)V(t, x(t)) + \int_{t-\tau}^{t} \|b(s)\|V(s, x(s))ds \\ &\le a(t)V(t, x(t)) + \int_{t-\tau}^{t} \|b(s)\|P(V(t, x(t)))ds \\ &= a(t)V(t, x(t)) + \int_{t-\tau}^{t} \|b(s)\|\lambda V(t, x(t))ds \\ &= \Big(a(t) + \lambda \int_{t-\tau}^{t} \|b(s)\|ds\Big)V(t, x(t)) \\ &= g(t)\omega(V(t, x(t))). \end{aligned}$$

By Theorem 3.1, the above property can be easily derived. □

Remark 4.1. It can be found that the system (4.1) is not asymptotically stable without impulsive effects. However, under the impulsive control, the system (4.1) can be asymptotically practically stable. Thus, our results are more useful and effective in practice.

Example 4.2. Consider the following impulsive functional differential system:

$$
\begin{cases}
\dot{x}(t) = -\cos(t)y(t) + \sin(t)x(t-\tau), & t \geq t_0, t \neq t_k, k \in \mathbb{Z}_+, \\
\dot{y}(t) = \cos(t)x(t) + \sin(t)y(t-\tau), & t \geq t_0, t \neq t_k, k \in \mathbb{Z}_+, \\
x(t_k) = qx(t_k^-), & k \in \mathbb{Z}_+, \\
y(t_k) = qy(t_k^-), & k \in \mathbb{Z}_+,
\end{cases} \tag{4.2}
$$

where $q < 1$, τ is a nonnegative constant. For convenience, we consider $h_0(x, y) = h(x, y) = x^2 + y^2$.

Property 4.2. We have given constants u, v satisfying $u < qv$, and a constant $\lambda > \frac{1}{q}$. Then the system (4.1) with respect to (u, v) is $(\tilde{h}_0, h)$-uniformly asymptotically practically stable if

$$(1+\lambda) \max_{k \in \mathbb{Z}_+} \{t_k - t_{k-1}\} < 1 - q.$$

Proof. Choose $V(x(t), y(t)) = x^2(t) + y^2(t)$, where $(x(t), y(t))$ is a solution of system (4.2). Let $\alpha(s) = \beta(s) = s$, $P(s) = \lambda s$, $\omega(s) = (1+\lambda)s$, $\psi(s) = qs$, $s > 0$, and then

$$
\begin{aligned}
D^+V(t, x(t)) &= 2x(t)\dot{x}(t) + 2y(t)\dot{y}(t) \\
&= 2x(t)\Big(-\cos(t)y(t) + \sin(t)x(t-\tau)\Big) \\
&\quad +2y(t)\Big(\cos(t)x(t) + \sin(t)y(t-\tau)\Big) \\
&= 2\sin(t)x(t)x(t-\tau) + 2\sin(t)y(t)y(t-\tau) \\
&= \sin(t)\Big(x^2(t) + y^2(t) + x^2(t-\tau) + y^2(t-\tau)\Big) \\
&= \sin(t)\Big(V(x(t), y(t)) + V(x(t-\tau), y(t-\tau))\Big) \\
&\leq \|\sin(t)\|\Big(V(x(t), y(t)) + \lambda V(x(t), y(t))\Big) \\
&= \|\sin(t)\|(1+\lambda)V(x(t), y(t)) \\
&= g(t)\omega(V(t, x(t))),
\end{aligned}
$$

where $g(t) = \|\sin(t)\|$, $\omega(V(t, x(t))) = (1+\lambda)V(x(t), y(t))$. By Theorem 3.1, the above property can be easily derived. □

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