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METHOD OF LINES FOR HAMILTON - JACOBI FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Initial boundary value problems for nonlinear first order partial functional differential equations are transformed by discretization in space variables into systems of ordinary functional differential equations. A method of quasi linearization is adopted. Sufficient conditions for the convergence of the method of lines and error estimates for approximate solutions are presented. The proof of the stability of the differential difference problems is based on a comparison technique. Nonlinear estimates of the Perron type with respect to the functional variable for given functions are used. Results obtained in the paper can be applied to differential integral problems and equations with deviated variables.

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1. Introduction

We are interested in establishing a method of approximation of solutions to first order partial functional differential equations with solutions of associated systems of ordinary functional differential equations and in estimating of the difference between the exact and approximate solutions. We investigate the question of under what conditions solutions of ordinary functional differential equations tend to a solutions of the original problem when a step size tends to zero. The systems of ordinary functional differential equations mentioned above are obtained in the paper by using a discretization in spatial variables of original problems, and they are called differential difference systems or method of lines. The advantage of the method of lines is that it allows to solve the problems for partial differential equations (often quite complicated) by using the general-purpose methods and software that have been developed for numerically integrating ordinary differential equations. It is easy to construct a differential difference system which satisfies consistency conditions with respect to the original problem on sufficiently regular solutions. The main question in these consideration is to find sufficient conditions for the stability of the method of lines. The method of

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differential inequalities and comparison techniques are used in investigations of the stability. There is an ample literature on the method of lines. The monographs [10], [12], [21], [23], [24], [28] contain a large bibliography on theoretical investigations and applications. The papers [5], [16] initiated a theory of the numerical method of lines for functional differential equations. Nonlinear parabolic functional differential equations with initial boundary value conditions were investigated in [15], [19], [29]. Results concerning the stability of the method of lines were obtained in these papers by using a comparison technique. The papers [1], [2], [6], [7], [14], [30] concern equations with first order partial derivatives. Initial problems with solutions defined on the Haar pyramid and initial boundary value problems were considered. Error estimates implying the convergence of the method are obtained by using a method of differential inequalities. It is assumed that given operators satisfy nonlinear estimates of the Perron type with respect to functional variables. The monograph [13] contains an exposition of the method of lines for hyperbolic functional differential problems. The method is also treated as a tool for proving existence theorems for differential problems corresponding to parabolic equations [22], [24]–[26] or hyperbolic problems [3], [4], [9], [18], [20].

The aim of the paper is to construct a method of lines for nonlinear first order partial functional differential equations with initial boundary conditions. Our results are based on the following idea. The original problem is transformed into a system of quasilinear functional differential equations for an unknown function and for their partial derivatives with respect to spatial variables. The numerical method of lines is constructed for systems such obtained. All the results on the numerical method of lines given in [1], [2], [5]–[7], [14], [15], [29], [30] have the following property. The authors have assumed that given operators satisfy the Lipschitz condition or satisfy nonlinear estimates of the Perron type with respect to functional variables and these conditions are global with respect to all variables. Our assumptions on regularity of given functions are more general. We assume nonlinear estimates of the Perron type and suitable inequalities are local with respect to functional variables. It is clear that there are differential equations with deviated variables and differential integral equations such that local estimates of the Perron type hold and global inequalities are not satisfied We use in the paper general ideas for functional differential equations and inequalities which were introduced in [13], [27].

We formulate our functional differential problems. For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Write $E=[0,a]\times[-b,b]$, $E_0=[-b_0,0]\times[-b,b]$ where $a>0$, $b_0\in\mathbb{R}_+$, $\mathbb{R}_+=[0,+\infty)$, $b=(b_1,\dots,b_n)\in\mathbb{R}^n$ and $b_i>0$ for $1\le i\le n$.

For $(t,x) \in E$ we define

$$D[t,x] = \{(\tau,s) \in \mathbb{R}^{1+n} : \tau \le 0,\ (t+\tau, x+s) \in E_0 \cup E\}.$$

It is clear that $D[t,x] = [-b_0 - t, 0] \times [-b - x, b - x]$. For a function $z : E_0 \cup E \to \mathbb{R}$ and for a point $(t,x) \in E$ we define a function $z_{(t,x)} : D[t,x] \to \mathbb{R}$ by

$$z_{(t,x)}(\tau,s) = z(t+\tau, x+s),\ \ (\tau,s) \in D[t,x].$$

Then $z_{(t,x)}$ is the restriction of z to the set $(E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n)$ and this restriction is shifted to the set $D[t,x]$.

Write $B = [-b_0 - a, 0] \times [-2b, 2b]$. Then $D[t,x] \subset B$ for $(t,x) \in E$. The maximum norm in $C(B,\mathbb{R})$ will be denoted by $\|\cdot\|_B$. Suppose that $\phi_0 : [0,a] \to \mathbb{R}$ and $\phi : E \to \mathbb{R}^n$, $\phi = (\phi_1, \ldots, \phi_n)$, are given functions. The requirements on ϕ_0 and ϕ are that $0 \le \phi_0(t) \le t$ for $t \in [0,a]$ and $\phi(t,x) \in [-b,b]$ for $(t,x) \in E$. Write $\varphi(t,x) = (\phi_0(t), \phi(t,x))$ for $(t,x) \in E$.

Suppose that $\Xi \subset \mathbb{R}^n$ is an open and bounded domain and $E \subset \Xi$. Set

$$\Omega = \Xi \times C(B,\mathbb{R}) \times C(B,\mathbb{R}) \times \mathbb{R}^n,\ \ \partial_0 E = [0,a] \times ([-b,b] \setminus (-b,b)),$$

and suppose that $F : \Omega \to \mathbb{R}$, $\psi : E_0 \cup \partial_0 E \to \mathbb{R}$ are given functions. Let z be an unknown function of the variables (t,x), $x = (x_1, \ldots, x_n)$. We consider the functional differential equation

$$\partial_t z(t,x) = F(t, x, z_{(t,x)}, z_{\varphi(t,x)}, \partial_x z(t,x)) \tag{1.1}$$

with the initial boundary condition

$$z(t,x) = (t,x) \text{ on } E_0 \cup \partial_0 E, \tag{1.2}$$

where $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$.

We will say that F satisfies condition (V) if for each $(t,x,q) \in \Xi \times \mathbb{R}^n$ and for v, w, $\tilde{v}$, $\tilde{w} \in C(B,R)$ such that $v(\tau,s) = \tilde{v}(\tau,s)$ for $(\tau,s) \in D[t,x]$ and $w(\tau,s) = \tilde{w}(\tau,s)$ for $(\tau,s) \in D[\varphi(t,x)]$ we have $F(t,x,v,w,q) = F(t,x,\tilde{v},\tilde{w},q)$. Condition (V) means that the value of F at $(t,x,v,w,q) \in \Omega$ depends on (t,x,q) and on the restrictions of v and w to the sets $D[t,x]$ and $D[\varphi(t,x)]$ only. We assume that F satisfies condition (V) and we consider classical solutions of (1.1), (1.2).

Sufficient conditions for the existence and uniqueness of classical or generalized solutions to initial boundary value problems can be found in [8] and [13], Chapter 5.

We give examples of functional differential equations which can be obtained from (1) by specializing given functions. Suppose that $G : \Xi \times \mathbb{R}^2 \times \mathbb{R}^n \to \mathbb{R}$ and

$$\vartheta_0,\ \gamma_0 : [0,a] \to \mathbb{R},\ \ (\vartheta_1, \ldots, \vartheta_n),\ (\gamma_1, \ldots, \gamma_n) : E \to \mathbb{R}^n$$

are given functions. Write

$$\vartheta(t,x) = (\vartheta_0(t), \vartheta_1(t,x), \ldots, \vartheta_n(t,x)) \text{ and } \gamma(t,x) = (\gamma_0(t), \gamma_1(t,x), \ldots, \gamma_n(t,x)).$$

We assume that $0 \le \vartheta_0(t) \le t$ and $0 \le \gamma_0(t) \le t$ for $t \in [0,a]$ and $\vartheta(t,x)$, $\gamma(t,x) \in E$ for $(t,x) \in E$. Then $\vartheta(t,x) - (t,x) \in B$ and $\gamma(t,x) - (t,x) \in B$ for $(t,x) \in E$. Consider the operator F defined by

$$F(t,x,v,w,q) = G\big(t,x,\int_{\vartheta(t,x)-(t,x)}^{\gamma(t,x)-(t,x)} v(\tau,s)dsd\tau, w(0,0_{[n]}), q\big) \text{ on } \Omega, \tag{1.3}$$

where $0_{[n]} = (0,\dots,0) \in \mathbb{R}^n$. Then (1.1) is equivalent to the functional differential equation

$$\partial_t z(t,x) = G\big(t,x,\int_{\vartheta(t,x)}^{(t,x)} z(\tau,s)dsd\tau, z(\varphi(t,x)), \partial_x z(t,x)\big). \tag{1.4}$$

For the above G we put

$$F(t,x,v,w,q) = G(t,x,v(0,0_{[n]}), w(0,0_{[n]}), q) \text{ on } \Omega. \tag{1.5}$$

Then (1.1) reduces to the differential equation with deviated variables

$$\partial_t z(t,x) = G(t,x,z(t,x), z(\varphi(t,x)), \partial_x z(t,x)). \tag{1.6}$$

Note that F given by (1.3) and F defined by (1.5) satisfy condition (V). It is clear that more complicated examples can be obtained from (1.1) by specializing F and φ.

2. **Differential difference problems**

We will be denote by $\mathbb{N}$ and $\mathbb{Z}$ the sets of natural numbers and integers respectively. Let $M_{n\times n}$ be the class of all $n \times n$ matrices with real elements. If $W \in M_{n\times n}$ then W^T is the transpose matrix. For $x, y \in \mathbb{R}^n$, $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n)$, and $W \in M_{n\times n}$, $W = [w_{ij}]_{i,j=1,\dots,n}$ we put

$$x \diamond y = (x_1y_1,\dots,x_ny_n) \in \mathbb{R}^n, \quad x \circ y = \sum_{i=1}^n x_iy_i, \quad \|x\| = \sum_{i=1}^n |x_i|,$$

$$\|x\|_\infty = \max\{|x_i| : 1 \le i \le n\}, \quad \|W\|_{n\times n} = \max\left\{\sum_{j=1}^n |w_{ij}| : 1 \le i \le n\right\}.$$

We denote by $CL(B,\mathbb{R})$ the set of all linear and continuous real functions defined on $C(B,\mathbb{R})$ and by $\|\cdot\|_\star$ the norm in $CL(B,\mathbb{R})$ generated by the maximum norm in $C(B,\mathbb{R})$. Write

$$\Delta_i^+ = \{(t,x) \in E : x_i = b_i\}, \quad \Delta_i^- = \{(t,x) \in E : x_i = -b_i\}, \quad i = 1,\dots,n.$$

Suppose that $\psi \in C(E_0 \cup \partial_0 E, \mathbb{R})$. Let us denote by U_ψ the set of all $z \in C(E_0 \cup E, \mathbb{R})$ such that $z(t,x) = \psi(t,x)$ on $E_0 \cup \partial_0 E$. The following assumptions on F, φ, ψ are needed in our considerations.

Assumption $H[\varphi]$. The functions $\phi_0 : [0,a] \to \mathbb{R}$ and $\phi : E \to \mathbb{R}^n$, $\phi = (\phi_1,\dots,\phi_n)$, are continuous and

1) $0 \le \phi_0(t) \le t$ for $t \in [0,a]$ and $\varphi(t,x) = (\phi_0(t), \phi(t,x)) \in E$ for $(t,x) \in E$,

2) there exist the derivatives $\partial_x \phi = [\partial_{x_j} \phi_i]_{i,j=1,\dots,n}$ and $\partial_x \phi \in C(E, M_{n\times n})$ and $Q \in \mathbb{R}_+$ is defined by the relation: $\|\partial_x \phi(t,x)\|_{n\times n} \le Q$ on E.

Assumption $H[F,\psi]$. The function $F : \Omega \to \mathbb{R}$ of the variables (t,x,v,w,q), $q = (q_1,\dots,q_n)$, satisfies condition (V) and

1) $F \in C(\Omega,\mathbb{R})$ and the partial derivatives $(\partial_{x_1} F,\dots,\partial_{x_n} F)\partial_x F$, $(\partial_x F, \partial_{q_1} F,\dots,\partial_{q_n} F) = \partial_q F$ exist on Ω and $\partial_x F$, $\partial_q F \in C(\Omega,\mathbb{R}^n)$,
2) there exist the Fréchet derivatives $\partial_v F(P)$, $\partial_w F(P)$ and $\partial_v F(P)$, $\partial_w F(P) \in CL(B,\mathbb{R})$ for $P = (t,x,v,w,q) \in \Omega$,
3) there is $\tilde{x} \in (-b,b)$, $\tilde{x} = (\tilde{x}_1,\dots,\tilde{x}_n)$, such that

$$(x - \tilde{x}) \diamond \partial_q F(t,x,v,w,q) > 0_{[n]} \text{ for } (t,x,q) \in \partial_0 E \times \mathbb{R}^n, \ \ v,w \in C(B,\mathbb{R}), \tag{2.1}$$

where $0_{[n]} = (0,\dots,0) \in \mathbb{R}^n$,

4) for each $x \in [-b,b]$ the the function

$$\text{sign } \partial_q F(\cdot,x,\cdot) : [0,a] \times C(B,\mathbb{R}) \times C(B,\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$$

is constant, where

$$\text{sign } \partial_q F(\cdot,x,\cdot) = \Big(\text{sign } \partial_{q_1} F(\cdot,x,\cdot),\dots,\text{sign } \partial_{q_n} F(\cdot,x,\cdot)\Big),$$

5) $\psi : E_0 \cup \partial_0 E \to \mathbb{R}$ is of class C^1 and for z, $\tilde{z} \in U_\psi$ we have

$$F(t,x,z_{(t,x)},z_{\varphi(t,x)},q) = F(t,x,\tilde{z}_{(t,x)},z_{\varphi(t,x)},q) \text{ for } (t,x,q) \in \partial_0 E \times \mathbb{R}^n,$$

6) the function $\chi : \partial_0 E \to \mathbb{R}^n$, $\chi = (\chi_1,\dots,\chi_n)$, satisfies the conditions:
   (i) for $(t,x) \in \Delta_i^+ \cup \Delta_i^-$ we have

$$\chi_j(t,x) = \partial_{x_j}(t,x) \text{ for } j \ne i \tag{2.2}$$

and

$$\partial_t \psi(t,x) = F\Big(t,x,z_{(t,x)},z_{\varphi(t,x)},\Pi_i(t,x)\Big) \tag{2.3}$$

where $z \in U_\psi$ and

$$\Pi_i(t,x) = \Big(\partial_{x_1}(t,x),\dots,\partial_{x_{i-1}}(t,x),\chi_i(t,x),\partial_{x_{i+1}}(t,x),\dots,\partial_{x_n}(t,x)\Big),$$

   (ii) $\chi \in C(\partial_0 E,\mathbb{R}^n)$.

Suppose that Assumption $H[F,\psi]$ is satisfied. Let $\Psi : E_0 \cup \partial_0 E \to \mathbb{R}$, $\Psi = (\Psi_1,\dots,\Psi_n)$, be defined by

$$\Psi(t,x) = \partial_x(t,x) \text{ on } E_0 \text{ and } \Psi(t,x) = \chi(t,x) \text{ on } \partial_0 E. \tag{2.4}$$

We give comments on Assumption $H[F,\psi]$.

Remark 2.1. If we assume that

$$(x - \tilde{x}) \diamond \partial_q F(t, x, v, w, q) \geq 0_{[n]} \text{ on } \Omega, \tag{2.5}$$

then condition 4) of Assumption $H[F, \psi]$ is satisfied. Assumption (2.5) is typical in theorems on the method of lines for Hamilton Jacobi functional differential equations, see [1], [2], [5]–[7], [14], [30].

Remark 2.2. Relations 5) and 6) of Assumption $H[F, \psi]$ are called the compatibility conditions for (1.1), (1.2). Formulas (2.2), (2.3) can be considered as the definitions of χ.

The above compatibility conditions appear in the theorem on the existence and uniqueness of solutions of (1.1), (1.2).

Two types of assumptions are needed in theorems on the existence and uniqueness of classical or generalized solutions for (1.1), (1.2). The first type conditions deal with the regularity of given functions. It is assumed in a theorem on the uniqueness of solutions that F is continuous on and it satisfies nonlinear estimates of Perron type with respect to the functional variables. The assumption of the second type are connected with the theory of bicharacteristics and condition (2.1) is needed. Suppose that $z \in C(E_0 \cup E, \mathbb{R})$ and $u \in C(E_0 \cup E, \mathbb{R}^n)$. Let us denote by $g[z, u](\cdot, t, x)$ the solution of the Cauchy problem

$$\omega'(\tau) = -\partial_q F\big(\tau, \omega(\tau), z_{(\tau,\omega(\tau))}, z_{\varphi(\tau,\omega(\tau))}, u(\tau, \omega(\tau))\big), \;\; \omega(t) = x,$$

where $(t, x) \in E$. The function $g[z, u](\cdot, t, x) = (g_1[z, u](\cdot, t, x), \ldots, g_n[z, u](\cdot, t, x))$ is the bicharacteristic of (1.1) corresponding to (z, u). Condition (2.1) states that the function $g_i[z, u](\cdot, t, x)$ is non decreasing if $(t, x) \in \Delta_i^-$and it is non increasing if $(t, x) \in \Delta_i^+$, where $1 \leq i \leq n$. The uniqueness condition is a consequence of a comparison theorem for functional differential inequalities (see [13], Chapter V) and condition (2.1) is needed in these considerations.

The existence theory of classical or generalized solutions for (1.1), (1.2) is based on the method of bicharacteristics. If Assumptions $H[\varphi]$, $H[F, \psi]$ are satisfied and the functions $\partial_x F$, $\partial_q F$, $\partial_v F$, $\partial_w F$ satisfy the Lipschitz condition with respect to (x, v, w, q) then under natural assumptions on there exists a solution of (1.1), (1.2). The solution is local with respect to t. Conditions (2.1), (2.5) are important in this result.

Thus we see that Assumption $H[F, \psi]$ is natural in the theory of initial boundary value problems (1.1), (1.2).

Remark 2.3. Suppose that F is given by (1.3). Then (1.1) reduces to (1.4). If we assume that ϑ, γ, ϕ satisfy the conditions

$$\vartheta_i(t, x) = b_i, \;\; \gamma_i(t, x) = b_i, \;\; \phi_i(t, x) = b_i \text{ for } (t, x) \in \Delta_i^+,$$

$$\vartheta_i(t,x) = -b_i, \ \ \gamma_i(t,x) = -b_i, \ \ \phi_i(t,x) = -b_i \ \text{ for } \ (t,x) \in \Delta_i^-,$$

where $1 \le i \le n$, then condition 5) of Assumption $H[F,\psi]$ is satisfied.

Suppose that F is given by (1.5). Then (1.1) is equivalent to (1.6). If we assume that

$$\phi_i(t,x) = b_i \quad \text{for} \ \ (t,x) \in \Delta_i^+, \ \ \phi_i(t,x) = -b_i \ \text{ for } \ (t,x) \in \Delta_i^-,$$

where $1 \le i \le n$, then condition 5) of Assumption $H[F,\psi]$ is satisfied.

Remark 2.4. Suppose that $f : \Xi \times C(B,\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$ is a given function. Let us consider the functional differential equation

$$\partial_t z(t,x) = f(t,x,z_{(t,x)},\partial_x z(t,x)). \tag{2.6}$$

The above equation is a particular case of (1.1). The functional differential problem consisting of (2.6), (1.2) is an initial boundary value problem.

There are the following motivation for investigation of (1.1), (1.2) instead of (2.6), (1.2). Differential equations with deviated variables can be obtained from (2.6) in the following way. Suppose that $G : \Xi \times \mathbb{R}^2 \times \mathbb{R}^n \to \mathbb{R}$ is a given function. Set

$$f(t,x,v,q) = G\big(t,x,v(0,0_{[n]}),v(\varphi(t,x)-(t,x)),q\big) \ \text{ on } \ \Xi \times C(B,\mathbb{R}) \times \mathbb{R}^n. \tag{2.7}$$

Then (2.6) is equivalent to (1.6). Note that Assumption $H[F,\psi]$ is not satisfied for f given by (2.7). More precisely, the derivatives $(\partial_{x_1} f,\dots,\partial_{x_n} f) = \partial_x f$ do not exist on $\Xi \times C(B,\mathbb{R}) \times \mathbb{R}^n$. With the above motivation we consider problem (1.1), (1.2) with F depending on two functional variables.

We define a mesh on $E_0 \cup E$ in the following way. Suppose that $(h_1,\dots,h_n) = h$, $h_i > 0$ for $1 \le i \le n$, stand for steps of the mesh. For $m = (m_1,\dots,m_n) \in \mathbb{Z}^n$ we put $x^{(m)} = (x_1^{(m_1)},\dots,x_n^{(m_n)}) = m \diamond h$ and

$$\mathbb{R}^{1+n}_{t.h} = \{(t,x^{(m)}) : \ t \in \mathbb{R}, \ m \in \mathbb{Z}^n\}.$$

Write

$$B_h = B \cap \mathbb{R}^{1+n}_{t.h}, \ \ E_h = E \cap \mathbb{R}^{1+n}_{t.h},$$
$$E_{0.h} = E_0 \cap \mathbb{R}^{1+n}_{t.h}, \ \ \partial_0 E_h = \partial_0 \cap \mathbb{R}^{1+n}_{t.h}.$$

Elements of the set $E_{0.h} \cup E_h$ will be denoted by $(t,x^{(m)})$ or (t,x). For functions $z : E_{0.h} \cup E_h \to \mathbb{R}$, $u : E_{0.h} \cup E_h \to \mathbb{R}^n$, $u = (u_1,\dots,u_n)$, we write $z^{(m)}(t) = z(t,x^{(m)})$ and $u^{(m)}(t) = u(t,x^{(m)})$.

Let us denote by $\mathbf{F}_c(E_{0.h} \cup E_h,\mathbb{R})$ the class of all $z : E_{0.h} \cup E_h \to \mathbb{R}$ such that $z(\cdot,x^{(m)}) \in C([-b_0,a],\mathbb{R})$ for $-K \le m \le K$. In a similar way we define the space $\mathbf{F}_c(E_{0.h} \cup E_h,\mathbb{R}^n)$. Solutions of differential difference equations corresponding to (1.1), (1.2) are defined on $E_{0.h} \cup E_h$. Since equation (1.1) contains the functional variables $z_{(t,x)}$ and $z_{\varphi(t,x)}$ which are elements of the spaces $C(D[t,x],\mathbb{R})$ and $C(D[\varphi(t,x),\mathbb{R})$ then we need an interpolating operator $T_h : \mathbf{F}_c(E_{0.h} \cup E_h,\mathbb{R}) \to C(E_0 \cup E,\mathbb{R})$. In the next part of the paper we adopt additional assumptions on T_h. For $z : E_{0.h} \cup E_h \to \mathbb{R}$ we write $(T_h z)_{[t,m]}$ instead of $(T_h z)_{(t,x^{(m)})}$ and we write $(T_h z)_{[t,m]}$ instead of $(T_h z)_{\varphi(t,x^{(m)})}$. Let us denote by Δ the set of all $h = (h_1, \dots, h_n)$ satisfying the conditions:

1) $\|h\|_\infty < \min\{b_\star, b^\star\}$ where $b_\star = \min\{b_i - \tilde{x}_i :\ 1 \le i \le n\}$ and $b^\star = \min\{b_i + \tilde{x}_i :\ 1 \le i \le n\}$,
2) there is $K = (K_1, \dots, K_n) \in \mathbb{N}^n$ such that $K \diamond h = b$.

Suppose that Assumption $H[F, \psi]$ is satisfied. For $x^{(m)} \in (-b, b)$ we put

$$I_+[m] = \{i \in \{1, \dots, n\} :\ \partial_{q_i} F(\cdot, x^{(m)}, \cdot) \ge 0\},$$

$$I_-[m] = \{1, \dots, n\} \setminus I_+[m].$$

Write $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$ with 1 standing on the i-th place. For functions $z : E_{0.h} \cup E_h \to \mathbb{R}$, $u : E_{0.h} \cup E_h \to \mathbb{R}^n$, $u = (u_1, \dots, u_n)$, and for $(t, x^{(m)}) \in [0, a] \times (-b, b)$ we write

$$\delta_i z^{(m)}(t) = \frac{1}{h_i}\big[z^{(m+e_i)}(t) - z^{(m)}(t)\big] \ \text{ if } \ i \in I_+[m],$$

$$\delta_i z^{(m)}(t) = \frac{1}{h_i}\big[z^{(m)}(t) - z^{(m-e_i)}(t)\big] \ \text{ if } \ i \in I_-[m],$$

and

$$\delta_i u^{(m)}(t) = \frac{1}{h_i}\big[u^{(m+e_i)}(t) - u^{(m)}(t)\big] \ \text{ if } \ i \in I_+[m],$$

$$\delta_i u^{(m)}(t) = \frac{1}{h_i}\big[u^{(m)}(t) - u^{(m-e_i)}(t)\big] \ \text{ if } \ i \in I_+[m],$$

and we put $i = 1, \dots, n$ in the above definitions. Set

$$\delta z^{(m)}(t) = \big(\delta_1 z^{(m)}(t), \dots, \delta_n z^{(m)}(t)\big), \delta u^{(m)}(t) = \big[\delta_j u_i^{(m)}\big]_{i,j=1,\dots,n}.$$

Write

$$P[z,u]^{(m)}(t) = \big(t, x^{(m)}, (T_h z)_{[t,m]}, (T_h z)_{\varphi[t,m]}, u^{(m)}(t)\big).$$

For $u : E_{0.h} \cup E_h \to \mathbb{R}^n$, $u = (u_1, \dots, u_n)$, and for $P \in \Omega$ we put

$$\partial_v F(P) \star (T_h u)_{[r,m]} = \big(\partial_v F(P)(T_h u_1)_{[r,m]}, \dots, \partial_v F(P)(T_h u_n)_{[r,m]}\big)$$

and

$$\begin{aligned}&\big[\partial_w F(P) \star (T_h u)_{\varphi[t,m]}\big]\partial_x \phi^{(m)}(t)\\ &\quad = \sum_{j=1}^n \partial_w F(P)(T_h u_j)_{\varphi[t,m]} \partial_{x_1} \phi_j^{(m)}(t), \dots, \left(\sum_{j=1}^n \partial_w F(P)(T_h u_j)_{\varphi[t,m]} \partial_{x_n} \phi_j^{(m)}(t)\right)\end{aligned}$$

Set

$$\mathbf{F}_{h.0}[z,u]^{(m)}(t) = F(P[z,u]^{(m)}(t)) + \partial_q F(P[z,u]^{(m)}(t)) \circ \big(\delta z^{(m)}(t) - u^{(m)}(t)\big)$$

and

$$\begin{aligned}\mathbf{F}_h[z,u]^{(m)}(t) &= \partial_x F(P[z,u]^{(m)}(t)) + \partial_v F(P[z,u]^{(m)}(t)) \star (T_h u)_{[r,m]} \\ &\quad + \big[\partial_w F(P[z,u]^{(m)}(t)) \star (T_h u)_{\varphi[t,m]}\big]\partial_x \phi^{(m)}(t) \\ &\quad + \partial_q F(P[z,u]^{(m)}(t))\big[\delta u^{(m)}(t)\big]^T.\end{aligned}$$

We consider the system of functional differential equations

$$\frac{d}{dt} z^{(m)}(t) = \mathbf{F}_{h.0}[z,u]^{(m)}(t), \tag{2.8}$$

$$\frac{d}{dt} u^{(m)}(t) = \mathbf{F}_h[z,u]^{(m)}(t) \tag{2.9}$$

with the initial boundary conditions

$$z^{(m)}(t) = \psi_h^{(m)}(t), \quad u^{(m)}(t) = \Psi_h^{(m)}(t) \ \text{ on } \ E_{0.h} \cup \partial_0 E_h \tag{2.10}$$

where $\psi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}$ and $\Psi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}^n$ are given functions. The above problem is obtained in the following way. We use a method of quasilinearization for (1.1), (1.2). It consists in replacing problem (1.1), (1.2) with the following one. Suppose that Assumption $H[F,\psi]$ is satisfied. Let (z,u), $u = (u_1, \ldots, u_n)$, be unknown functions of the variables $(t,x) \in E_0 \cup E$. We introduce an additional unknown function $u = \partial_x z$ in (1.1) and we consider the following linearization of (1.1) with respect u:

$$\partial_t z(t,x) = F(\Upsilon(t,x)) + \partial_q F\big(\Upsilon(t,x)\big) \circ \big(\partial_x z(t,x) - u(t,x)\big) \tag{2.11}$$

where $\Upsilon(t,x) = \big(t, x, z_{(t,x)}, z_{\varphi(t,x)}, u(t,x)\big)$. We get functional differential equations for u by differentiating equation (1.1). The result is the following

$$\begin{aligned}\partial_t u(t,x) &= \partial_x F(\Upsilon(t,x)) + \partial_v F(\Upsilon(t,x)) \star u_{(t,x)} \\ &\quad + \big[\partial_w F(\Upsilon(t,x)) \star u_{\varphi(t,x)}\big]\partial_x \phi(t,x) + \partial_q F(\Upsilon(t,x))\big[\partial_x u(t,x)\big]^T.\end{aligned} \tag{2.12}$$

where

$$\partial_v F(\Upsilon(t,x)) \star u(t,x) = \big(\partial_v F(\Upsilon(t,x))(u_1)_{(t,x)}, \ldots, \partial_v F(\Upsilon(t,x))(u_n)_{(t,x)}\big),$$

$$\begin{aligned}&\big[\partial_w F(\Upsilon(t,x)) \star u_{\varphi(t,x)}\big]\partial_x \phi(t,x) \\ &= \Big(\sum_{j=1}^n \partial_w F(\Upsilon(t,x))(u_j)_{\varphi(t,x)} \partial_{x_1} \phi_j(t,x), \ldots, \sum_{j=1}^n \partial_w F(\Upsilon(t,x))(u_j)_{\varphi(t,x)} \partial_{x_n} \phi_j(t,x)\Big).\end{aligned}$$

It is natural to consider the following initial boundary conditions for (2.11), (2.12):

$$z(t,x) = \psi(t,x), \quad u(t,x) = \Psi(t,x) \ \text{ on } \ E_0 \cup \partial_0 E \tag{2.13}$$

where Ψ is given by (2.4). There are the following relations between (1.1), (1.2) and (2.11)–(2.13). Under natural assumptions on F, φ, ψ, we have

(I) If $(\mathbf{z}, \mathbf{u})$ is a solution of (2.11)–(2.13) then $\partial_x \mathbf{z} = \mathbf{u}$ and $\mathbf{z}$ is a solution of (1.1), (1.2).

(II) If $\mathbf{z}$ is a solution of (1.1), (1.2) and $\mathbf{u} = \partial_x \mathbf{z}$ then $(\mathbf{z}, \mathbf{u})$ is a solution of (2.11)–(2.13).

Existence results for (1.1), (1.2) are obtained by using the above method of quasi linearization (see [8] and [13], Chapter 5). Differential difference problem (2.8)–(2.10) is obtained by the discretization of (2.11)–(2.13) with respect to the spatial variable x.

3. **Solutions of functional differential problems**

In this Section we prove that there is a solution to (2.8)–(2.10) and we give estimates of solutions of (1.1), (1.2) and (2.8)–(2.10). For functions $z \in C(E_0 \cup E, \mathbb{R})$, $u \in C(E_0 \cup E, \mathbb{R}^n)$ and $z_h \in \mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R})$, $u_h \in \mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R}^n)$ we define the seminorms

$$\|z\|_t = \max\{|z(\tau, s)| : \ (\tau, s) \in E_0 \cup E, \ \tau \le t\},$$

$$[|u|]_t = \max\{\|u(\tau, s)\|_\infty : \ (\tau, s) \in E_0 \cup E, \ \tau \le t\},$$

$$\|z_h\|_{h.t} = \max\{|z_h(\tau, s)| : \ (\tau, s) \in E_{0.h} \cup E_h, \ \tau \le t\},$$

$$[|u_h|]_{h.t} = \max\{\|u_h(\tau, s)\|_\infty : \ (\tau, s) \in E_{0.h} \cup E_h, \ \tau \le t\},$$

where $t \in [0, a]$.

Assumption $H[T_h]$. The operator $T_h : \mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R}) \to C(E_0 \cup E, \mathbb{R})$ satisfies the conditions:

1) for $z, \ \tilde{z} \in \mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R})$ we have

$$\|T_h z - T_h \tilde{z}\|_t \le \|z - \tilde{z}\|_{h.t}, \quad t \in [0, a],$$

2) if $\theta_h \in \mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R})$ is given by $\theta_h(\tau, s) = 0$ for $(\tau, s) \in E_{0.h} \cup E_h$ then $(T_h \theta_h)(\tau, s) = 0$ for $(\tau, s) \in E_0 \cup E$,

3) if $z : E_0 \cup E \to \mathbb{R}$ is of class C^1 and z_h is the restriction of z to $E_{0.h} \cup E_h$ then there is $\gamma_\star : \Delta \to \mathbb{R}_+$ such that

$$\|T_h z_h - z\|_t \le \gamma_\star(h) \ \text{ for } \ t \in [0, a] \ \text{ and } \ \lim_{h \to 0_{[n]}} \gamma_\star(h) = 0.$$

Remark 3.1. The interpolating operator T_h given in [13] (Chapter VI) satisfies Assumption $H[T_h]$.

Assumption $H[F, \varrho, A]$. The functions φ and F, ψ satisfy Assumptions $H[\varphi]$ and $H[F, \psi]$ and

1) there is $\rho \in C([0,a] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ such that

$$\|\partial_x F(t,x,v,w,q)\|_\infty \le \varrho\big(t, \max\{\|v\|_B, \|w|_B\}, \|q\|\big) \text{ on } \Omega$$

and the function ρ is nondecreasing with respect to the last two variables,

2) there is $A \in \mathbb{R}_+$ such that for $P = (t,x,v,w,q) \in \Omega$ we have

$$\|\partial_q F(P)\|_\infty, \ \ \|\partial_v F(P)\|_\star, \ \ \|\partial_w F(P)\|_\star \le A,$$

3) the constant $A_0 \in \mathbb{R}_+$ is defined by the relation

$$|F(t,x,\theta,\theta,0_{[n]})| \le A_0, \ \ (t,x) \in E,$$

where $\theta \in C(B,\mathbb{R})$ is given by $\theta(\tau,s) = 0$ for $(\tau,s) \in B$,

4) for each $(\mu,\nu) \in \mathbb{R}_+ \times \mathbb{R}_+$ there is on $[0,a]$ the maximal solution $\big(\omega_0(\cdot,\mu,\nu), \omega(\cdot,\mu,\nu)\big)$ of the Cauchy problem

$$\xi'(t) = A_0 + 2A\big(\xi(t) + \kappa(t)\big), \tag{3.1}$$

$$\kappa'(t) = \varrho(t,\xi(t),\kappa(t)) + A(1+Q)\kappa(t), \tag{3.2}$$

$$\xi(0) = \mu, \ \ \kappa(0) = \nu, \tag{3.3}$$

5) $\psi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}$, $\Psi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}^n$ and there are α_0, $\alpha : \Delta \to \mathbb{R}_+$ such that

$$|\psi(t,x) - \psi_h(t,x)| \le \alpha_0(h), \ \ \|\Psi(t,x) - \Psi_h(t,x)\|_\infty \le \alpha(h) \text{ on } E_{0.h} \cup \partial_0 E_h, \tag{3.4}$$

and

$$\lim_{h \to 0_{[n]}} \alpha_0(h) = 0, \ \ \lim_{h \to 0_{[n]}} \alpha(h) = 0. \tag{3.5}$$

Suppose that Assumption $H[F,\varrho,A]$ is satisfied. Let the constants $\bar\mu, \bar\nu \in \mathbb{R}_+$ are defined by the relations

$$|\psi(t,x)| \le \bar\mu, \ \ \|\Psi(t,x)\|_\infty \le \bar\nu \text{ on } E_0 \cup \partial_0 E, \tag{3.6}$$

$$|\psi_h(t,x)| \le \bar\mu, \ \ \|\Psi_h(t,x)\|_\infty \le \bar\nu \text{ on } E_{0.h} \cup \partial_0 E_h. \tag{3.7}$$

Lemma 3.2. *If Assumptions $H[F,\varrho,A]$, $H[T_h]$ are satisfied then there is a solution $(\mathbf{z}_h, \mathbf{u}_h) : E_{0.h} \cup E_h \to \mathbb{R}^{1+n}$, $\mathbf{u}_h = (\mathbf{u}_{h.1}, \ldots, \mathbf{u}_{h.n})$, of* (2.8)–(2.10) *and*

$$\|\mathbf{z}_h\|_{h.t} \le \omega_0(t,\bar\mu,\bar\nu), \ \ [|\mathbf{u}_h|]_{h.t} \le \omega(t,\bar\mu,\bar\nu), \ \ t \in [0,a], \tag{3.8}$$

where $\big(\omega_0(\cdot,\bar\mu,\bar\nu), \omega(\cdot,\bar\mu,\bar\nu)\big)$ is the maximal solution of (3.1)–(3.3) *for $(\mu,\nu) = (\bar\mu,\bar\nu)$.*

Proof. From classical theorems on ordinary functional differential equations [11] it follows that there is $\tilde{\varepsilon} > 0$ such that the solution $(\mathbf{z}_h, \mathbf{u}_h)$ of (2.8)–(2.10) is defined on $(E_{0.h} \cup E_h) \subset ([-b_0, \tilde{\varepsilon}] \times \mathbb{R}^n)$. Suppose that $(\mathbf{z}_h, \mathbf{u}_h)$ is defined on $(E_{0.h} \cup E_h) \subset ([-b_0, \tilde{a}) \times \mathbb{R}^n)$, $\tilde{a} > 0$, and it is non continuable. We prove that

$$\|\mathbf{z}_h\|_{h.t} \le \omega_0(t, \bar{\mu}, \bar{\nu}) \text{ and } [|\mathbf{u}_h|]_{h.t} \le \omega(t, \bar{\mu}, \bar{\nu}) \text{ for } t \in [0, \tilde{a}).$$

For $\varepsilon > 0$ we denote by $\big(\omega_0(\cdot, \bar{\mu}, \bar{\nu}; \varepsilon), \omega(\cdot, \bar{\mu}, \bar{\nu}; \varepsilon)\big)$ the maximal solution of the Cauchy problem

$$\xi'(t) = A_0 + 2A(\xi(t) + \kappa(t)) + \varepsilon, \tag{3.9}$$

$$\kappa'(t) = \varrho(t, \xi(t), \kappa(t)) + A(1 + Q)\kappa(t) + \varepsilon, \tag{3.10}$$

$$\xi(0) = \bar{\mu} + \varepsilon, \quad \kappa(0) = \bar{\nu} + \varepsilon. \tag{3.11}$$

There is $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the functions $\big(\omega_0(\cdot, \bar{\mu}, \bar{\nu}; \varepsilon), \omega(\cdot, \bar{\mu}, \bar{\nu}; \varepsilon)\big)$ are defined on $[-b_0, \tilde{a})$ and

$$\lim_{\varepsilon \to 0} \omega_0(\cdot, \bar{\mu}, \bar{\nu}; \varepsilon) = \omega_0(\cdot, \bar{\mu}, \bar{\nu}), \quad \lim_{\varepsilon \to 0} \omega(\cdot, \bar{\mu}, \bar{\nu}; \varepsilon) = \omega(\cdot, \bar{\mu}, \bar{\nu}) \tag{3.12}$$

uniformly on $[0, \tilde{a})$. Set

$$\xi_h(t) = \|\mathbf{z}_h\|_{h.t}, \quad \kappa_h(t) = [|\mathbf{u}_h|]_{h.t}, \quad t \to [0, a).$$

We prove that

$$\xi_h(t) < \omega_0(t, \bar{\mu}, \bar{\nu}; \varepsilon) \text{ and } \kappa_h(t) < \omega(t, \bar{\mu}, \bar{\nu}; \varepsilon) \tag{3.13}$$

where $t \to [0, \tilde{a})$. It is clear that there is $\tilde{t} > 0$ such that estimates (3.13) are satisfied on $[0, \tilde{t})$. Suppose by contradiction that (3.13) fails to be true on $[0, \tilde{a})$. Then there is $t \in (0, \tilde{a})$ such that

$$\xi_h(\tau) < \omega_0(\tau, \bar{\mu}, \bar{\nu}; \varepsilon) \text{ and } \kappa_h(\tau) < \omega(\tau, \bar{\mu}, \bar{\nu}; \varepsilon) \text{ for } \tau \in [0, t)$$

and

$$\xi_h(t) = \omega_0(t, \bar{\mu}, \bar{\nu}; \varepsilon) \text{ or } \kappa_h(t) = \omega(\cdot, \bar{\mu}, \bar{\nu}; \varepsilon).$$

Let us consider the case when $\kappa_h(t) = \omega(t, \bar{\mu}, \bar{\nu}; \varepsilon)$. Then we have

$$D_- \kappa_h(t) \ge \omega(t, \bar{\mu}, \bar{\nu}; \varepsilon), \tag{3.14}$$

where D_- is the left hand lower Dini derivative. There are $m \in \mathbb{Z}^n$, $-K \le m \le K$, and $\bar{t} \le t$ and j, $1 \le j \le n$, such that $\kappa_h(t) = |\mathbf{u}_{h.j}^{(m)}(\bar{t})|$. If $\bar{t} < t$ then $D_- \kappa_h(t) = 0$ which contradicts (3.14). Suppose that $\bar{t} = t$. Then we have (i) $\kappa_h(t) = \mathbf{u}_{h.j}^{(m)}(t)$ or (ii) $\kappa_h(t) = -\mathbf{u}_{h.j}^{(m)}(t)$. Let us consider the first case. We deduce from (3.7) that $x^{(m)} \in (-b, b)$. Then we have

$$D_- \kappa_h(t) \le \frac{d}{dt} \mathbf{u}_{h.j}^{(m)}(t) = \partial_{x_j} F(P[\mathbf{z}_h, \mathbf{u}_h]^{(m)}(t)) + \partial_v F(P[\mathbf{z}_h, \mathbf{u}_h]^{(m)}(t)) \big(T_h \mathbf{u}_{h.j}\big)_{[r,m]}$$

$$+\sum_{i=1}^{n}\partial_w F(P[\mathbf{z}_h,\mathbf{u}_h]^{(m)}(t))\big(T_h\mathbf{u}_{h.i}\big)_{\varphi[t,m]}\partial_{x_j}\phi_i^{(m)}(t)+\partial_q F(P[\mathbf{z}_h,\mathbf{u}_h]^{(m)}(t))\circ\delta\mathbf{u}_{h.j}^{(m)}(t).$$

It follows from conditions 3), 4) of Assumption $H[F,\psi]$ and from the definition of $\delta\mathbf{u}_h^{(m)}(t)$ that

$$\partial_q F(P[\mathbf{z}_h,\mathbf{u}_h]^{(m)}(t))\circ\delta\mathbf{u}_{h.j}^{(m)}(t)\le 0.$$

This gives

$$D_-\kappa_h(t)\le\varrho\big(t,\omega_0(t,\bar\mu,\bar\nu;\varepsilon),\omega(t,\bar\mu,\bar\nu;\varepsilon)\big)+A(1+Q)\omega(t,\bar\mu,\bar\nu;\varepsilon)+\varepsilon=\omega'(t,\bar\mu,\bar\nu;\varepsilon),$$

which contradicts (3.14). The case (ii) can be treated in a similar way.

The same proof remains valid for the case when $\xi_h(t)=\omega_0(t,\bar\mu,\bar\nu;\varepsilon)$. Then inequalities (3.13) are satisfied on $[0,\tilde a)$. From (3.13) we obtain in the limit, letting ε tend to zero, inequalities (3.8) on $(E_{0.h}\cup E_h)\setminus([-b_0,\tilde a)\times\mathbb{R}^n)$.

We prove that there are the limits

$$\lim_{\substack{t\to\tilde a\\ t<\tilde a}}\mathbf{z}_h^{(m)}(t),\quad \lim_{\substack{t\to\tilde a\\ t<\tilde a}}\mathbf{u}_h^{(m)}(t)\ \text{ for }\ -K<m<K.$$

Write

$$\tilde\omega_{h.0}(t,\bar t)=\max\{|\mathbf{z}_h^{(m)}(\bar t)-\mathbf{z}_h^{(m)}(t)|:\ -K<m<K\}$$
$$\tilde\omega_h(t,\bar t)=\max\{\|\mathbf{u}_h^{(m)}(\bar t)-\mathbf{u}_h^{(m)}(t)\|_\infty:\ -K<m<K\}$$

where $t,\bar t\in[0,\tilde a)$. We prove that

$$\tilde\omega_{h.0}(t,\bar t)\le\big|\omega_0(\bar t,\bar\mu,\bar\nu)-\omega_0(t,\bar\mu,\bar\nu)\big| \tag{3.15}$$

$$\tilde\omega_h(t,\bar t)\le\big|\omega(\bar t,\bar\mu,\bar\nu)-\omega(t,\bar\mu,\bar\nu)\big|. \tag{3.16}$$

We consider (3.16). Suppose that $\bar t>t$. The are j, $1\le j\le n$, and $m\in\mathbb{Z}^n$, $-K<m<K$, such that $\tilde\omega_h(t,\bar t)=\mathbf{u}_{h.j}^{(m)}(\bar t)-\mathbf{u}_{h.j}^{(m)}(t)$ or $\tilde\omega_h(t,\bar t)=-[\mathbf{u}_{h.j}^{(m)}(\bar t)-\mathbf{u}_{h.j}^{(m)}(t)]$. Let us consider the first case. Then we have

$$\begin{aligned}\tilde\omega_h(t,\bar t)&=\int_t^{\bar t}\partial_{x_j}F(P[\mathbf{z}_h,\mathbf{u}_h]^{(m)}(\tau))d\tau+\int_t^{\bar t}\partial_\nu F(P[\mathbf{z}_h,\mathbf{u}_h]^{(m)}(\tau))\big(T_h\mathbf{u}_{h.j}\big)_{[\tau,m]}d\tau\\&+\sum_{i=1}^{n}\int_t^{\bar t}\partial_w F(P[\mathbf{z}_h,\mathbf{u}_h]^{(m)}(\tau))\big(T_h\mathbf{u}_{h.i}\big)_{\varphi[\tau,m]}\partial_{x_j}\phi_i^{(m)}d\tau\\&+\int_t^{\bar t}\partial_q F(P[\mathbf{z}_h,\mathbf{u}_h]^{(m)}(\tau))\circ\delta\mathbf{u}_{h.j}^{(m)}d\tau.\end{aligned}$$

It follows from conditions 3), 4) of Assumption $H[F,\psi]$ and from the definition of $\delta\mathbf{u}_h^{(m)}(t)$ that

$$\int_t^{\bar t}\partial_q F(P[\mathbf{z}_h,\mathbf{u}_h]^{(m)}(\tau))\circ\delta\mathbf{u}_{h.j}^{(m)}d\tau\le 0.$$

We thus get

$$\begin{aligned}\omega_h(t,\bar{t}) &\le \int_t^{\bar{t}} \rho\big(\tau, \omega_0(\tau,\bar{\mu},\bar{\nu}), \omega(\tau,\bar{\mu},\bar{\nu})\big)d\tau + A(1+Q)\int_t^{\bar{t}} \omega(\tau,\bar{\mu},\bar{\nu})d\tau \\ &= \int_t^{\bar{t}} \omega'(\tau,\bar{\mu},\bar{\nu})d\tau = \omega(\bar{t},\bar{\mu},\bar{\nu}) - \omega(t,\bar{\mu},\bar{\nu}),\end{aligned}$$

which proves (3.16). The case $\omega_h(t,\bar{t}) = -[\mathbf{u}^{(m)}_{h.j}(\bar{t}) - \mathbf{u}^{(m)}_{h.j}(t)]$ can be treated in a similar way. The same considerations apply to (3.15). We omit details. It follows from (3.15), (3.16) that there are the limits

$$\lim_{\substack{t\to\tilde{a}\\ t<\tilde{a}}} \mathbf{z}^{(m)}_h(t) = \mathbf{z}^{(m)}_h(\tilde{a}), \quad \lim_{\substack{t\to\tilde{a}\\ t<\tilde{a}}} \mathbf{u}^{(m)}_h(t) = \mathbf{u}^{(m)}_h(\tilde{a}) \quad \text{for} \quad -K < m < K$$

Then the solution $(\mathbf{z}_h, \mathbf{u}_h)$ is defined on $(E_{0.h}\cup E_h)\setminus([-b_0,\tilde{a}]\times\mathbb{R}^n)$. If $\tilde{a} < a$ then there is $\bar{a} > \tilde{a}$ such that $(\mathbf{z}_h, \mathbf{u}_h)$ is defined on $(E_{0.h}\cup E_h)\setminus([-b_0,\bar{a}]\times\mathbb{R}^n)$. This contradicts our assumption that $(\mathbf{z}_h, \mathbf{u}_h)$ is defined on $(E_{0.h}\cup E_h)\setminus([-b_0,\tilde{a})\times\mathbb{R}^n)$ and it is non continuable.

It follows from the above considerations that $(\mathbf{z}_h, \mathbf{u}_h)$ is defined on $E_{0.h}\cup E_h$ and estimates (3.8) are satisfied. The proof of the lemma is completed. □

Now we give estimates of solutions of (2.11)–(2.13).

Lemma 3.3. *If Assumption $H[F,\rho,A]$ is satisfied and $(\bar{\mathbf{z}},\bar{\mathbf{u}}) : E_0\cup E \to \mathbb{R}^{1+n}$, $\bar{\mathbf{u}} = (\bar{\mathbf{u}}_1,\dots,\bar{\mathbf{u}}_n)$, is a solution of (2.11))–(2.13) then*

$$\|\bar{\mathbf{z}}\|_t \le \omega_0(t,\bar{\mu},\bar{\nu}), \quad [|\bar{\mathbf{u}}|]_t \le \omega(t,\bar{\mu},\bar{\nu}), \quad t\in[0,a], \tag{3.17}$$

where $\big(\omega_0(\cdot,\bar{\mu},\bar{\nu}), \omega(\cdot,\bar{\mu},\bar{\nu})\big)$ is the maximal solution of (3.1)–(3.3) with $(\mu,\nu) = (\bar{\mu},\bar{\nu})$.

Proof. Let us denote by $\big(\omega_0(\cdot,\bar{\mu},\bar{\nu};\varepsilon), \omega(\cdot,\bar{\mu},\bar{\nu};\varepsilon)\big)$ the maximal solution of (3.9)–(3.11) where $\varepsilon > 0$. There is $\varepsilon_0 > 0$ such that for $0<\varepsilon<\varepsilon_0$ the above solutions are defined on $[0,a]$ and conditions (3.12) are satisfied on $[0,a]$. Set

$$\tilde{\xi}(t) = \|\bar{\mathbf{z}}\|_t, \quad \tilde{\kappa}(t) = [|\bar{\mathbf{u}}|]_t, \quad t\in[0,a].$$

We prove that for $0<\varepsilon<\varepsilon_0$ we have

$$\tilde{\xi}(t) < \omega_0(t,\bar{\mu},\bar{\nu};\varepsilon) \quad \text{and} \quad \tilde{\kappa} < \omega(t,\bar{\mu},\bar{\nu};\varepsilon), \tag{3.18}$$

where $t\in[0,a]$. It is clear that there is $\tilde{t}\in(0,a]$ such that inequalities (3.18) hold on $[0,\tilde{t})$. Suppose by contradiction that estimates (3.18) are not satisfied on $[0,a]$. Then there is $t\in(0,a]$ such that

$$\tilde{\xi}(\tau) < \omega_0(\tau,\bar{\mu},\bar{\nu};\varepsilon) \quad \text{and} \quad \tilde{\kappa}(\tau) < \omega(\tau,\bar{\mu},\bar{\nu};\varepsilon) \quad \text{for} \quad \tau\in[0,t)$$

and

$$\tilde{\xi}(t) = \omega_0(t,\bar{\mu},\bar{\nu};\varepsilon) \quad \text{or} \quad \tilde{\kappa}(t) = \omega(t,\bar{\mu},\bar{\nu};\varepsilon).$$

Suppose that $\tilde{\kappa}(t) = \omega(t, \bar{\mu}, \bar{\nu}; \varepsilon)$. Then we have

$$D_{-}\tilde{\kappa}(t) \geq \omega'(t, \bar{\mu}, \bar{\nu}; \varepsilon). \tag{3.19}$$

There are $(\bar{t}, x) \in E$, $\bar{t} \leq t$, and j, $1 \leq j \leq n$, such that $\tilde{\kappa}(t) = |\mathbf{u}_j(\bar{t}, x)|$. If $\bar{t} < t$ then $D_{-}\tilde{\kappa}(t) = 0$ which contradicts (3.19). Suppose that $\bar{t} = t$. We deduce from (3.6) that $x \in (-b, b)$. This gives $\partial_x \mathbf{u}_j(t, x) = 0_{[n]}$. It follows from (2.12) and from Assumption $H[F, \varrho, A]$ that

$$\begin{aligned} D_{-}\tilde{\kappa}(t) \leq |\partial_t \mathbf{u}_j(t, x)| &< \varrho(t, \omega_0(t, \bar{\mu}, \bar{\nu}; \varepsilon), \omega(t, \bar{\mu}, \bar{\nu}; \varepsilon)) \\ &\quad + A(1 + Q)\omega(t, \bar{\mu}, \bar{\nu}; \varepsilon) + \varepsilon = \omega'(t, \bar{\mu}, \bar{\nu}; \varepsilon), \end{aligned}$$

which contradicts (3.19).

The case when $\tilde{\xi}(t) = \omega_0(t, \bar{\mu}, \bar{\nu}; \varepsilon)$ can be treated in a similar way. Then estimates (3.18) are satisfied on $[0, a]$. From (3.18) we obtain in the limit, letting ε tend to zero, inequalities (3.17). This is the desired conclusion. □

4. **Convergence of the method of lines**

We will assume nonlinear estimates of Perron type for $\partial_x F$, $\partial_v F$, $\partial_w F$, $\partial_q F$ on a subspace of Ω. Now we construct this subspace. Suppose that Assumptions $H[F, \varrho, A]$ and $H[T_h]$ are satisfied and $(\bar{\mu}, \bar{\nu})$ are defined by (3.6), (3.7). Set $\bar{c} = \omega_0(a, \bar{\mu}, \bar{\nu})$, $\tilde{c} = \omega(a, \bar{\mu}, \bar{\nu})$, $C = (\bar{c}, \tilde{c})$ and

$$\Omega[C] = \{(t, x, v, w, q) \in \Omega : \ \|v\|_B \leq \bar{c}, \ \|w\|_B \leq \bar{c}, \ \|q\|_\infty \leq \tilde{c}\}.$$

Write

$$E'_h = \{(t, x^{(m)}) \in E_h : -K < m < K\}.$$

Assumption $H[F, \sigma]$. The functions φ and F, ψ satisfy Assumption $H[F, \varrho, A]$ and

1) $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and it is nondecreasing with respect to the second variable,
2) for each $c \geq 1$ the maximal solutions of the Cauchy problem

$$\omega'(t) = c\big[\omega(t) + \sigma(t, \omega(t))\big], \ \ \omega(0) = 0,$$

is $\tilde{\omega}(t) = 0$ for $t \in [0, a]$,
3) the expressions

$$\|\partial_x F(t, x, v, w, q) - \partial_x F(t, x, \bar{v}, \bar{w}, \bar{q})\|, \ \ \|\partial_q F(t, x, v, w, q) - \partial_q F(t, x, \bar{v}, \bar{w}, \bar{q})\|,$$

$$\|\partial_v F(t, x, v, w, q) - \partial_v F(t, x, \bar{v}, \bar{w}, \bar{q})\|_\star, \ \ \|\partial_w F(t, x, v, w, q) - \partial_w F(t, x, \bar{v}, \bar{w}, \bar{q})\|_\star,$$

are estimated on $\Omega[C]$ by $\sigma\big(t, \max\{\|v - \bar{v}\|_B, \ \|w - \bar{w}\|_B\} + \|q - \bar{q}\|\big)$.

Remark 4.1. It is important in Assumption $H[F,\sigma]$ that we have assumed nonlinear estimates for $\partial_x F$, $\partial_v F$, $\partial_w F$, $\partial_q F$ on $\Omega[C]$. There are differential equations with deviated variables and differential integral equations such that Assumption $H[F,\sigma]$ is satisfied and global estimates for $\partial_x F$, $\partial_v F$, $\partial_w F$, $\partial_q F$ are not satisfied. We give comments on such equations.

Set $\Omega_0 = \Xi \times \mathbb{R}^2 \times \mathbb{R}^n$ and suppose that the function $G : \Omega_0 \to \mathbb{R}$ of the variables (t, x, p, r, q) satisfies the conditions:

1) G is continuous and for each $t \in [0, a]$ the function $G(t, \cdot)$ is of class C^2,
2) there is $A \in \mathbb{R}^+$ such that
$$|\partial_p G(P)|, \ \ |\partial_r G(P)|, \ \ \|\partial_q G(P)\|_\infty \le A, \ \ P = (t, x, p, r, q) \in \Omega_0,$$
3) $A_0 \in \mathbb{R}_+$ is defined by the relation
$$|G(t, x, 0, 0, 0_{[n]})| \le A_0 \ \text{ for } \ (t, x) \in E,$$
and there is $\varrho : [0, a] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that condition 4) of assumption $H[F, \varrho, A]$ is satisfied and
$$\|\partial_x G(t, x, p, r, q)\|_\infty \le \varrho(t, \max\{|p|, |r|\}, \|q\|_\infty) \ \text{ on } \ \Omega_0.$$

Then there is $L \in \mathbb{R}_+$ such that

(i) the operator F given by (1.3) satisfies Assumption $H[F,\sigma]$ for $\sigma(t, p) = Lp$,
(ii) the operator F given by (1.5) satisfies Assumption $H[F,\sigma]$ for $\sigma(t, p) = Lp$.

It is important that we do not assume that the partial derivatives of the second order of $G(t, \cdot)$ are bounded on Ω_0. It follows that the main assumption is satisfied for a large class of differential equations with deviated variables and differential integral equations

Remark 4.2. It is assumed in [1], [6], [7], [30] that right hand sides of differential function equations satisfy global estimates of Perron type. It follows from our considerations that local estimates are sufficient for the convergence of the method of lines.

Lemma 4.3. *If Assumptions $H[T_h]$ and $H[F,\sigma]$ are satisfied then the solution $(\mathbf{z}_h, \mathbf{u}_h)$ of (2.8)–(2.10) is unique.*

Proof. Suppose that $(\mathbf{z}_h, \mathbf{u}_h)$ and $(\tilde{\mathbf{z}}_h, \tilde{\mathbf{u}}_h)$ are solutions of (2.8)–(2.10). Set
$$\tilde{\lambda}_h(t) = \|\mathbf{z}_h - \tilde{\mathbf{z}}_h\|_{h.t}, \ \ \tilde{\zeta}_h(t) = [|\mathbf{u}_h - \tilde{\mathbf{u}}_h|]_{h.t}, \ \ t \in [0, a],$$
and $\tilde{\omega}_h = \tilde{\lambda}_h + \tilde{\zeta}_h$. It is easy to prove that there is $c_h \ge 1$ such that the function $\tilde{\omega}_h$ satisfies the differential inequality
$$D_-\tilde{\omega}_h(t) \le c_h\big[\tilde{\omega}_h(t) + \sigma(t, \tilde{\omega}_h(t))\big], \ \ t \in (0, a],$$

and $\tilde{\omega}_h(0) = 0$. It follows from condition 2) of Assumption $H[F,\sigma]$ and from classical theorems on differential inequalities [17] that $\tilde{\omega}_h(t) = 0$ for $t \in [0,a]$ and the lemma follows. □

Now we formulate the main theorem of the paper.

Theorem 4.4. *Suppose that Assumption $H[T_h]$ and $H[F,\sigma]$ are satisfied and*

1) $\bar{\mathbf{z}} : E_0 \cup E \to \mathbb{R}$ *is a solution of* (1.1), (1.2) *and* $\bar{\mathbf{z}}$ *is of class* C^2,
2) $\bar{\mathbf{u}} = \partial_x \bar{\mathbf{z}}$ *and* $(\bar{\mathbf{z}}_h, \bar{\mathbf{u}}_h)$ *is the restriction of* $(\bar{\mathbf{z}}, \bar{\mathbf{u}})$ *to* $E_{0.h} \cup E_h$.

Then

(i) *there is exactly one solution* $(\mathbf{z}_h, \mathbf{u}_h) : E_{0.h} \cup E_h \to \mathbb{R}^{1+n}$ *of* (2.8)–(2.10),
(ii) *there are* β_0, $\beta : \Delta \to \mathbb{R}_+$ *such that*

$$\|\bar{\mathbf{z}}_h - \mathbf{z}_h\|_{h.t} \le \beta_0(h), \quad [|\bar{\mathbf{u}}_h - \mathbf{u}_h|]_{h.t} \le \beta(h) \ \ \text{for} \ \ t \in [0,a], \tag{4.1}$$

and

$$\lim_{h \to 0_{[n]}} \beta_0(h) = 0, \quad \lim_{h \to 0_{[n]}} \beta(h) = 0. \tag{4.2}$$

Proof. The existence and uniqueness of a solution of (2.8)–(2.10) follows from Lemmas 3.2 and 4.3. Let $\Gamma_{h.0} : E'_h \to \mathbb{R}$, $\Gamma_h : E'_h \to \mathbb{R}^n$ be defined by the relations

$$\frac{d}{dt}\bar{\mathbf{z}}_h^{(m)}(t) = \mathbf{F}_{h.0}[\bar{\mathbf{z}}_h, \bar{\mathbf{u}}_h]^{(m)}(t) + \Gamma_{h.0}^{(m)}(t),$$

$$\frac{d}{dt}\bar{\mathbf{u}}_h^{(m)}(t) = \mathbf{F}_h[\bar{\mathbf{z}}_h, \bar{\mathbf{u}}_h]^{(m)}(t) + \Gamma_h^{(m)}(t),$$

There are γ_0, $\gamma : \Delta \to \mathbb{R}_+$ such that

$$|\Gamma_{h.0}^{(m)}(t)| \le \gamma_0(h), \quad \|\Gamma_h^{(m)}(t)\|_\infty \le \gamma(h) \ \ \text{on} \ \ E'_h$$

and

$$\lim_{h \to 0_{[n]}} \gamma_0(h) = 0, \quad \lim_{h \to 0_{[n]}} \gamma(h) = 0.$$

There is $c_\star \to \mathbb{R}_+$ such that

$$\|\partial_{xx}\bar{\mathbf{z}}(t,x)\|_{n\times n} \le c^\star \ \ \text{on} \ \ E \ \ \text{where} \ \ \partial_{xx}\bar{\mathbf{z}} = \left[\partial_{x_i x_j}\bar{\mathbf{z}}\right]_{i,j=1,\ldots,n}.$$

It follows from Lemmas 3.2 and 3.3 and from Assumption $H[T_h]$ that for $(t, x^{(m)}) \in E'_h$ we have

$$|(T_h \mathbf{z}_h)_{[t,m]}(\tau,s)| \le \bar{c}, \quad |(T_h \bar{\mathbf{z}}_h)_{[t,m]}(\tau,s)| \le \bar{c}, \quad (\tau,s) \in D[t, x^{(m)}], \tag{4.3}$$

and

$$|(T_h \mathbf{z}_h)_{\varphi[t,m]}(\tau,s)| \le \bar{c}, \quad |(T_h \bar{\mathbf{z}}_h)_{\varphi[t,m]}(\tau,s)| \le \bar{c}, \quad (\tau,s) \in D[\varphi(t, x^{(m)})], \tag{4.4}$$

and

$$\|\mathbf{u}_h^{(m)}(t)\|_\infty \le \tilde{c}, \quad \|\bar{\mathbf{u}}_h^{(m)}(t)\|_\infty \le \tilde{c}. \tag{4.5}$$

Let us denote by $\big(\omega_{h.0}(\cdot,\varepsilon),\omega_h(\cdot,\varepsilon)\big)$ the maximal solution of the Cauchy problem

$$\xi'(t) = 2A(\kappa(t)+\xi(t)) + 2\tilde{c}\sigma\big(t,\xi(t)+\kappa(t)\big) + \gamma_0(h) + \varepsilon, \tag{4.6}$$

$$\kappa'(t) = \bar{a}\sigma\big(t,\xi(t)+\kappa(t)\big) + A(1+Q)\kappa(t) + \gamma(h) + \varepsilon, \tag{4.7}$$

$$\xi(0) = \alpha_0(h) + \varepsilon, \quad \kappa(0) = \alpha(h) + \varepsilon, \tag{4.8}$$

where $\bar{a} = 1 + \tilde{c}(1+Q) + c^\star$ and α_0, $\alpha : \Delta \to \mathbb{R}_+$ are given by (3.4), (3.5). Note that the function $\omega_{h.\star}(\cdot,\varepsilon) = \omega_{h.0}(\cdot,\varepsilon) + \omega_h(\cdot,\varepsilon)$ is a solution of the initial problem

$$\omega'(t) = \hat{c}\big[\omega(t) + \sigma(t,\omega))\big] + \gamma_0(h) + \gamma(h) + 2\varepsilon, \tag{4.9}$$

$$\omega(0) = \alpha_0(h) + \alpha(h) + 2\varepsilon, \tag{4.10}$$

where $\hat{c} = \max\{A(3+Q), 2\tilde{c}+\bar{a}\}$. It follows from condition 2) of Assumption $H[F,\sigma]$ that there is $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the functions $\big(\omega_{h.0}(\cdot,\varepsilon),\omega_h(\cdot,\varepsilon)\big)$ are defined on $[0,a]$ and

$$\lim_{\varepsilon\to 0}\omega_{h.0}(t,\varepsilon) = \omega_{h.0}(t), \quad \lim_{\varepsilon\to 0}\omega_h(t,\varepsilon) = \omega_h(t),$$

where $(\omega_{h.0},\omega_h)$ is the maximal solution of (4.6)–(4.8) with $\varepsilon = 0$. Set

$$\lambda_h(t) = \|\bar{\mathbf{z}}_h - \mathbf{z}_h\|_{h.t}, \quad \zeta_h(t) = [|\bar{\mathbf{u}}_h - \mathbf{u}_h|]_{h.t}, \quad t \in [0,a].$$

We prove that for each $0 < \varepsilon < \varepsilon_0$ we have

$$\lambda_h(t) < \omega_{h.0}(t,\varepsilon) \ \text{ and } \ \zeta_h(t) < \omega_h(t,\varepsilon) \tag{4.11}$$

where $t \in [0,a]$. It is clear that there is $\tilde{t} \in (0,a]$ such that estimates (4.11) are satisfied on $[0,\tilde{t})$. Suppose by contradiction that (4.11) fails to be true on $[0,a]$. Then there is $t \in (0,a]$ such that

$$\lambda_h(\tau) < \omega_{h.0}(\tau,\varepsilon) \ \text{ and } \ \zeta_h(\tau) < \omega_h(\tau,\varepsilon) \ \text{ for } \ \tau \in [0,t)$$

and

$$\lambda_h(t) = \omega_{h.0}(t,\varepsilon) \ \text{ or } \ \zeta_h(t) = \omega_h(t,\varepsilon).$$

Suppose that $\lambda_h(t) = \omega_{h.0}(t,\varepsilon)$. Then we have

$$D_-\lambda_h(t) \geq \omega'_{h.0}(t,\varepsilon). \tag{4.12}$$

There are $m \in \mathbb{Z}^n$, $-K \leq m \leq K$, and $\bar{t} \leq t$ such that $\lambda_h(t) = |\mathbf{z}_h^{(m)}(\bar{t}) - \bar{\mathbf{z}}_h^{(m)}(\bar{t})|$. If $\bar{t} < t$ then $D_-\lambda_h(t) = 0$ which contradicts (4.12). Suppose that $\bar{t} = t$. Then we have (i) $\lambda_h(t) = \mathbf{z}_h^{(m)}(t) - \bar{\mathbf{z}}_h^{(m)}(t)$ or (ii) $\lambda_h(t) = -\big[\mathbf{z}_h^{(m)}(t) - \bar{\mathbf{z}}_h^{(m)}(t)\big]$. Let us consider the first case.

It follows from (3.4) that $x^{(m)} \in (-b,b)$. We deduce from condition 3) of Assumption $H[F,\sigma]$ and from (4.3)–(4.5) that

$$\|\partial_q F(P[\mathbf{z}_h,\mathbf{u}_h]^{(m)}(t)) - \partial_q F(P[\bar{\mathbf{z}}_h,\bar{\mathbf{u}}_h]^{(m)}(t))\| \leq \sigma\big(t,\omega_{h.0}(t,\varepsilon)+\omega_h(t,\varepsilon)\big).$$

Then we have

$$D_-\lambda_h(t) \le \frac{d}{dt}\big[\mathbf{z}_h^{(m)}(t) - \bar{\mathbf{z}}_h^{(m)}(t)\big]$$
$$\le 2A\big(\omega_{h.0}(t,\varepsilon) + \omega_h(t,\varepsilon)\big) + 2\tilde{c}\sigma\big(t, \omega_{h.0}(t,\varepsilon) + \omega_h(t,\varepsilon)\big)$$
$$+ \partial_q F(P[\mathbf{z}_h, \mathbf{u}_h]^{(m)}(t)) \circ \delta(\mathbf{z}_h - \bar{\mathbf{z}}_h)^{(m)}(t) + \gamma_0(h) + \varepsilon.$$

It follows from conditions 3), 4) of Assumption $H[F,\psi]$ that

$$\partial_q F(P[\mathbf{z}_h, \mathbf{u}_h]^{(m)}(t)) \circ \delta(\mathbf{z}_h - \bar{\mathbf{z}}_h)^{(m)}(t) \le 0.$$

Then we obtain

$$D_-\lambda_h(t) < 2A\big(\omega_{h.0}(t,\varepsilon)+\omega_h(t,\varepsilon)\big)+2\tilde{c}\sigma\big(t,\omega_{h.0}(t,\varepsilon)+\omega_h(t,\varepsilon)\big)+\gamma_0(h)+\varepsilon = \omega'_{h.0}(t,\varepsilon),$$

which contradicts (4.12). The case (ii) can be treated in a similar way.

Suppose that $\zeta_h(t) = \omega_h(t,\varepsilon)$ Then we have

$$D_-\zeta_h(t) \ge \omega'_h(t,\varepsilon). \tag{4.13}$$

There are $m \in \mathbb{Z}^n$, $-K \le m \le K$ and $\bar{t} \in (0,t]$ and $j \in \{1,\dots,n\}$ such that $\zeta_h(t) = |\mathbf{u}_{h.j}^{(m)}(\bar{t}) - \bar{\mathbf{u}}_{h.j}^{(m)}(\bar{t})|$. If $\bar{t} < t$ then $D_-\zeta_h(t) = 0$ which contradicts (4.13). Suppose that $\bar{t} = t$. Then we have (i) $\zeta_h(t) = \mathbf{u}_{h.j}^{(m)}(t) - \bar{\mathbf{u}}_{h.j}^{(m)}(t)$ or (ii) $\zeta_h(t) = -\big[\mathbf{u}_{h.j}^{(m)}(t) - \bar{\mathbf{u}}_{h.j}^{(m)}(t)\big]$. Let us consider the first case. It follows that $x^{(m)} \in (-b,b)$. We deduce from condition 3) of Assumption $H[F,\sigma]$ and from (4.3)–(4.5) that the expressions

$$\|\partial_x F(P[\mathbf{z}_h, \mathbf{u}_h]^{(m)}(t)) - \partial_x F(P[\bar{\mathbf{z}}_h, \bar{\mathbf{u}}_h]^{(m)}(t))\|,$$
$$\|\partial_v F(P[\mathbf{z}_h, \mathbf{u}_h]^{(m)}(t)) - \partial_v F(P[\bar{\mathbf{z}}_h, \bar{\mathbf{u}}_h]^{(m)}(t))\|_\star,$$
$$\|\partial_w F(P[\mathbf{z}_h, \mathbf{u}_h]^{(m)}(t)) - \partial_w F(P[\bar{\mathbf{z}}_h, \bar{\mathbf{u}}_h]^{(m)}(t))\|_\star$$

may be estimated by $\sigma\big(t, \omega_{h.0}(t,\varepsilon) + \omega_h(t,\varepsilon)\big)$. Then we have

$$D_-\zeta_h(t) \le \frac{d}{dt}\big[\mathbf{u}_{h.j}^{(m)}(t) - \bar{\mathbf{u}}_{h.j}^{(m)}(t)\big] \le \bar{a}\sigma\big(t, \omega_{h.0}(t,\varepsilon) + \omega_h(t,\varepsilon)\big) + A(1+Q)\omega_h(t,\varepsilon)$$
$$+\partial_q F(P[\mathbf{z}_h, \mathbf{u}_h]^{(m)}(t)) \circ \delta(\mathbf{u}_{h.j} - \bar{\mathbf{u}}_{h.j})^{(m)}(t) + \gamma(h).$$

It follows from conditions 3), 4) of Assumption $H[F,\psi]$ that

$$\partial_q F(P[\mathbf{z}_h, \mathbf{u}_h]^{(m)}(t)) \circ \delta(\mathbf{u}_{h.j} - \bar{\mathbf{u}}_{h.j})^{(m)}(t) \le 0.$$

Then we obtain

$$D_-\zeta_h(t) < \bar{a}\sigma\big(t, \omega_{h.0}(t,\varepsilon) + \omega_h(t,\varepsilon)\big) + A(1+Q)\omega_h(t,\varepsilon) + \gamma(h) + \varepsilon = \omega'_h(t,\varepsilon)$$

which contradicts (4.13). The case (ii) can be treated in a similar way.

Then inequalities (4.11) are satisfied on $[0,a]$. From (4.11) we obtain in the limit, letting ε tend to zero, the following estimates

$$\lambda_h(t) \le \omega_{h.0}(t) \;\text{ and }\; \zeta_h(t) \le \omega_h(t), \quad t \in [0,a],$$

where $(\omega_{h.0}, \omega_h)$ is the maximal solution of (4.6)–(4.8) with $\varepsilon = 0$. It follows that conditions (4.1), (4.2) are satisfied for $\beta_0(h) = \omega_{h.0}(a)$ and $\beta(h) = \omega_h(a)$. This completes the proof. □

Remark 4.5. Suppose that all the assumptions of Theorem 4.4 are satisfied with $\sigma(t, p) = Lp$, $(t, p) \in [0, a] \times \mathbb{R}_+$, where $L \in \mathbb{R}_+$. Then we have

$$\|\bar{\mathbf{z}}_h - \mathbf{z}_h\|_{h.t} + [\|\bar{\mathbf{u}}_h - \mathbf{u}_h\|]_{h.t} \leq \tilde{\beta}(h) \;\text{ for }\; t \in [0, a],$$

where

$$\tilde{\beta}(h) = \big(\alpha_0(h) + \alpha(h)\big) \exp\{\hat{c}(1 + L)a\} + \frac{\gamma_0(h) + \gamma(h)}{\hat{c}(1 + L)}\big[\exp\{\hat{c}(1 + L)a\} - 1\big].$$

The above estimates is obtained by solving problem (4.9), (4.10) with $\sigma(t, p) = Lp$ and $\varepsilon = 0$.

5. **Examples**

Put $n = 2$ and $E = [0, 0.5] \times [-1, 1] \times [-1, 1]$, $E_0 = 0 \times [-1, 1] \times [-1, 1]$. We consider initial boundary value problems for functional differential equations with solutions defined on E. We apply the Euler difference method or the Lax difference scheme to solve numerically ordinary functional differential problems. Nodal point on $[0, 0.5]$ are defined by $t^{(r)} = rh_0$, $r = 0, 1, \ldots, N_0$.

Example 5.1. Consider the differential equation with deviated variable

$$\begin{aligned}\partial_t z(t, x, y) = x\big\{&2\partial_x z(t, x, y) + \sin\big[\partial_x z(t, x, y) - z(t, 0.5x, 0.5y)\big]\big\}\\ &+ y\big\{2\partial_y z(t, x, y) + \cos\big[\partial_y z(t, x, y) + z(t, 0.5(x + y), 0.5(x - y))\big]\big\}\\ &+ f(t, x, y).\end{aligned}$$

with initial boundary conditions

$$z(0, x, y) = 1, \;\; (x, y) \in [-1, 1] \times [-1, 1],$$

$$z(t, -1, y) = z(t, 1, y) = \exp\{t(1 - y^2)\}, \;\; (t, y) \in [0, 0.5] \times [-1, 1],$$

$$z(t, x, -1) = z(t, x, 1) = \exp\{t(x^2 - 1)\}, \;\; (t, x) \in ([0, 0.5] \times [-1, 1].$$

where

$$f(t, x, y) = \exp\{t(x^2 - y^2)\}(x^2 - y^2)(1 - 4t) - y\cos\big\{\exp\{txy\} - 2yt\exp\{t(x^2 - y^2)\}\big\}$$

$$-x\sin\big\{\exp\{t(x^2 - y^2)\}\big[2xt - \exp\{-\frac{3}{4}t(x^2 - y^2)\}\big]\big\}.$$

The solution of the above problem is known, it is $\bar{z}(t,x,y) = \exp\{t(x^2 - y^2)\}$. Let us denote by $(\tilde{z}_h, \tilde{z}_{h.x}, \tilde{z}_{h.y})$ approximate solutions of ordinary differential equation corresponding to the above problem. They are obtained by using the explicit Euler difference method. Set

$$\varepsilon_h^{(r)} = \max\{|(\bar{z} - \tilde{z}_h)(t^{(i)}, x^{(m_1)}, y^{(m_2)})| : \ x^{(m1)}, y^{(m2)} \in [-1,1], \ 0 \le i \le r\}, \tag{5.1}$$

and

$$\varepsilon_{h.x}^{(r)} = \max\{|(\partial_x \bar{z} - \tilde{z}_{h.x})(t^{(i)}, x^{(m_1)}, y^{(m_2)})| : \ x^{(m1)}, y^{(m2)} \in [-1,1], \ 0 \le i \le r\}, \tag{5.2}$$

$$\varepsilon_{h.y}^{(r)} = \max\{|(\partial_y \bar{z} - \tilde{z}_{h.y})(t^{(i)}, x^{(m_1)}, y^{(m_2)})| : \ x^{(m1)}, y^{(m2)} \in [-1,1], \ 0 \le i \le r\}. \tag{5.3}$$

Let us denote by $\hat{z}_h$ an approximate solution of (1.1), (1.2) which is obtained by using the Lax difference scheme. Set

$$\hat{\varepsilon}_h^{(r)} = \max\{|(\bar{z} - \hat{z}_h)(t^{(i)}, x^{(m_1)}, y^{(m_2)})| : \ x^{(m1)}, y^{(m2)} \in [-1,1], \ 0 \le i \le r\}.$$

In the Table we give experimental values of the errors $(\varepsilon_h, \varepsilon_{h.x}, \varepsilon_{h.y})$ and $\hat{\varepsilon}_h$. Note

TABLE 1. Table of errors, $h_0 = 0.0001$, $h_1 = h_2 = 0.01$

$t^{(r)}$	$\varepsilon_h^{(r)}$	$\varepsilon_{h.x}^{(r)}$	$\varepsilon_{h.y}^{(r)}$	$\hat{\varepsilon}_h^{(r)}$
0.30	0.000353	0.000836	0.001217	0.003378
0.25	0.000464	0.001092	0.002091	0.007177
0.40	0.000545	0.003375	0.003255	0.005060
0.45	0.000698	0.005012	0.005301	0.007177
0.50	0.000712	0.006413	0.005956	0.012831

that the errors of the classical difference method $\hat{\varepsilon}_h^{(r)}$ are larger than the errors obtained by discretization of the numerical method of lines $\varepsilon_h^{(r)}$. This is due to the fact that the Lax difference scheme has the following property: we approximate partial derivatives of z with respect to spatial variables by difference expressions which are calculated by means of previous values of the approximate solution. In our approach we approximate the partial derivatives for the unknown function in (1.1), (1.2) by using difference equations which are generated by the original problem.

Example 5.2. Consider the differential integral equation

$$\partial_t z(t,x,y) = x \arctan\left[\partial_x z(t,x,y) + \frac{\pi^2}{4}\int_0^x z(t,s,y)ds\right]$$

$$+y \arctan\left[\partial_y z(t,x,y) - \frac{\pi^2}{4}\int_0^{-y} z(t,s,y)ds\right] - \int_0^t z(\tau,x,y)d\tau + \cos\frac{\pi x}{2}\cos\frac{\pi y}{2}$$

with the initial boundary condition

$$z(t,x,y) = 0 \ \text{ for } \ (t,x,y) \in E_0 \cup \partial_0 E.$$

The solution of the above problem is known. It is $\bar{z}(t,x,y) = \sin t \cos \frac{\pi x}{2} \cos \frac{\pi y}{2}$. Let us denote by $(\tilde{z}_h, \tilde{z}_{h.x}, \tilde{z}_{h.y})$ approximate solutions of ordinary differential equation corresponding to the above problem. They are obtained by using the implicit Euler difference method.

Let $(\varepsilon_h, \varepsilon_{h.x}, \varepsilon_{h.y})$ be defined by (5.1)–(5.3). In the Table we give experimental values of the above defined errors.

TABLE 2. Table of errors, $h_0 = 0.01$, $h_1 = h_2 = 0.005$

$t^{(r)}$	$\varepsilon_h^{(r)}$	$\varepsilon_{h.x}^{(r)}$	$\varepsilon_{h.y}^{(r)}$
0.30	0.001972	0.004665	0.004187
0.25	0.002213	0.006204	0.005626
0.40	0.002472	0.007900	0.006244
0.45	0.002792	0.008731	0.007028
0.50	0.002931	0.009243	0.008965

Two types of assumptions are needed in theorems on the convergence of functional difference schemes for (1.1), (1.2). The first type conditions the regularity of given functions. The second type conditions concern the mesh and they are known as (CFL) conditions (see [13], Theorem 3.21)

The (CFL) conditions for the differential integral equation considered here have the form:

$$2h_0 \le h_i, \quad i = 1, 2.$$

Note the the steps $h_0 = 0.01$, $h_1 = h_2 = 0.005$ do not satisfy the above condition and the classical Lax difference scheme is not applicable.

Results on the method of lines presented here have the potential for applications in the numerical solving of first order partial functional differential equations.

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