Dynamic Systems and Applications 23 (2014) 1-14

GENERALIZED EMDEN-FOWLER-NEUTRAL TYPE EQUATIONS WITH PROPERTIES A AND B

A. K. TRIPATHY, S. PANIGRAHI, AND K. SRILAKSHMI

Department of Mathematics, Sambalpur University, Sambalpur-768 019, INDIA
School of Mathematics and Statistics, University of Hyderabad
Hyderabad-500 046, INDIA
arun_tripathy70@rediffmail.com
panigrahi2008@gmail.com
kolaganis@yahoo.co.in

ABSTRACT. In this paper, a class of n-th order functional differential equations is considered for which the generalized Emden-Fowler-neutral type equation

$$(E) \qquad (y(t) + p(t)y(t-\tau))^{(n)} + q(t)|y(t-\sigma)|^{\mu(t)} sgn\ y(t-\sigma) = 0$$

can be considered as a nonlinear model, where $n \geqslant 2$, $q \in L_{loc}(\mathbb{R}_+, \mathbb{R})$, $p \in L_{loc}(\mathbb{R}_+, \mathbb{R})$, and $\mu \in C(\mathbb{R}_+, (0,1])$ is a nondecreasing function. It has been proved that the oscillation properties of (E) substantially depend on the rate at which the function $\mu^+ - \mu(t)$ tends to zero as $t \to \infty$, where $\mu^+ = \lim_{t\to\infty} \mu(t)$. New sufficient conditions for (E) to have Properties A and B are established.

AMS (MOS) Subject Classification. 34C10, 34C15.

Key Words: Functional differential equations, neutral, nonlinear, higher order, oscillation, Properties A and B.

1. INTRODUCTION

Properties A and B are introduced in [8]. The problem of determining criteria for nonlinear differential equations of the second and higher order to have that each solution oscillates or converges to zero (or be oscillatory, converges to zero or diverges to ∞) has been interest to researchers even before the now commonly used names of Properties A and B. Actually, it has its roots in the pioneering paper of Atkinson [2] for second order equations and the work of Kiguradze [7], who gave sufficient conditions for this behaviour in case n is even, and Licko and Svec [12], who gave necessary and sufficient conditions for n both even and odd. In the present paper, the authors have made an attempt to establish Properties A and B for a general class of nonlinear neutral differential equations of the form

$$(1.1) \qquad (y(t) + p(t)y(t-\tau))^{(n)} + q(t)|y(t-\sigma)|^{\mu(t)} sgn\ y(t-\sigma)) = 0,$$

Received May 4, 2013

where $n \geqslant 2$, $q \in L_{loc}(\mathbb{R}_+, \mathbb{R})$, $p \in L_{loc}(\mathbb{R}_+, \mathbb{R})$, $\mu \in C(\mathbb{R}_+, (0, 1])$, and $\sigma \geqslant 0$, $\tau > 0$ are constants. In case $\mu(t) \equiv$ constant > 0, the oscillatory properties of (1.1) have been extensively studied (see for e.g. [10, 11, 13, 14, 15]) classifying as linear, sublinear and superlinear, where as $\mu(t) \not\equiv$ constant, to the extent of authors' knowledge, the analogous questions have not been examined. If $p(t) \equiv 0$ and $\sigma = 0$, then (1.1) reduces to

$$y^{(n)}(t) + q(t)|y(t)|^{\mu(t)} sgn\ y(t)) = 0. \tag{1.2}$$

Graef et. al [4] have studied (1.2) and established sufficient conditions under which (1.2) admits Properties A and B. It reveals that their work is motivated by the recent work of Grammatikopulos et. al [5]. However, nothing is known about (1.1) to have Properties A and B. Clearly, (1.2) is a particular case of (1.1). Hence, the study of (1.1) subject to the Properties A and B is more interesting. In this direction, we refer the reader to some of the works [8, 9, 10, 11] and the references cited therein.

Let $t_0 \in \mathbb{R}_+$, $t_0 > \max\{\sigma, \tau\}$. A function $y : [t_0, +\infty) \to \mathbb{R}$ is said to be a proper solution of (1.1) if it is locally continuous along with its derivatives of order up to and including $n - 1$, $\sup\{|y(t)| : t \in [t_0, +\infty)\} > 0$, for $t \geqslant t_0$, there exists a function $u \in C(\mathbb{R}_+, \mathbb{R})$ such that $u(t) = y(t)$ on $[t_0, +\infty)$, and the equality

$$(u(t) + p(t)u(t - \tau))^{(n)} + q(t)|u(t - \sigma)|^{\mu(t)} sgn\ u(t - \sigma) = 0$$

holds for $t \in [t_0, +\infty)$. A proper solution of (1.1) is said to be oscillatory, if it has a sequence of zeros tending to $+\infty$. Otherwise, the solution is said to be nonoscillatory.

Definition 1.1 ([4, 8])**.** We say that (1.1) has Property A, if any proper solution y is oscillatory if n is even and is either oscillatory or satisfies

$$|y^{(i)}(t)| \downarrow 0 \text{ as} t \uparrow +\infty \quad (i = 0, 1, 2, \dots, n - 1), \tag{1.3}$$

if n is odd.

Definition 1.2 ([4, 8])**.** We say that (1.1) has Property B if any proper solution y is either oscillatory, satisfies (1.3) or satisfies

$$|y^{(i)}(t)| \uparrow +\infty \text{ as } t \uparrow +\infty \quad (i = 0, 1, 2, \dots, n - 1), \tag{1.4}$$

if n is even, and is either oscillatory or satisfies (1.4), if n is odd.

The higher order nonlinear neutral delay differential equation

$$(y(t) + p(t)y(t - \tau))^{(n)} + q(t)|y(t - \sigma)|^{\lambda} sgn\ y(t - \sigma) = 0, \tag{1.5}$$

where $p \in L_{loc}(\mathbb{R}_+, \mathbb{R})$, $q \in L_{loc}(\mathbb{R}_+, \mathbb{R}_+)$, $\lambda > 0$, and $\lambda \neq 1$ is a special case of (1.1) if we let $\lambda = \lim_{t\to\infty} \mu(t)$ and $\mu(t) \neq \lambda$ for $t \in \mathbb{R}_+$. It may turn out to be that in certain cases, (1.1) may not have Property A(B), but the 'limiting' equation (1.5) does have this property.

2. SOME AUXILIARY LEMMAS

In the sequel, $\bar{C}^{n-1}_{loc}([t_0,+\infty))$ denotes the set of all functions $y:[t_0,+\infty)\to\mathbb{R}$ that are absolutely continuous on any finite subinterval of $[t_0,+\infty)$ along with their derivatives of order up to and including $n-1$.

Lemma 2.1 ([8]). *Let $u\in\bar{C}^{n-1}_{loc}([t_0,+\infty))$ satisfy $u(t)>0$ and $u^{(n)}(t)\leqslant 0$ ($u^{(n)}(t)\geqslant 0$), for $t\geqslant t_0$ and $u^{(n)}(t)\not\equiv 0$ in any neighbourhood of $+\infty$. Then there exist $t_1>t_0$ and $l\in\{0,1,2,\dots,n\}$ such that $(l+n)$ is odd (even) and*

$$u^{(i)}(t)>0,\quad \textit{for } t\geqslant t_1\quad (i=0,1,2,\dots,l-1),$$

$$(-1)^{i+l}u^{(i)}(t)>0,\quad \textit{for } t\geqslant t_1\quad (i=l,l+1,\dots,n-1). \tag{2.1}$$

NOTE. In case $l=0$, we mean that the second inequality in (2.1) holds, while if $l=n$, the first one holds.

Lemma 2.2 ([4, 5]). *Let $u\in\bar{C}^{n-1}_{loc}([t_0,+\infty))$ and (2.1) be satisfied for some $l\in\{1,2,\dots,n-1\}$ with $(l+n)$ odd (even). Then*

$$\int^{+\infty} t^{n-l-1}|u^{(n)}(t)|dt<+\infty.$$

If, moreover,

$$\int^{+\infty} t^{n-l-1}|u^{(n)}(t)|dt=+\infty, \tag{2.2}$$

then there exists $t^*\geqslant t_0$ such that

$$\frac{u^{(i)}(t)}{t^{l-i}}\downarrow,\ \frac{u^{(i)}(t)}{t^{l-i-1}}\uparrow\quad (i=0,1,2,\dots,n-1), \tag{2.3}$$

$$u(t)\geqslant\frac{t^{l-1}}{l!}u^{(l-1)}(t),\qquad \textit{for } t\geqslant t^*, \tag{2.4}$$

and

$$u^{(l-1)}(t)\geqslant\frac{t}{(n-l)!}\int_t^{+\infty}s^{n-l-1}|u^{(n)}(s)|ds + \frac{1}{(n-l)!}\int_{t^*}^{t}s^{n-l}|u^{(n)}(s)|ds,\quad t\geqslant t^*. \tag{2.5}$$

Lemma 2.3 ([3]). *Let $u\geq 0$ and $v\geq 0$. Then*

$$u^\gamma+v^\gamma\geqslant\delta(u+v)^\gamma,\quad \gamma>0,$$

where $\delta=1$, if $0<\gamma\leq 1$ and $\delta=2^{1-\gamma}$, if $\gamma>1$.

3. Eq. (1.1) WITH PROPERTIES A AND B

In this section sufficient conditions are obtained in order for (1.1) to have no solutions of the type (2.1), which then implies that (1.1) has either Property A or Property B. For convenience, we view (1.1) as

$$(E_1) \qquad (y(t)+p(t)y(t-\tau))^{(n)}+q(t)|y(t-\sigma)|^{\mu(t)}sgn\ y(t-\sigma))=0$$

and

$$(E_2) \qquad (y(t)+p(t)y(t-\tau))^{(n)}-q(t)|y(t-\sigma)|^{\mu(t)}sgn\ y(t-\sigma))=0,$$

where $q \in L_{loc}(\mathbb{R}_+,\mathbb{R}_+)$.

Proposition 3.1. *Let $0 \leqslant p(t) \leqslant a < \infty$ and $\mu^+ = \lim_{t\to\infty}\mu(t)$. For every $0<\epsilon<\mu^+$, let $l \in \{1,2,\dots,n-1\}$ with $(l+n)$ odd and*

$$\begin{aligned}\limsup_{t\to+\infty}\Big[(t-\sigma)^{(\mu^+-\epsilon)}\int_t^{+\infty}(s-\sigma)^{n-l-1+(l-1)(\mu^+-\epsilon)}Q(s)ds \\ +\frac{1}{t}\int_0^t(s-\sigma)^{n-l+l(\mu^+-\epsilon)}Q(s)ds\Big] \\ (H_1) \qquad > l!(n-l)!\Omega(\mu^+-\epsilon)(1+a^{\mu^+-\epsilon})\end{aligned}$$

holds,where $Q(t)=\min\{q(t),q(t-\tau)\}$, $t \geqslant \tau$, and $\Omega(s)=\begin{cases}1, & s=1\\ 0, & 0<s<1.\end{cases}$ Then (E_1) has no solution of the type (2.1).

Proof. We will first show that (H_1) implies

$$(H_2) \qquad \int^{+\infty} t^{n-l-1+l(\mu^+-\epsilon)}Q(t)dt=+\infty.$$

Indeed, if this is not the case, then it is easy to verify that

$$(t-\sigma)^{(\mu^+-\epsilon)}\int_t^{+\infty}(s-\sigma)^{n-l-1+(l-1)(\mu^+-\epsilon)}Q(s)ds \to 0 \text{ as } t\to\infty$$

and

$$\begin{aligned}&\frac{1}{t}\int_0^t(s-\sigma)^{n-l+l(\mu^+-\epsilon)-l}Q(s)ds\\ &=\frac{1}{t}\int_0^{t^*}(s-\sigma)^{n-l+l(\mu^+-\epsilon)}Q(s)ds+\frac{1}{t}\int_{t^*}^t(s-\sigma)^{n-l+l(\mu^+-\epsilon)}Q(s)ds\\ &<\frac{1}{t}\int_0^{t^*}(s-\sigma)^{n-l+l(\mu^+-\epsilon)}Q(s)ds+\int_{t^*}^t(s-\sigma)^{n-l-1+l(\mu^+-\epsilon)}Q(s)ds\\ &\to 0 \text{ as } t\to\infty,\end{aligned}$$

implies a contradiction to (H_1). Thus our assertion holds, that is, Lemma 2.2 can be applied in this case.

Suppose that (E_1) has a proper nonoscillatory solution $y(t)$ on $[t_0, +\infty)$ with $l \in \{1, 2, \ldots, n-1\}$ and $(l+n)$ is odd. Setting

$$z(t) = y(t) + p(t)y(t-\tau), \tag{3.1}$$

we obtain from (E_1) that

$$z^{(n)}(t) + q(t)|y(t-\sigma)|^{\mu(t)} sgn\ y(t-\sigma) = 0.$$

Without loss of generality, we may assume that $y(t) > 0$, for $t \geqslant t_0$. Hence for $t_1 \geqslant t_0$,

$$z^{(n)}(t) = -q(t)(y(t-\sigma))^{\mu(t)} \leqslant 0, \quad \not\equiv 0, \tag{3.2}$$

for $t \geqslant t_1$. Consequently, $z(t)$ is a monotonic function on $[t_1, +\infty)$. Since $y(t)$ is a proper nonoscillatory solution, then $z(t)$ is nonoscillatory and hence it satisfies (2.1). By Lemma 2.1, there exist $c > 0$ and $t_2 \geqslant t_1$ such that $z(t) \geqslant ct^{l-1}$, for $t \geqslant t_2$. Therefore, from (H_1), we see that $z(t)$ satisfies the hypotheses of Lemma 2.2, that is, (2.3) holds. Because $\lim_{t\to\infty} \mu(t) = \mu^+$, then there exists $T > 0$ such that for every $0 < \epsilon < \mu^+$, $\mu(t) \geqslant \mu^+ - \epsilon$ for $t \geqslant T$. Let $t_3 = \max\{T, t_2\}$. Due to (3.2), it is to verify that

$$z^{(n)}(t) + q(t)(y(t-\sigma))^{\mu(t)} + a^{(\mu^+-\epsilon)} z^{(n)}(t-\tau) + a^{(\mu^+-\epsilon)} q(t-\tau)(y(t-\tau-\sigma))^{\mu(t-\tau)} = 0$$

for $t \geqslant t_3$. Hence, for $t \geqslant t_3 + \tau$, using Lemma 2.3, we obtain

$$\begin{aligned}
0 &\geqslant z^{(n)}(t) + a^{(\mu^+-\epsilon)} z^{(n)}(t-\tau) \\
&\quad + Q(t)[(y(t-\sigma))^{\mu(t)} + a^{(\mu^+-\epsilon)}(y(t-\tau-\sigma))^{\mu(t-\tau)}] \\
&\geqslant z^{(n)}(t) + a^{(\mu^+-\epsilon)} z^{(n)}(t-\tau) \\
&\quad + Q(t)[(y(t-\sigma))^{(\mu^+-\epsilon)} + a^{(\mu^+-\epsilon)}(y(t-\tau-\sigma))^{(\mu^+-\epsilon)}] \\
&\geqslant z^{(n)}(t) + a^{(\mu^+-\epsilon)} z^{(n)}(t-\tau) + Q(t)(z(t-\sigma))^{(\mu^+-\epsilon)},
\end{aligned}$$

where $0 < z(t) \leqslant y(t) + ay(t-\tau)$. Thus, for $t \geqslant t_4 > t_3 + \tau$,

$$[z(t) + a^{(\mu^+-\epsilon)} z(t-\tau)]^{(n)} \leqslant -Q(t)(z(t-\sigma))^{(\mu^+-\epsilon)}. \tag{3.3}$$

We note that, when (3.2) holds so does (3.3) for $t \geqslant t_4$. By Lemma 2.2 and in view of (3.3), we have

$$\begin{aligned}
[z(t)+a^{(\mu^+-\epsilon)}z(t-\tau)]^{(l-1)} &\geqslant \frac{t}{(n-l)!}\int_t^{+\infty} s^{n-l-1}|[z(s)+a^{(\mu^+-\epsilon)}z(s-\tau)]^{(n)}|ds \\
&+\frac{1}{(n-l)!}\int_{t^*}^{t} s^{n-l}|[z(s)+a^{(\mu^+-\epsilon)}z(s-\tau)]^{(n)}|ds \\
&\geqslant \frac{t}{(n-l)!}\int_t^{+\infty} s^{n-l-1}Q(s)(z(s-\sigma))^{(\mu^+-\epsilon)}ds \\
&+\frac{1}{(n-l)!}\int_{t^*}^{t} s^{n-l}Q(s)(z(s-\sigma))^{(\mu^+-\epsilon)}ds \\
&\geqslant \frac{t}{l!(n-l)!}\int_t^{+\infty} s^{n-l-1}(s-\sigma)^{(l-1)(\mu^+-\epsilon)}Q(s)[z^{(l-1)}(s-\sigma)]^{(\mu^+-\epsilon)}ds \\
&+\frac{1}{l!(n-l)!}\int_{t^*}^{t} s^{n-l}(s-\sigma)^{(l-1)(\mu^+-\epsilon)}Q(s)[z^{(l-1)}(s-\sigma)]^{(\mu^+-\epsilon)}ds \\
&\geqslant \frac{t[z^{(l-1)}(t-\sigma)]^{(\mu^+-\epsilon)}}{l!(n-l)!}\int_t^{+\infty} s^{n-l-1}(s-\sigma)^{(l-1)(\mu^+-\epsilon)}Q(s)ds \\
&+\frac{1}{l!(n-l)!}\int_{t^*}^{t} s^{n-l}(s-\sigma)^{l(\mu^+-\epsilon)}Q(s)\left[\frac{z^{(l-1)}(s-\sigma)}{(s-\sigma)}\right]^{(\mu^+-\epsilon)}ds \\
&\geqslant \frac{t[z^{(l-1)}(t-\sigma)]^{(\mu^+-\epsilon)}}{l!(n-l)!}\int_t^{+\infty} (s-\sigma)^{n+(l-1)(\mu^+-\epsilon)-l-1}Q(s)ds \\
&+\frac{1}{l!(n-l)!}\left[\frac{z^{(l-1)}(t-\sigma)}{(t-\sigma)}\right]^{(\mu^+-\epsilon)}\int_{t^*}^{t}(s-\sigma)^{n+l(\mu^+-\epsilon)-l}Q(s)ds,
\end{aligned}$$

that is, since $z^{(l-1)}(t)$ is nondecreasing,

$$\begin{aligned}
&z^{(l-1)}(t)+a^{\mu^+-\epsilon}z^{(l-1)}(t) \\
&\quad\geqslant \frac{t[z^{(l-1)}(t-\sigma)]^{(\mu^+-\epsilon)}}{l!(n-l)!(t-\sigma)^{(\mu^+-\epsilon)}}(t-\sigma)^{(\mu^+-\epsilon)}\int_t^{+\infty}(s-\sigma)^{n+(l-1)(\mu^+-\epsilon)-l-1}Q(s)ds \\
&\quad+\frac{1}{l!(n-l)!}\left[\frac{z^{(l-1)}(t-\sigma)}{(t-\sigma)}\right]^{(\mu^+-\epsilon)}\int_{t^*}^{t}(s-\sigma)^{n+l(\mu^+-\epsilon)-l}Q(s)ds.
\end{aligned}$$

Consequently, for $t > t^* > t_4$ the last inequality reduces to

$$\begin{aligned}
(1+a^{\mu^+-\epsilon})\frac{z^{(l-1)}(t)}{t} &\geqslant \frac{1}{l!(n-l)!}\left[\frac{z^{(l-1)}(t-\sigma)}{(t-\sigma)}\right]^{(\mu^+-\epsilon)} \\
&\times\left[(t-\sigma)^{(\mu^+-\epsilon)}\int_t^{+\infty}(s-\sigma)^{n+(l-1)(\mu^+-\epsilon)-l-1}Q(s)ds\right. \\
&\left.+\frac{1}{t}\int_{t_4}^{t}(s-\sigma)^{n+l(\mu^+-\epsilon)-l}Q(s)ds\right]. \qquad (3.4)
\end{aligned}$$

Since $\frac{z^{(l-1)}(t)}{t}$ is nonincreasing for large t,

$$\limsup_{t\to+\infty}\left[\frac{z^{(l-1)}(t)}{t}\right]^{1-(\mu^+-\epsilon)}\leqslant \Omega(\mu^+-\epsilon),$$

and hence (3.4) becomes

$$\begin{aligned}&(1+a^{\mu^+-\epsilon})\left[\frac{z^{(l-1)}(t)}{t}\right]^{1-(\mu^+-\epsilon)}\\ &\geqslant \frac{1}{l!(n-l)!}\Big[(t-\sigma)^{(\mu^+-\epsilon)}\int_t^{+\infty}(s-\sigma)^{n+(l-1)(\mu^+-\epsilon)-l-1}Q(s)ds\\ &\quad+\frac{1}{t}\int_{t_4}^{t}(s-\sigma)^{n+l(\mu^+-\epsilon)-l}Q(s)ds\Big],\end{aligned}$$

that is,

$$\begin{aligned}\limsup_{t\to+\infty}&\Big[(t-\sigma)^{(\mu^+-\epsilon)}\int_t^{+\infty}(s-\sigma)^{n+(l-1)(\mu^+-\epsilon)-l-1}Q(s)ds\\ &\quad+\frac{1}{t}\int_{t_4}^{t}(s-\sigma)^{n+l(\mu^+-\epsilon)-l}Q(s)ds\Big]\\ &\leqslant (1+a^{\mu^+-\epsilon})\Omega(\mu^+-\epsilon)l!(n-l)!,\end{aligned}$$

a contradiction to (H_1). This completes the proof of the proposition. □

Proposition 3.2. *Let $-\infty<-b\leqslant p(t)\leqslant 0$, $b>0$, and $\mu^+=\lim_{t\to\infty}\mu(t)$. Assume that*

(H_3) $l\in\{1,2,\dots,n-1\}$ *with $(l+n)$ odd and*

$$\limsup_{t\to+\infty}\left[(t-\sigma)^{\mu(t)}\int_t^{+\infty}(s-\sigma)^{n+(l-1)\mu(s)-l-1}q(s)ds+\frac{1}{t}\int_0^t(s-\sigma)^{n+l\mu(s)-l}q(s)ds\right]$$
$$>l!(n-l)!\Omega(\mu^+).$$

Furthermore, assume that $\tau<\sigma$,

(H_4) $l\in\{1,2,\dots,n-2\}$ *with $(l+n)$ even and*

$$\begin{aligned}\limsup_{t\to+\infty}&\Big[(t+\tau-\sigma)^{\mu(t)}\int_t^{+\infty}(s+\tau-\sigma)^{n+(l-1)\mu(s)-l-1}q(s)b^{-\mu(s)}ds\\ &+\frac{1}{t}\int_0^t(s+\tau-\sigma)^{n+l\mu(s)-l}q(s)b^{-\mu(s)}ds\Big]>l!(n-l)!\Omega(\mu^+)\end{aligned}$$

holds, where $\Omega(s)$ is defined in Proposition 3.1. Then (E_1) has no solution of the type (2.1).

Proof. Proceeding as in the proof of Proposition 3.1, it is easy to verify that (H_4) implies

(H_5) $\int^{+\infty}t^{n+l\mu(t)-l-1}q(t)dt=+\infty$,

and (H_3) implies

(H_6) $\int^{+\infty}t^{n+l\mu(t)-l-1}q(t)dt=+\infty$.

Let $y(t)$ be a proper nonoscillatory solution of (1.1) on $[t_0,\infty)$ and $l\in\{1,2,\ldots,n-1\}$ such that $(l+n)$ odd/(even). Defining z as in (3.1), we get (3.2) for $t\geqslant t_1$. Consequently, $z(t)$ is monotonic on $[t_1,\infty)$. Here, we have either $z(t)>0$ or $z(t)<0$, for $t\geqslant t_1$. Suppose the former holds. Then $z(t)\leqslant y(t)$ implies that $|z^{(n)}(t)|\geqslant q(t)(z(t-\sigma))^{\mu(t)}>0$ for $t\geqslant t_1+\sigma$. Therefore, from (H_3) it follows that $z(t)$ satisfies the hypotheses of Lemma 2.2, that is, (2.3) holds and hence

$$\begin{aligned}
z^{(l-1)}(t) &\geqslant \frac{t}{(n-l)!}\int_t^{+\infty} s^{n-l-1}|z^{(n)}(s)|ds+\frac{1}{(n-l)!}\int_{t^*}^{t} s^{n-l}|z^{(n)}(s)|ds\\
&\geqslant \frac{t}{(n-l)!}\int_t^{+\infty} s^{n-l-1}q(s)(z(s-\sigma))^{\mu(s)}ds\\
&\quad+\frac{1}{(n-l)!}\int_{t^*}^{t} s^{n-l}q(s)(z(s-\sigma))^{\mu(s)}ds
\end{aligned}$$

for $t\geqslant t^*>t_2>t_1+\sigma$. Using (2.4), we obtain from the last inequality that

$$\begin{aligned}
z^{(l-1)}(t) &\geqslant \frac{t}{l!(n-l)!}\int_t^{+\infty} s^{n-l-1}q(s)(s-\sigma)^{(l-1)\mu(s)}(z^{(l-1)}(s-\sigma))^{\mu(s)}ds\\
&\quad+\frac{1}{l!(n-l)!}\int_{t^*}^{t} s^{n-l}q(s)(s-\sigma)^{(l-1)\mu(s)}(z^{(l-1)}(s-\sigma))^{\mu(s)}ds\\
&\geqslant \frac{t(z^{(l-1)}(t-\sigma))^{\mu(t)}}{l!(n-l)!}\int_t^{+\infty} s^{n-l-1}q(s)(s-\sigma)^{(l-1)\mu(s)}ds\\
&\quad+\frac{1}{l!(n-l)!}\int_{t_2}^{t} s^{n-l}q(s)(s-\sigma)^{l\mu(s)}\left[\frac{z^{(l-1)}(s-\sigma)}{(s-\sigma)}\right]^{\mu(s)}ds\\
&\geqslant \frac{t[z^{(l-1)}(t-\sigma)]^{\mu(t)}}{l!(n-l)!}\int_t^{+\infty}(s-\sigma)^{n+(l-1)\mu(s)-l-1}q(s)ds\\
&\quad+\frac{1}{l!(n-l)!}\left[\frac{z^{(l-1)}(t-\sigma)}{(t-\sigma)}\right]^{\mu(t)}\int_{t_2}^{t}(s-\sigma)^{n+l\mu(s)-l}q(s)ds
\end{aligned}$$

due to (2.3). Hence,

$$\begin{aligned}
\frac{z^{(l-1)}(t)}{t} &\geqslant \frac{1}{l!(n-l)!}\left[\frac{z^{(l-1)}(t-\sigma)}{(t-\sigma)}\right]^{\mu(t)}\\
&\quad\times\left[(t-\sigma)^{\mu(t)}\int_t^{+\infty}(s-\sigma)^{n+(l-1)\mu(s)-l-1}q(s)ds\right.\\
&\quad\left.+\frac{1}{t}\int_{t_2}^{t}(s-\sigma)^{n+l\mu(s)-l}q(s)ds\right]. \qquad (3.5)
\end{aligned}$$

Since $\frac{z^{(l-1)}(t)}{t}$ is nonincreasing for large t,

$$\limsup_{t\to+\infty}\left[\frac{z^{(l-1)}(t)}{t}\right]^{1-\mu(t)}\leqslant \Omega(\mu^+),$$

and hence (3.5) becomes

$$\left[(t-\sigma)^{\mu(t)}\int_t^{+\infty}(s-\sigma)^{n+(l-1)\mu(s)-l-1}q(s)ds+\frac{1}{t}\int_{t_2}^t(s-\sigma)^{n+l\mu(s)-l}q(s)ds\right]$$
$$\leqslant \Omega(\mu^+)l!(n-l)!,$$

a contradiction to (H_3).

Now, assume that $z(t)<0$ for $t\geq t_1$. Setting $x(t)=-z(t)>0$, (E_1) can be written as

$$x^{(n)}(t)-q(t)(y(t-\sigma))^{\mu(t)}=0.$$

Consequently, $x^{(n)}(t)\geqslant 0$, for $t\geqslant t_1+\sigma$. Because $x(t)\leqslant by(t-\tau)$, $y(t-\sigma)\geqslant \frac{1}{b}x(t+\tau-\sigma)$. It follows from Lemma 2.1 that $x^{(n-1)}(t)\leqslant 0$, for $t\geqslant t_2>t_1+\sigma$. Hence, applying Lemma 2.2, when $l\in\{1,2,\dots,n-2\}$ with $(l+n)$ even, we get

$$\begin{aligned}
x^{(l-1)}(t)\geqslant{}& \frac{t}{(n-l)!}\int_t^{+\infty}s^{n-l-1}q(s)(y(s-\sigma))^{\mu(s)}ds\\
&+\frac{1}{(n-l)!}\int_{t^*}^t s^{n-l}q(s)(y(s-\sigma))^{\mu(s)}ds\\
\geqslant{}& \frac{t}{(n-l)!}\int_t^{+\infty}s^{n-l-1}q(s)b^{-\mu(s)}(x(s+\tau-\sigma))^{\mu(s)}ds\\
&+\frac{1}{(n-l)!}\int_{t^*}^t s^{n-l}q(s)b^{-\mu(s)}(x(s+\tau-\sigma))^{\mu(s)}ds\\
\geqslant{}& \frac{t}{l!(n-l)!}\int_t^{+\infty}s^{n-l-1}q(s)(s+\tau-\sigma)^{(l-1)\mu(s)}b^{-\mu(s)}(x^{(l-1)}(s+\tau-\sigma))^{\mu(s)}ds\\
&+\frac{1}{l!(n-l)!}\int_{t_2}^t s^{n-l}q(s)(s+\tau-\sigma)^{(l-1)\mu(s)}b^{-\mu(s)}(x^{(l-1)}(s+\tau-\sigma))^{\mu(s)}ds,
\end{aligned}$$

for $t\geqslant t^*\geqslant t_2>t_1+\sigma$. Using the same type of reasoning as in the former case yields

$$\limsup_{t\to+\infty}\left[(t+\tau-\sigma)^{\mu(t)}\int_t^{+\infty}(s+\tau-\sigma)^{n+(l-1)\mu(s)-l-1}b^{-\mu(s)}q(s)ds\right.$$
$$\left.+\frac{1}{t}\int_{t_2}^t(s+\tau-\sigma)^{n+l\mu(s)-l}b^{-\mu(s)}q(s)ds\right]\leqslant\Omega(\mu^+)l!(n-l)!,$$

a contradiction to (H_4). Hence, the proposition is proved. □

Theorem 3.3. *Let $0\leqslant p(t)\leqslant a<\infty$ and $\mu^+=\lim_{t\to\infty}\mu(t)$. For every $0<\epsilon<\mu^+$, let $l\in\{1,2,\dots,n-1\}$ with $(l+n)$ odd. If (H_1) and*

(H_7) $\int^{+\infty}t^{n-1}Q(t)dt=+\infty$

hold, then (E_1) has the Property A.

Proof. Suppose that (E_1) has a proper nonoscillatory solution $y(t)$ on $[t_0,+\infty)$ such that $y(t)>0$ for $t\geqslant t_0$. The case $y(t)<0$ on $[t_0,+\infty)$ can similarly be dealt with. Then by Lemma 2.1, there exists $l\in\{0,1,2,\dots,n\}$ such that $l+n$ is odd and condition (2.1) holds. In view of (H_2) and Proposition 3.1, $l\notin\{1,2,\dots,n-1\}$.

Hence n is odd and $l = 0$. We assert that (1.3) holds. If this is not the case, then there exist $c > 0$ and $t_1 > t_0$ such that $z(t) > c$ for $t \geqslant t_1$, where $z(t)$ defined by (3.1) satisfying the inequality

$$z^{(n)}(t) + a^{\mu^+ - \epsilon} z^{(n)}(t-\tau) + Q(t)(z(t-\sigma))^{\mu^+ - \epsilon} \leqslant 0 \tag{3.6}$$

due to Lemma 2.3, for $t \geqslant t_3 + \tau$. Therefore, for $t \geqslant t_4 > t_3 + \tau + \sigma$, (3.6) implies

$$c^{\mu^+ - \epsilon} Q(t) \leqslant -z^{(n)}(t) - a^{\mu^+ - \epsilon} z^{(n)}(t-\tau). \tag{3.7}$$

Using the fact that (2.1) holds, and then integrating (3.7), n-times from t_4 to $+\infty$, we obtain

$$c^{\mu^+ - \epsilon} \int_{t_4}^{+\infty} s^{n-1} Q(s) ds < +\infty,$$

a contradiction to (H_7). Thus our assertation holds and (E_1) has the Property A. □

Theorem 3.4. *Let $-\infty < -b \leqslant p(t) \leqslant 0$, $b > 0$ and $\mu^+ = \lim_{t\to\infty} \mu(t)$. If (H_3) holds and n is odd, let*

(H_8) $\int^{+\infty} t^{n-1} q(t) dt = +\infty$

hold; then (E_1) has the Property A. If (H_4) holds, n is even,

(H_9) $\int^{+\infty} t^{n-2} q(t) dt = +\infty$

and

(H_{10}) $\int^{+\infty} (s + \tau - \sigma)^{(n-1)\mu(s)} q(s) ds = +\infty,$

hold, then (E_1) has the Property B.

Proof. Let $y(t)$ be a proper nonoscillatory solution of (E_1) on $[t_0, +\infty)$. Without loss of generality, we may assume that $y(t) > 0$, for $t \geqslant t_0$. Since $z(t)$ is monotonic, then $z(t) > 0$ or < 0 for $t \geqslant t_1$. Let $z(t) > 0$ for $t \geqslant t_1$. Then by Lemma 2.1, there exists $l \in \{0, 1, 2, \ldots, n\}$ such that $l + n$ is odd and condition (2.1) holds. In view of (H_4) and Proposition 3.2, $l \notin \{1, 2, \ldots, n-1\}$. Thus n is odd and $l = 0$. We claim that (1.3) holds. If not, there exist $c > 0$ and $t_2 > t_1$ such that $z(t) \geqslant c$, for $t \geqslant t_2$, where $z(t)$ is defined by (3.1) satisfying the inequality

$$z^{(n)}(t) + q(t)(z(t-\sigma))^{\mu(t)} \leqslant 0, \quad t \geqslant t_2,$$

that is, for $t \geqslant t_2 + \sigma$,

$$z^{(n)}(t) + c^{\mu(t)} q(t) \leqslant z^{(n)}(t) + q(t)(z(t-\sigma))^{\mu(t)} \leqslant 0. \tag{3.8}$$

Integrating (3.8), n-times from $t_2 + \sigma$ to t and then using (2.1), we get

$$\int_{t_2+\sigma}^{t} s^{n-1} q(s) dt < \infty,$$

a contradiction to (H_8). Hence (E_1) has the Property A. If $z(t) < 0$, for $t \geqslant t_1$, then from Proposition 3.2 it follows that $x^{(n)}(t) \geqslant 0$ with $(l + n)$ even. Since $l \notin$

$\{1, 2, 3, \dots, n-2\}$, then either $l = 0$ and n is even or $l = n$. Consider the case $l = 0$ and n is even. Proceeding as in Proposition 3.2, we obtain

$$x^{(n)}(t) - \left(\frac{1}{b}\right)^{\mu(t)} (x(t+\tau-\sigma))^{\mu(t)} q(t) \geqslant 0, \quad t \geqslant t_2. \tag{3.9}$$

Using the same type of reasoning as above and integrating (3.9) $(n-1)$ times from t_2 to t, we obtain

$$\lim_{t\to\infty} \int_{t_2}^{t} s^{n-2} b^{-\mu(s)} q(s) ds < \infty,$$

a contradiction to (H_9). Hence (1.3) holds. When $l = n$, it follows from Lemma 2.1 that

$$x^{(i)}(t) > 0, \text{ for } t \geqslant t_2 \quad (i = 0, 1, 2, \dots, n-1).$$

Consequently, there exist $c > 1$ and $t_3 > t_2$ such that $x(t) \geqslant ct^{n-1}$ for $t \geqslant t_3$. Integrating (3.9) from $t_3 + \sigma - \tau$ to t, it yields,

$$\begin{aligned} x^{(n-1)}(t) &\geqslant x^{(n-1)}(t_3+\sigma-\tau) + c\int_{t_3+\sigma-\tau}^{t} b^{-\mu(s)} q(s)(s+\sigma-\tau)^{(n-1)\mu(s)} ds \\ &\to +\infty \end{aligned}$$

as $t \to \infty$ due to (H_{10}). Thus, if n is even and $l = 0$, then (1.3) holds, while if $l = n$, then (1.4) holds. This means that (E_1) has the Property B. This completes the proof of the theorem. □

Theorem 3.5. *Let* $0 \leqslant p(t) \leqslant a < \infty$ *and* $\mu^+ = \lim_{t\to\infty} \mu(t)$. *For every* $\epsilon > 0$, *let* $\epsilon < \mu^+$, *and* $l \in \{1, 2, \dots, n-2\}$ *with* $(l+n)$ *even. If* (H_1), (H_7) *and*

(H_{11}) $\quad \int^{+\infty} (t-\sigma)^{(n-1)(\mu^+-\epsilon)} Q(t) dt = +\infty$

hold, then (E_2) *has the Property B.*

Proof. Let $y(t)$ be a proper nonoscillatory solution of (E_2) on $[t_0, +\infty)$. Without loss of generality, we may assume that $y(t) > 0$, for $t \geqslant t_0$. Proceeding as in Proposition 3.1, we get

$$\begin{aligned} z^{(n)}(t) + a^{(\mu^+-\epsilon)} z^{(n)}(t-\tau) - q(t)(y(t-\sigma))^{\mu(t)} \\ - q(t-\tau) a^{(\mu^+-\epsilon)} (y(t-\tau-\sigma))^{\mu(t-\tau)} = 0, \end{aligned}$$

that is,

$$z^{(n)}(t) + a^{(\mu^+-\epsilon)} z^{(n)}(t-\tau) \geqslant Q(t)(z(t-\sigma))^{(\mu^+-\epsilon)} \tag{3.10}$$

for $t \geqslant t_4$. It follows from Lemma 2.1 that there exists $l \in \{0, 1, 2, \dots, n\}$ such that $(l+n)$ even and condition (2.1) holds. In view of (H_2) and Proposition 3.1, $l \notin \{1, 2, \dots, n-2\}$. Hence either $l = n$ or n is even and $l = 0$. In the later case, using (H_7) as in the proof of Theorem 3.3, we can easily show that (1.3) holds. On the other hand, if $l = n$, then by (2.1), there exist $c > 1$ and $t_5 > t_4$ such that

$z(t) \geqslant ct^{n-1}$, for $t \geqslant t_5$. Therefore, by (2.1) for $l = n$ and $t \geqslant t_6 > t_5 + \sigma$, inequality (3.10) yields

$$\begin{aligned}[z(t) + a^{(\mu^+ - \epsilon)} z(t-\tau)]^{(n-1)} &\geqslant [z(t_6) + a^{(\mu^+ - \epsilon)} z(t_6 - \sigma)]^{(n-1)} \\ &\quad + c\int_{t_6}^{t} (s-\sigma)^{(n-1)(\mu^* - \epsilon)} Q(s) ds \\ &\to +\infty \quad \text{as } t \to \infty\end{aligned}$$

due to (H_{11}). Thus, if n is even and $l = 0$, then (1.3) holds, while if $l = n$, (1.4) holds. Hence we conclude that (E_2) has the Property B, and the theorem is proved. ☐

Theorem 3.6. *Let* $-\infty < -b \leqslant p(t) \leqslant 0$, $b > 0$ *and* $\mu^+ = \lim_{t\to\infty} \mu(t)$. *If* (H_3), (H_6), *and*

(H_{12}) $\int_{\sigma}^{+\infty} (t-\sigma)^{\mu(t)(n-1)} q(t) dt = +\infty$,

hold, n *is even, and* (H_9) *holds, then* (E_2) *has the Property B. If* (H_4) *and* (H_5) *hold,* n *is odd,* $\tau < \sigma$, *and* (H_8) *hold, then* (E_2) *has the Property A.*

Proof. Suppose that $y(t)$ is a proper nonoscillatory solution of (E_2) on $[t_0, +\infty)$. Without loss of generality, we may assume that $y(t) > 0$ for $t \geqslant t_0$. Since $z(t)$ is monotonic, then $z(t) > 0$ or < 0, for $t \geqslant t_1$. Let $z(t) > 0$, for $t \geqslant t_1 > t_0$. Then by Lemma 2.1, there exists $l \in \{0, 1, 2, \ldots, n\}$ such that $(l + n)$ even and (2.1) holds. In view of (H_6) and Proposition 3.2, $l \notin \{1, 2, \ldots, n-2\}$. Thus either $l = n$ or $l = 0$ and n is even. Proceeding as in Theorem 3.4, we get contradictions to (H_9) and (H_{12}). Hence, (E_2) has the Property B.

Let $z(t) < 0$, for $t \geqslant t_1$. Setting $x(t) = -z(t) > 0$ and $y(t-\sigma) = \frac{1}{b}(x(t+\tau-\sigma))$ for $t \geqslant t_2 > t_1$, (E_2) yields

$$x^{(n)}(t) + b^{-\mu(t)} q(t)(x(t+\tau-\sigma))^{\mu(t)} \leqslant 0, \ \tau < \sigma. \tag{3.11}$$

In view of our supposition, (H_5) and Proposition 3.2, $l \notin \{1, 2, \ldots, n-1\}$. Consequently, $l = 0$ and n is odd. Integrating (3.11) n-times from $t_2 + \tau - \sigma$ to t and then using (2.1), we have a contradiction to (H_8). This completes the proof of the theorem. ☐

Example 3.7. Consider

$$(y(t) + p(t)y(t-\tau))''' + q(t)|y(t-\sigma)|^{\mu(t)} sgn\ y(t-\sigma) = 0, \tag{3.12}$$

where $t > \max\{\tau, \sigma\}$. Let the function $\mu \in C(R_+; (0,1))$ be nondecreasing, $\lim\limits_{t\to\infty} \mu(t) = 1$, $\limsup_{t\to\infty} (t-\sigma)^{\mu(t)-1} = \gamma > 0$, $q \in L_{loc}(R_+; R_+)$ and for sufficiently large t

$$q(t) \geq \frac{c}{(t-\sigma)^{1+2\mu(t)}}, \quad t > \sigma, \quad c > 0.$$

Clearly for $l = 2$, (H_8) holds. Then in order for (3.12) to have the Property A, it is sufficient by Theorem 3.4 that $c(1+\gamma) > 2$ for (H_3).

Acknowledgement: The authors are thankful to the referees for their helpful suggestions and necessary corrections in the completion of this paper.

This work was supported by Department of Science and Technology, New Delhi, through the letter no. SR/S4/MS : 541/08, dated 30th, September 2008.

REFERENCES

[1] R. P. Agarwal, S. R. Grace, D. O' Regan, *Oscillation Theory for Second Order Linear, Half-linear, Super-linear and Sublinear Dynamic Equations,* Kluwer Academic Dordrecht, 2002.

[2] F. V. Atkinson, On second order nonlinear oscillations, *Pacific J. Math.* 5(1): 643–647, 1955.

[3] B. Baculikova, J. Dzurina, Oscillation theorems for second-order nonlinear neutral differential equations, *Comp. Math. Appl.,* 62:4472–4478, 2011.

[4] J. R. Graef, R. Koplatadze, G. Kvinikadze, Nonlinear functional differential equations with Properties A and B, *J. Math. Anal. Appl.* 306:136–160, 2005.

[5] M. K. Grammatikopulos, R. Koplatadze, G. Kvinikadze, Linear functional differential equations with Property A, *J. Math. Anal. Appl.* 284:294 - 314, 2003.

[6] I. Gyori, G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications,* Clarendon Press, Oxford, 1991.

[7] I. T. Kiguradze, On oscillatory solutions of some ordinary differential equations, *Sov. Math. Dokl.* 144:33–36, 1962.

[8] I. T. Kigradze, T. A. Chanturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Kluwer Academic, Dordrecht, 1993.

[9] R. Koplatadze, On oscillatory properties of solutions of functional differential equations, *Mem. Differential Equations Math. Phys.* 3:1–179, 1994.

[10] R. Koplatadze, G. Kvinikadze, I. P. Stavroulakis, Properties A and B of n-th order linear differential equations with deviating argument, *Georgian. Math. J.* 6:553–566, 1999.

[11] R. Koplatadze, On higher order functional differential equations with Property A, *Georgian. Math. J.* 11:307–336, 2004.

[12] I. Licko, M. Svec, Le caractere oscillatorie des solutions de l'equation $y^{(n)} + f(x)y^{\alpha} = 0$, $n > 1$, *Czech. Math. J.* 13:481–489, 1963.

[13] N. Parhi, A. K. Tripathy, On oscillatory fourth order nonlinar neutral differential equations - I, *Math. Slovaca*, 54:389–410, 2004.

[14] N. Parhi, A. K. Tripathy, On oscillatory fourth order nonlinear neutral differential equations - II, *Math. Slovaca*, 55:183–202, 2005.

[15] M. K. Yildiz, Oscillation results of higher order nonlinear neutral delay differential equations with oscillating coefficients, *Adv. Dyn. Syst. Appl.* 3:297–303, 2008.