Dynamic Systems and Applications 23 (2014) 15-30

FREQUENCY DOMAIN APPROACH FOR THE STABILIZATION OF TIMOSHENKO-TYPE SYSTEM OF THERMOELASTICITY OF TYPE III

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ABSTRACT. In this paper, we study the energy decay rate for the one-dimensional linear thermoelastic system of Timoshenko type. This system models the transverse vibration of a thick beam, taking into account the heat conduction given by Green and Naghdi's theory [6, 7]. First, we establish a polynomial energy decay rate in the case of unequal speeds. Second, when the wave speeds are equal, an exponetial type decay is obtained (similar to the result obtained by [20] for other boundary conditions). Our proof is based on the frequency domain approach introduced in [15].

AMS (MOS) Subject Classification. 35B37, 35L55, 93D15.

1. INTRODUCTION

In 1921, Timoshenko [28] introduced the following system:

$$(1.1)\qquad \begin{cases} \rho_1 u_{tt} = \kappa_1 (u_x - \varphi)_x & \text{in } (0, L) \times \mathbb{R}^+ \\ \rho_2 \varphi_{tt} = \kappa_2 \varphi_{xx} + \kappa_1 (u_x - \varphi) & \text{in } (0, L) \times \mathbb{R}^+, \end{cases}$$

to describe the transverse vibration of a thick beam, where t denotes the time variable, x is the space variable along the beam of length L, in its equilibrium configuration, u is the transverse displacement of the beam, and φ is the rotation angle of the filament of the beam. The coefficients ρ_1, ρ_2, κ_1 and κ_2 are positive constants and denote, respectively, the density (the mass per unit length), the polar moment of inertia of a cross section, the shear modulus and Young's modulus of elasticity times the moment of inertia of a cross section.

During the last few years, an important amount of research has been devoted to the issue of the stabilization of system (1.1) and search the minimum dissipation by which solutions of (1.1) decay uniformly to the stable state. To achieve this goal,

Received Novemeber 17, 2013 1056-2176 $15.00 Dynamic Publishers, Inc.

several types of dissipative mechanisms have been introduced and several stability results have been obtained. We mention some of these results.

In the case of a one feedback acting only on the rotation angle, the rate of decay depends on the constants ρ_1, ρ_2, κ_1 and κ_2. Precisely, if $\frac{\kappa_1}{\rho_1} = \frac{\kappa_2}{\rho_2}$, the results show that we obtain similar decay rates as in the presence of two controls. We quote in this regard [1, 2, 4, 8, 9, 18, 22, 23, 24, 27]. However, if $\frac{\kappa_1}{\rho_1} \neq \frac{\kappa_2}{\rho_2}$, a situation which is more interesting from the physics point of view, then it has been shown that (1.1) is not exponentially stable even for exponentially decaying relaxation functions.

For Timoshenko systems coupled with the heat equation, we mention the pioneer work of Rivera and Racke [21], where they considered the following system:

$$
\begin{cases}
\rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0, & \text{in } (0, L) \times \mathbb{R}^+ \\
\rho_2\psi_{tt} - b\psi_{xx} + K\left(\varphi_x + \psi\right) + \gamma\theta_x = 0 & \text{in } (0, L) \times \mathbb{R}^+ \\
\rho_3\theta_t - \kappa\theta_{xx} + \gamma\psi_{tx} = 0 & \text{in } (0, L) \times \mathbb{R}^+,
\end{cases}
\tag{1.2}
$$

for θ denoting the temperature difference. Under appropriate conditions of $\sigma, \rho_i, b, K, \gamma$, they established well posedness and exponential decay results for the linearized system with several boundary conditions. They also proved a non exponential stability result for the case of different wave speeds. In addition, the nonlinear case was discussed and an exponential decay was established.

In the above system, the heat flux is given by Fourier's law. Using the new theory developed by Green and Naghdi [6, 7], Messaoudi and Said-Houari [20] considered a Timoshenko-type system of the form

$$
\begin{cases}
\rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, & \text{in } (0,1) \times \mathbb{R}^+ \\
\rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \beta\theta_x = 0, & \text{in } (0,1) \times \mathbb{R}^+ \\
\rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - \kappa\theta_{txx} = 0, & \text{in } (0,1) \times \mathbb{R}^+ \\
\varphi(\cdot, 0) = \varphi_0,\ \varphi_t(\cdot, 0) = \varphi_1, \psi(\cdot, 0) = \psi_0, & \text{on } (0,1) \\
\psi_t(\cdot, 0) = \psi_1, \theta(\cdot, 0) = \theta_0, \theta_t(\cdot, 0) = \theta_1, & \text{on } (0,1) \\
\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, & \text{on } \mathbb{R}^+
\end{cases}
\tag{1.3}
$$

they proved an exponential decay in the case of equal-speed propagation. This system models the transverse vibration of a thick beam, taking in account the heat conduction given by Green and Naghdi's theory where the dissipation is given by the heat conduction, of type III. This result was later established for system (1.3), in the presence of a viscoelastic damping of the form $\int_0^\infty g(s)\psi_{xx}(x, t - s)ds$ acting in the second equation by Messaoudi and Said-Houari [19]. Recently Ma et al. [16] proved the exponential stability of (1.3) using the semigroup method.

In this paper, we study the stability of the following system

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, & \text{in } (0,1)\times\mathbb{R}^+ \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \beta\theta_x = 0, & \text{in } (0,1)\times\mathbb{R}^+ \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - \kappa\theta_{txx} = 0, & \text{in } (0,1)\times\mathbb{R}^+ \\ \varphi(\cdot,0) = \varphi_0,\ \varphi_t(\cdot,0) = \varphi_1, \psi(\cdot,0) = \psi_0, & \text{on } (0,1) \\ \psi_t(\cdot,0) = \psi_1, \theta(\cdot,0) = \theta_0, \theta_t(\cdot,0) = \theta_1 & \text{on } (0,1) \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = \theta(0,t) = \theta(1,t) = 0 & \text{on } \mathbb{R}^+ \end{cases} \tag{1.4}$$

In order to exhibit the dissipative nature of the above system, we introduce the new variables

$$\phi = \varphi_t, \quad \Psi = \psi_t$$

and using the following notation $(\phi', \phi', \theta') := (\phi_x, \phi_x, \theta_x)$, then we obtain:

$$\begin{cases} \rho_1\phi_{tt} - K(\phi' + \Psi)' = 0, \\ \rho_2\Psi_{tt} - b\Psi'' + K(\phi' + \Psi) + \beta\theta_t' = 0, \\ \rho_3\theta_{tt} - \delta\theta'' + \gamma\Psi_t' - \kappa\theta_t'' = 0, \\ \phi(\cdot,0) = \phi_0, \quad \phi_t(\cdot,0) = \phi_1, \Psi(\cdot,0) = \Psi_0, \\ \Psi_t(\cdot,0) = \Psi_1, \theta(\cdot,0) = \theta_0, \theta_t(.\cdot 0) = \theta_1 \\ \phi(0,t) = \phi(1,t) = \Psi(0,t) = \Psi(1,t) = \theta(0,t) = \theta(1,t) = 0. \end{cases} \tag{1.5}$$

This paper is organized as follows. Well-posedness of the problem is analyzed in section 2. Sections 3 and 4 are devoted to polynomial and exponential decay rate of the system energy, respectively.

2. EXISTENCE AND UNIQUENESS

In this section, we prove the existence of the solution of (1.5). We define the energy space $\mathcal{H}$ associated to problem (1.5)

$$\mathcal{H} = \left(H_0^1(0,1)\right)^3 \times \left(L^2(0,1)\right)^3. \tag{2.1}$$

The space $\mathcal{H}$ is equipped with the inner product which induces the energy norm

$$\|z\|_{\mathcal{H}}^2 := \gamma\rho_1\|u\|^2 + \gamma\rho_2\|v\|^2 + \gamma b\|\Psi'\|^2 + \gamma K\|\phi' + \Psi\|^2 + \beta\rho_3\|w\|^2 + \beta\delta\|\theta'\|^2, \tag{2.2}$$

for all $z = (\phi, \Psi, \theta, u, v, w) \in \mathcal{H}$, and $\|\cdot\|$ is the $L^2(0,1)$ norm.

Next, we define the linear operator $\mathcal{A}$ by

$$\mathcal{A}z = \begin{pmatrix} u \\ v \\ w \\ \frac{K}{\rho_1}(\phi' + \Psi)' \\ \frac{b}{\rho_2}\Psi'' - \frac{K}{\rho_2}(\phi' + \Psi) - \frac{\beta}{\rho_2}w' \\ (\frac{b}{\rho_3}\theta + \frac{\kappa}{\rho_3}w)'' - \frac{\gamma}{\rho_3}v' \end{pmatrix}, \tag{2.3}$$

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{c} (\phi, \Psi, \theta, u, v, w) \in \mathcal{H} : (\phi, \Psi) \in (H^2(0.1))^2, \\ (u, v, w) \in (H_0^1(0.1))^3 \text{ and } (\delta\theta + \kappa w) \in H^2(0.1) \end{array} \right\}.$$

Now, let $u = \phi_t$, $v = \Psi_t$, and $w = \theta_t$. Then we can formulate the system (1.5) as an evolution equation of the form

$$z_t = \mathcal{A}z, \quad z(0) = z^0 \in \mathcal{H},$$

where $z = (\phi, \Psi, \theta, u, v, w)$.

Proposition 1. *$\mathcal{A}$ generates a C^0-semigroup $e^{\mathcal{A}t}$ of contractions in $\mathcal{H}$, and $0 \in \rho(\mathcal{A})$.*

Proof. First, we prove that $\mathcal{A}$ is a maximal dissipative operator on the energy space $\mathcal{H}$, and the conclusion will follow by Lummer-Phillips theorem (see [25]).

a- $\forall\ z \in \mathcal{D}(\mathcal{A})$, it is easy to see that

$$\operatorname{Re}(\mathcal{A}z, z)_{\mathcal{H}} = -\beta\kappa \left\| w' \right\|^2, \tag{2.4}$$

which implies that $\mathcal{A}$ is a dissipative linear operator on $\mathcal{H}$.

b- Let $F = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathcal{H}$.

$$\text{Find } z \in \mathcal{D}(\mathcal{A}), \text{ such that } \mathcal{A}z = F, \tag{2.5}$$

which implies that

$$\begin{cases} u = f_1, v = f_2, w = f_3, \\ \dfrac{K}{\rho_1}(\phi' + \Psi)' = f_4, \\ \dfrac{b}{\rho_2}\Psi'' - \dfrac{K}{\rho_2}(\phi' + \Psi) - \dfrac{\beta}{\rho_2}w' = f_5, \\ \left(\dfrac{\delta}{\rho_3}\theta + \dfrac{\kappa}{\rho_3}\right)'' - \dfrac{\gamma}{\rho_3}v' = f_6. \end{cases} \tag{2.6}$$

Let $(\phi_*, \Psi_*) \in \left(H_0^1(0,1)\right)^2$ and using the above system (2.6) we obtain:

$$\begin{aligned} &K\int_0^1 (\phi' + \Psi)(\overline{\phi}_*' + \overline{\Psi}_*)dx + b\int_0^1 \Psi'\overline{\Psi}_*'dx \\ &= \rho_1 \int_0^1 f_4\overline{\phi}_*dx - \rho_2\int_0^1 f_5\overline{\Psi}_*dx - \beta\int_0^1 f_3\overline{\Psi}_*dx. \end{aligned} \tag{2.7}$$

Let us denote by

$$\left\{ \begin{array}{c} a((\phi, \Psi), (\phi_*, \Psi_*)) = K\int_0^1 (\phi' + \Psi)(\overline{\phi}_*' + \overline{\Psi}_*)dx + b\int_0^1 \Psi'\overline{\Psi}_*'dx \\ L((\phi_*, \Psi_*)) = \rho_1\int_0^1 f_4\overline{\phi}_*dx - \rho_2\int_0^1 f_5\overline{\Psi}_*dx - \beta\int_0^1 f_3\overline{\Psi}_*dx. \end{array} \right. \tag{2.8}$$

Then (2.7) is equivalent to

$$a((\phi, \Psi), (\phi_*, \Psi_*)) = L((\phi_*, \Psi_*)). \tag{2.9}$$

Applying Lax-Milgram's theorem we conclude that there exists a unique solution $(\phi,\psi)\in\left(H_0^1(0,1)\right)^2$.

Using $(2.6)_2$ and $(2.6)_3$ we have

$$\phi'' = \frac{\rho_1}{K} f_4 - \Psi' \in L^2(0,1), \tag{2.10}$$

$$\Psi'' - K\Psi = \frac{1}{b}\left(\rho_2 f_5 + \beta f_3' + K\phi'\right) \in L^2(0,1), \tag{2.11}$$

subject to the boundary conditions

$$\phi(0) = \phi(1) = \Psi(0) = \Psi(1) = 0. \tag{2.12}$$

Using the classical elliptic theory [11] we obtain that

$$(\phi,\Psi)\in\left(H_0^1(0,1)\cap H^2(0,1)\right)^2.$$

Now, rewrite $(2.6)_4$ as:

$$\begin{cases} (\delta\theta+\kappa w)'' = \rho_3 f_6 + \gamma v' \in L^2(0,1) \\ \theta(0) = \theta(1) = w(0) = w(1) = 0. \end{cases} \tag{2.13}$$

Using again the classical elliptic theory, we deduce that (2.13) admits a unique solution:

$$(\delta\theta+\kappa w)\in H_0^1(0,1)\cap H^2(0,1), \tag{2.14}$$

which implies that $\theta\in H_0^1(0,1)$. Using Lumer-Phillips theorem [25] we conclude that $\mathcal{A}$ generates a C^0-semigroup $e^{\mathcal{A}t}$ of contractions in $\mathcal{H}$. □

3. LACK OF EXPONENTIAL STABILITY AND POLYNOMIAL STABILITY IF $\frac{K}{\rho_1}\neq\frac{b}{\rho_2}$

In this section, we will prove the lack of exponential decay and the polynomial decay rate of the solutions of (1.5). Following the method introduced in [15] we will obtain the polynomial energy decay rate (3.16).

3.1. Lack of exponential stability.

To prove the lack of exponential stability in the case of $\frac{K}{\rho_1}\neq\frac{b}{\rho_2}$, we use the following result.

Theorem 1 (see [14] and [26]). *Let $e^{\mathcal{A}t}$ be a C^0 semigroup of contractions in H. Then $e^{\mathcal{A}t}$ is exponentially stable if and only if,*

$$i\mathbb{R}\subset\rho(\mathcal{A}), \tag{3.1}$$

and

$$\left\|(\lambda_n I-\mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})}\leq C, \quad \textit{for every } \lambda_n\in i\mathbb{R} \tag{3.2}$$

Our result in this subsection is stated in the following theorem

Theorem 2. *If $\frac{\kappa}{\rho_1} \neq \frac{b}{\rho_2}$, then the semigroup corresponding to system* (1.5) *is not exponentially stable.*

Proof. For $\lambda_n \in \rho(\mathcal{A})$ and $\mathbf{F}_n = (0,0,0,f_4,0,0)^T \in \mathcal{H}$, we consider

$$\lambda_n z_n - \mathcal{A} z_n = \mathbf{F}_n, \tag{3.3}$$

where $z_n = (\phi_n, \Psi_n, \theta_n, u_n, v_n, w_n)$. To prove our result we need to show that $\|z_n\|_{\mathcal{H}}$ is unbounded.

Using (2.3), then (3.3) will be written as

$$\lambda_n^2 \phi_n - \frac{K}{\rho_1}\left(\phi_n^{'} + \Psi_n\right)^{'} = f_4, \tag{3.4}$$

$$\lambda_n^2 \Psi_n - \frac{b}{\rho_2}\Psi_n^{''} + \frac{K}{\rho_2}\left(\phi_n^{'} + \Psi_n\right) + \frac{\lambda_\mu \beta}{\rho_2}\theta_n^{'} = 0, \tag{3.5}$$

$$\lambda_n^2 \theta_n - \left(\frac{\delta}{\rho_3} + \frac{\kappa}{\rho_3}\right)\theta_n^{\prime\prime} + \frac{\lambda_n \gamma}{\rho_3}\Psi_n^{'} = 0. \tag{3.6}$$

Differentiating (3.5) and using (3.4), we get

$$b\phi_n^{''''} - \lambda_n^2\left[\rho_2 + \frac{b\rho_1}{\kappa}\right]\phi_n^{''} + \frac{\lambda_n^2\rho_1}{K}\left(\rho_2\lambda_n^2 + K\right)\phi_n + \lambda\beta\theta_n^{''} = -\frac{b\rho_1}{K}f_4^{''} \tag{3.7}$$

$$\lambda_n^2\rho_3\theta_n - (\delta + \lambda_n\kappa)\,\theta_n^{\prime\prime} + \lambda_n\frac{\lambda_n^2\rho_1\gamma}{K}\phi_n - \lambda_n\gamma\phi_n^{''} = \frac{\lambda_n\rho_1\gamma}{K}f_4. \tag{3.8}$$

Let us take

$$f_4(x) = \frac{1}{\pi n} - \frac{\cos(\pi n)}{\pi n} + \sin(\pi n x). \tag{3.9}$$

Taking into account the boundary conditions in (1.5), one can assume that

$$\phi_n(x) = A_n \sin(n\pi x), \text{ and } \theta_n(x) = D_n \sin(n\pi x), \tag{3.10}$$

then (3.7)–(3.8) will be written as

(3.11)

$$\begin{cases} (\lambda_n^4\frac{\rho_1\rho_2}{bK} + \lambda_n^2\left(\frac{\rho_1}{b} + (\frac{\rho_2}{b} + \frac{\rho_1}{K})(\pi n)^2\right) + (\pi n)^4)A_n - \frac{\lambda_n\beta(\pi n)^2}{b}D_n = \frac{(\pi n)^2\rho_1}{K}, \\ \lambda_n\gamma\left(\lambda_n^2\frac{\rho_1}{K} + (\pi n)^2\right)A_n + \left[\lambda_n^2\rho_3 + (\pi n)^2(\delta + \kappa\lambda_n)\right]D_n = \lambda_n\frac{\rho_1\gamma}{K}. \end{cases}$$

Now, we choose λ_n solution of

$$\lambda_n^4\frac{\rho_1\rho_2}{bK} + \lambda_n^2\left(\frac{\rho_2}{b} + \frac{\rho_1}{K}\right)(\pi n)^2 + (\pi n)^4 = 0, \tag{3.12}$$

since $\frac{K}{\rho_1} \neq \frac{b}{\rho_2}$, we obtain

$$\lambda_n^2 = -(\pi n)^2\frac{K}{\rho_1}, \tag{3.13}$$

as a solution of (3.12).

Using the value of λ_n in (3.11), we obtain

$$\left\{\begin{aligned} A_n &= -\frac{b\rho_1}{K^2} + \frac{(\delta\rho_1 - K\rho_3) - i\kappa\sqrt{K\rho_1}\,(\pi n)}{(\delta\rho_1 - K\rho_3)^2 + K\rho_1\kappa^2\,(\pi n)^2}\rho_1\beta\gamma \\ D_n &= \frac{\gamma\rho_1\sqrt{K\rho_1}\left(\kappa\sqrt{K\rho_1}\,(\pi n) + i\,(\delta\rho_1 - K\rho_3)^2\right)}{(\delta\rho_1 - K\rho_3)^2 + K\rho_1\kappa^2\,(\pi n)^2} \end{aligned}\right. \tag{3.14}$$

as $n \to \infty$, we get

$$A_n \to -\frac{b\rho_1}{K^2} \text{ and } D_n \to 0. \tag{3.15}$$

Finally we have

$$\|z_n\|_{\mathcal{H}} \geq \gamma\rho_1 \|u_n\|^2 = \gamma\rho_1 \|\lambda_n\phi_n\|^2 = \gamma\,(\pi n)^2\,K\,\|A_n \sin(n\pi x)\|^2 \to \infty,$$

which yields the conclusion. □

3.2. **Polynomial Stability.** In this subsection we will prove the polynomial decay of the solutions of system (1.5) using the method introduced in [15]. Our main result in this subsection is stated as follow:

Theorem 3. *If $\frac{\kappa}{\rho_1} \neq \frac{b}{\rho_2}$, then, for all positive $m \in N^*$, there exists a constant $C_m \succ 0$ such that:*

$$\forall\, U_0 \in D(\mathcal{A}^m), \quad \forall t > 0, \quad \left\|e^{t\mathcal{A}}U_0\right\| \leq C_m \left(\frac{\ln t}{t}\right)^{\frac{m}{j}} \ln t\, \|U_0\|_{D(\mathcal{A}^m)}\,. \tag{3.16}$$

Proof. Following Liu and Rao in [15], the following conditions are necessary and sufficient for the polynomial energy decay rate (3.16).

$$i\mathbb{R} \subset \rho\,(\mathcal{A})\,, \tag{3.17}$$

and

$$\sup_{|\alpha|\geq 1} \frac{\left\|(i\alpha I - \mathcal{A})^{-1}\right\|}{\alpha^8} \prec +\infty \tag{3.18}$$

First, we will check the condition (3.17). To do this, let $b \neq 0$ be a real number and let $z \in \mathcal{D}\,(\mathcal{A})$, with

$$\mathcal{A}z = ibz \tag{3.19}$$

Taking the inner product with z in $\mathcal{H}$, we deduce that

$$w' = 0,$$

using Poincare's inequality we obtain

$$w = 0.$$

Now, (3.19) is equivalent to

$$(3.20)\qquad \begin{cases} u = ib\phi \\ v = ib\Psi \\ w = ib\theta \\ \frac{K}{\rho_1}(\phi' + \Psi)' = ibu \\ \frac{b}{\rho_2}\Psi'' - \frac{K}{\rho_2}(\phi' + \Psi) - \frac{\beta}{\rho_2}w' = ibv \\ (\frac{b}{\rho_3}\theta + \frac{\kappa}{\rho_3}w)'' - \frac{\gamma}{\rho_3}v' = ibw \end{cases}$$

From $(3.20)_3$ and the fact that $b \neq 0$, we deduce that $\theta = 0$, and then from the last equation in (3.20) we obtain $v' = 0$. Using Poincare's inequality, we have $v = 0$. Using $(3.20)_2$ then we have $\Psi = 0$.

From $(3.20)_5$ and applying again Poincare's inequality, we obtain $\phi = 0$, and from $(3.20_1$ we get $u = 0$. Hence $z = 0$, which implies that $i\mathbb{R} \subset \rho(\mathcal{A})$.

Now, suppose that the condition (3.18) is false. Then there exist a sequence $\alpha_n \in \mathbb{R}$ and a sequence $z_n = (\phi_n, \Psi_n, \theta_n, u_n, v_n, w_n) \in \mathcal{D}(\mathcal{A})$ such that

$$(3.21)\qquad \|z_n\|_{\mathcal{H}} = 1,$$

$$(3.22)\qquad \lim_{n\to+\infty} \alpha_n = 0,$$

$$(3.23)\qquad \lim_{n\to+\infty} \alpha_n^8 \|(i\alpha_n I - \mathcal{A}) z_n\|_{\mathcal{H}} = 0,$$

i.e. in $L^2(0,1)$, we have the following convergence:

$$(3.24)\qquad \alpha_n^8 \left[i\alpha_n u_n - \frac{K}{\rho_1}\left(\phi_n' + \Psi_n\right)' \right] \longrightarrow 0,$$

$$(3.25)\qquad \alpha_n^8 \left[i\alpha_n v_n - \frac{1}{\rho_2}\left(b\Psi_n'' - K\left(\phi_n' + \Psi_n\right) - \beta w_n'\right) \right] \longrightarrow 0,$$

$$(3.26)\qquad \alpha_n^8 \left[(i\alpha_n\phi_n - u_n)' + i\alpha_n\Psi_n - v_n \right] \longrightarrow 0,$$

$$(3.27)\qquad \alpha_n^8 \left[(i\alpha_n\Psi_n - v_n)' \right] \longrightarrow 0,$$

$$(3.28)\qquad \alpha_n^8 \left[i\alpha_n w_n - \frac{1}{\rho_3}\left(\delta\theta_n'' - \gamma v_n' + \kappa w_n''\right) \right] \longrightarrow 0,$$

$$(3.29)\qquad \alpha_n^8 \left[(i\alpha_n\theta_n - w_n)' \right] \longrightarrow 0.$$

Our goal is to derive a contradiction with (3.21). For clarity, we divide the proof into several steps.

Step 1. Taking the inner product of $\alpha_n^8(i\alpha_n I - \mathcal{A}) z_n$ with z_n in $\mathcal{H}$, we get:

$$(3.30)\qquad \mathbf{Re}\left(\alpha_n^8 \langle (i\alpha I - \mathcal{A}) z_n, z_n \rangle_{\mathbb{H}}\right) = -\beta\kappa \left\|\alpha_n^4 w_n'\right\|^2,$$

from (3.23) and (3.30), we have

$$(3.31)\qquad \left\|\alpha_n^4 w_n'\right\| \longrightarrow 0,$$

applying Poincare's inequality, we get

$$\left\|\alpha_n^4 w_n\right\| \longrightarrow 0. \tag{3.32}$$

Step 2. Using triangular inequality, we obtain that

$$\left\|\alpha_n^5 \theta_n'\right\| \leq \left\|\alpha_n^4 \left(i\alpha_n\theta_n - w_n\right)'\right\| + \left\|\alpha_n^4 w_n'\right\| \tag{3.33}$$

From (3.29) and (3.31), we have

$$\left\|\alpha_n^5 \theta_n'\right\| \longrightarrow 0. \tag{3.34}$$

Step 3. Multiplying (3.25) by $\frac{1}{\alpha_n^9}$, we obtain that

$$iv_n - \frac{1}{\rho_2}\left(b\frac{\Psi_n''}{\alpha_n} - \frac{K}{\alpha_n}\left(\phi_n' + \Psi_n\right) - \frac{\beta}{\alpha_n} w_n'\right) \longrightarrow 0. \tag{3.35}$$

Now, from (3.21) and (3.31) and using again the triangular inequality, we deduce that

$$\left(\left\|\frac{\Psi_n''}{\alpha_n}\right\|\right)_{n\in\mathbb{N}} \text{ is uniformly bounded.} \tag{3.36}$$

Multiplying (3.28) by $\frac{1}{\alpha_n^9}$ and using (3.27), we get

$$iw_n - \frac{1}{\rho_3}\left(\frac{1}{\alpha_n}\left(\delta\theta_n + \kappa w_n\right)'' - i\gamma\Psi_n'\right) \longrightarrow 0, \tag{3.37}$$

using the triangular inequality and the fact that $\|z_n\|_{\mathcal{H}} = 1$, we deduce that

$$\left(\|\frac{1}{\alpha_n}\left[\delta\theta_n + \kappa w_n\right]''\|\right)_{n\in\mathbb{N}} \text{ is uniformly bounded.} \tag{3.38}$$

Step 3. Taking the inner product of (3.37) with $\alpha_n^2 \Psi_n'$ in $L^2(0,1)$, we obtain

$$\begin{aligned}
&\left\langle i\alpha_n^2 w_n - \frac{\alpha_n}{\rho_3}\left(\left[\delta\theta + \kappa w\right]'' - i\alpha_n\gamma\Psi_n'\right), \Psi_n'\right\rangle \\
&= i\left\langle \alpha_n^2 w_n, \Psi_n'\right\rangle + \frac{1}{\rho_3}\left\langle \alpha_n^2\left[\delta\theta_n + \kappa w_n\right]', \frac{\Psi_n''}{\alpha_n}\right\rangle + \frac{i\gamma}{\rho_3}\|\alpha_n\Psi_n'\|^2 \\
&- \frac{1}{\rho_3}\left[\alpha_n\left[\delta\theta_n + \kappa w_n\right]'\overline{\Psi_n}'\right]_0^1 \longrightarrow 0.
\end{aligned} \tag{3.39}$$

To estimate the boundary term in (3.39), we proceed as follow

$$\begin{aligned}
&\left|\alpha_n\left[\delta\theta_n + \kappa w_n\right]'\overline{\Psi_n}'(x)\right| \preceq C\left[\|\frac{\Psi_n'}{\alpha_n^{\frac{1}{2}}}\|^2 + \|\Psi_n'\|\|\frac{\Psi_n''}{\alpha_n}\|\right]^{\frac{1}{2}} \\
&\left[\|\alpha_n^{\frac{3}{2}}\left(\delta\theta_n + \kappa w_n\right)'\|^2 + \|\alpha_n^4\left(\delta\theta_n + \kappa w_n\right)'\|\|\frac{\left(\delta\theta_n + \kappa w_n\right)''}{\alpha_n}\|\right]^{\frac{1}{2}}
\end{aligned} \tag{3.40}$$

now, using (3.31), (3.34), (3.36) and (3.38), we obtain

$$\left|\alpha_n\left[\delta\theta_n + \kappa w_n\right]'\overline{\Psi_n}'(x)\right| \longrightarrow 0. \tag{3.41}$$

Then using (3.41) into (3.40) we get

$$\|\alpha_n\Psi_n'\| \longrightarrow 0, \tag{3.42}$$

applying the triangular inequality, we have

$$\|v_n^{'}\| \preceq \|v_n^{'} - i\alpha_n \Psi_n\| + \|i\alpha_n \Psi_n\|, \tag{3.43}$$

then (3.42) implies that

$$\|v_n^{'}\| \longrightarrow 0, \text{ and } \|v_n\| \longrightarrow 0. \tag{3.44}$$

Step 4. Taking the inner product (3.25) with $\frac{v_n}{\alpha_n^7}$ in $L^2(0,1)$ and using (3.24), we get

$$i\|\alpha_n v_n\|^2 + \frac{b}{\rho_2}\left\langle \alpha_n \Psi_n^{'}, v_n' \right\rangle + \frac{K}{\rho_2}\left\langle \alpha_n \phi_n^{'} + \alpha_n \Psi_n, v_n \right\rangle + \frac{\beta}{\rho_2}\left\langle \alpha_n w_n^{'}, v_n \right\rangle \longrightarrow 0. \tag{3.45}$$

But from (3.26) and (3.27), we have

$$i\alpha_n \phi_n' - u_n' \longrightarrow 0, \tag{3.46}$$

then (3.45) becomes

$$i\|\alpha_n v_n\|^2 + \left\langle \frac{b}{\rho_2}\alpha_n \Psi_n^{'} - \frac{iK}{\rho_2} u_n, v_n' \right\rangle + \frac{K}{\rho_2}\left\langle \alpha_n \Psi_n, v_n \right\rangle + \frac{\beta}{\rho_2}\left\langle \alpha_n w_n^{'}, v_n \right\rangle \longrightarrow 0. \tag{3.47}$$

Using (3.31), (3.42), (3.44), and taking into account that $\|z_n\|_{\mathcal{H}} = 1$, we obtain

$$\|\alpha_n v_n\| \longrightarrow 0, \tag{3.48}$$

from (3.27) and (3.48), we get

$$\|\alpha_n^2 \Psi_n\| \longrightarrow 0. \tag{3.49}$$

Now, from (3.25) and (3.48), we obtain

$$\left(\|\Psi_n^{''}\|\right)_{n\in\mathbb{N}} \text{ is uniformly bounded.} \tag{3.50}$$

Step 5. Taking the inner product of (3.25) with $\frac{\phi_n^{'}}{\alpha_n^8}$ in $L^2(0,1)$, we deduce that

$$\begin{aligned} &\left\langle i\alpha_n v_n, \phi_n^{'} \right\rangle - \frac{b}{\rho_2}\left\langle \Psi_n^{''}, \phi_n^{'} \right\rangle - \frac{K}{\rho_2}\left\langle \phi_n^{'} + \Psi_n, \phi_n^{'} \right\rangle + \frac{\beta}{\rho_2}\left\langle w_n^{'}, \phi_n^{'} \right\rangle \\ &= i\left\langle \alpha_n v_n, \phi_n^{'} \right\rangle + \frac{b}{\rho_2}\left\langle \alpha_n \Psi_n^{'}, \frac{\phi_n^{''}}{\alpha_n} \right\rangle - \frac{b}{\rho_2}\left[\Psi_n^{'}\overline{\phi_n}^{'}\right]_0^1 \\ &\quad + \frac{K}{\rho_2}\|\phi_n^{'}\|^2 + \frac{K}{\rho_2}\left\langle \Psi_n, \phi_n^{'} \right\rangle + \frac{\beta}{\rho_2}\left\langle w_n^{'}, \phi_n^{'} \right\rangle \longrightarrow 0. \end{aligned} \tag{3.51}$$

However, from (3.24) and (3.42), we have

$$\left(\|\frac{\phi_n^{''}}{\alpha_n}\|\right)_{n\in\mathbb{N}} \text{ is uniformly bounded.} \tag{3.52}$$

Now, the estimation of the boundary term in (3.51) is given by

$$\left|\left(\Psi_n^{'}\overline{\phi_n}^{'}\right)(x)\right| \preceq 2\left[\|\frac{\phi_n^{'}}{\alpha_n^{\frac{1}{2}}}\|^2 + \|\phi_n^{'}\|\|\frac{\phi_n^{''}}{\alpha_n}\|\right]^{\frac{1}{2}}\left[\|\alpha_n^{\frac{1}{2}}\Psi_n^{'}\|^2 + \|\alpha_n \Psi_n^{'}\|\|\Psi_n^{''}\|\right]^{\frac{1}{2}}. \tag{3.53}$$

Then from (3.42), (3.50) and (3.52), we have

$$\left|\left(\Psi'_n \phi'_n\right)(x)\right| \longrightarrow 0, \tag{3.54}$$

using (3.42), (3.48), (3.51), (3.52), and (3.53), we deduce that

$$\|\phi'_n\| \longrightarrow 0, \text{ and } \|\phi'_n + \Psi_n\| \longrightarrow 0. \tag{3.55}$$

Step 6. From (3.26) and (3.27), we get

$$(i\alpha_n^2 \phi_n - \alpha_n u_n)' \longrightarrow 0, \tag{3.56}$$

by Poincare's inequality, we deduce

$$i\alpha_n^2 \phi_n - \alpha_n u_n \longrightarrow 0. \tag{3.57}$$

Using (3.24) and (3.57), we obtain

$$i\alpha_n^2 \phi_n - i\frac{K}{\rho_1}\left(\phi'_n + \Psi_n\right)' \longrightarrow 0. \tag{3.58}$$

Taking the inner product of (3.58) with ϕ_n in $L^2(0,1)$, we have

$$i\|\alpha_n \phi_n\|^2 + i\frac{K}{\rho_1}\left\langle \left(\phi'_n + \Psi_n\right), \phi'_n \right\rangle \longrightarrow 0, \tag{3.59}$$

From (3.55) and (3.59), we have

$$\|\alpha_n \phi_n\| \longrightarrow 0. \tag{3.60}$$

Step 7. Finally from (3.57), we have

$$i\alpha_n \phi_n - u_n \longrightarrow 0. \tag{3.61}$$

Taking the inner product of (3.61) with u_n in $L^2(0,1)$, we get

$$i\left\langle \alpha_n \phi_n, u_n \right\rangle - \|u_n\|^2 \longrightarrow 0. \tag{3.62}$$

By using (3.60) and (3.62), we obtain

$$\|u_n\| \longrightarrow 0. \tag{3.63}$$

Now, using (3.32), (3.34), (3.42), (3.44), (3.55) and (3.63), we deduce that

$$\|z_n\| \longrightarrow 0. \tag{3.64}$$

Hence, we obtain the contradiction with (3.18). □

4. EXPONENTIAL DECAY RATE IF $\frac{K}{\rho_1}=\frac{b}{\rho_2}$

In this section, we will prove the exponential stability of the system (1.5) in the case of the same speed of propagation $\frac{K}{\rho_1}=\frac{b}{\rho_2}$. Our method of proof is based on [10] and [26] instead of using the multipliers techniques as in [20].

Theorem 4. *If $\frac{\kappa}{\rho_1}=\frac{b}{\rho_2}$, then the C^0-semigroup $e^{t\mathcal{A}}$ is exponentially stable, i.e. there exist constant M and $\alpha>0$ independent of z^0 such that*

$$\left\|e^{t\mathcal{A}}z^0\right\|_{\mathcal{H}}\leq Me^{-\alpha t}\left\|z^0\right\|_{\mathcal{H}},\quad t>0. \tag{4.1}$$

Proof. Following the results in [10] and in [26], the following two conditions

$$i\mathbb{R}\subset\rho\left(\mathcal{A}\right), \tag{4.2}$$

$$\sup_{\alpha\in\mathbb{R}}\left\|\left(i\alpha I-\mathcal{A}\right)^{-1}\right\|\prec+\infty, \tag{4.3}$$

are necessary and sufficient for the exponential stability.

The condition (4.2) was already verified in the last section. Now, assume that the condition (4.3) is false. Then there is a real sequence $(\alpha_n)_{n\in\mathbb{N}}$ and a sequence $(z_n)_{n\in\mathbb{N}}\in\mathcal{D}\left(\mathcal{A}\right)$, such that

$$\|z_n\|_{\mathcal{H}}=1, \tag{4.4}$$

$$\alpha_n\longrightarrow+\infty, \tag{4.5}$$

$$\lim_{n\to+\infty}\left\|\left(i\alpha_nI-\mathcal{A}\right)z_n\right\|_{\mathcal{H}}=0, \tag{4.6}$$

i.e. in $L^2\left(0,1\right)$, we have the following convergence:

$$\left[i\alpha_nu_n-\frac{K}{\rho_1}\left(\phi_n'+\Psi_n\right)'\right]\longrightarrow0 \tag{4.7}$$

$$\left[i\alpha_nv_n-\frac{1}{\rho_2}\left(b\Psi_n''-K\left(\phi_n'+\Psi_n\right)-\beta w_n'\right)\right]\longrightarrow0 \tag{4.8}$$

$$\left[\left(i\alpha_n\phi_n-u_n\right)'+i\alpha_n\Psi_n-v_n\right]\longrightarrow0 \tag{4.9}$$

$$\left[\left(i\alpha_n\Psi_n-v_n\right)'\right]\longrightarrow0 \tag{4.10}$$

$$\left[i\alpha_nw_n-\frac{1}{\rho_3}\left(\delta\theta_n''-\gamma v_n'+\kappa w_n''\right)\right]\longrightarrow0 \tag{4.11}$$

$$\left[\left(i\alpha_n\theta_n-w_n\right)'\right]\longrightarrow0. \tag{4.12}$$

In the following, we will check the condition (4.3) by finding a contradiction with (4.4). Our proof is divided into several steps.

Step 1. Taking the inner product of $\left(i\alpha_nI-\mathcal{A}\right)z_n$ with z_n in $\mathcal{H}$, we get:

$$\operatorname{Re}\left\langle\left(i\alpha_nI-\mathcal{A}\right)z_n,z_n\right\rangle_{\mathcal{H}}=-\beta\kappa\left\|w_n'\right\|^2$$

Using (4.6), we deduce that

$$\|w_n'\| \longrightarrow 0 \text{ and } \|w_n\| \longrightarrow 0. \tag{4.13}$$

Using the triangular inequality, we get

$$\|\alpha_n \theta_n'\| \leq \left\|(i\alpha_n\theta_n - w_n)'\right\| + \|w_n'\|.$$

From (4.12) and (4.13), we deduce that

$$\left\|\alpha_n \theta_n'\right\| \longrightarrow 0. \tag{4.14}$$

Step 2. Multiplying (4.8) by $\frac{1}{\alpha_n}$, we obtain

$$iv_n - \frac{1}{\rho_2}\left(b\frac{\Psi_n''}{\alpha_n} - \frac{K}{\alpha_n}\left(\phi_n' + \Psi_n\right) - \frac{\beta}{\alpha_n}w_n'\right) \longrightarrow 0,$$

now, using (4.13) and $\|z_n\|_{\mathcal{H}} = 1$, then we deduce that

$$\left(\left\|\frac{\Psi_n''}{\alpha_n}\right\|\right)_{n\in\mathbb{N}} \text{ is uniformly bounded.} \tag{4.15}$$

Using (4.10) and multiplying (4.11) by $\frac{1}{\alpha_n}$, we obtain

$$iw_n - \frac{1}{\rho_3}\left(\left(\frac{\delta}{\alpha_n}\theta_n + \frac{\kappa}{\alpha_n}w_n\right)'' - i\gamma\Psi_n'\right) \to 0.$$

Applying the triangular inequality and $\|z_n\|_{\mathcal{H}} = 1, \ \forall\, n \succeq 0$, we deduce that

$$\left(\|\frac{1}{\alpha_n}\left[\delta\theta_n + \kappa w_n\right]''\|\right)_{n\in\mathbb{N}} \text{ is uniformly bounded.} \tag{4.16}$$

Replacing v_n' by $i\alpha_n\Psi_n'$ in (4.11), and then taking the inner product of the obtained equation with $\frac{\Psi_n'}{\alpha_n}$ in $L^2(0.1)$, we get

$$i\left\langle w_n, \Psi_n'\right\rangle + \frac{1}{\rho_3}\left\langle \delta\theta_n' + \kappa w_n', \frac{\Psi_n''}{\alpha_n}\right\rangle + \frac{i\gamma}{\rho_3}\left\|\frac{\Psi_n'}{\alpha_n}\right\|^2 - \frac{1}{\rho_3}\left|\left(\left(\delta\theta_n' + \kappa w_n'\right)\cdot\frac{\overline{\Psi}_n'}{\alpha_n}\right)(x)\right|_{x=0}^{x=1} \to 0,$$

using (4.13), (4.14), (4.15), (4.16), and the triangular inequality, we obtain

$$\left|\left(\left(\delta\theta_n' + \kappa w_n'\right)\cdot\frac{\overline{\Psi}_n'}{\alpha_n}\right)(x)\right|_{x=0}^{x=1} \to 0. \tag{4.17}$$

Using the above estimates, we deduce that

$$\|\Psi_n'\| \longrightarrow 0. \tag{4.18}$$

Step 3. Taking the inner product (4.7) with $\Psi^{'}_n$ and (4.8) with $\phi^{'}_n + \Psi_n$ in $L^2(0.1)$, and using the fact that $\frac{K}{\rho_1} = \frac{b}{\rho_2}$, we get

$$\begin{aligned} & i\alpha_n \left\langle u_n, \Psi^{'}_n \right\rangle + i\alpha_n \left\langle v_n, \phi^{'}_n + \Psi_n \right\rangle + \frac{K}{\rho_2} \left\| \phi^{'}_n + \Psi_n \right\|^2 \\ & \quad + \frac{\beta}{\rho_2} \left\langle w'_n, \phi^{'}_n + \Psi_n \right\rangle - \frac{K}{\rho_1} \left| ((\Psi'_n) \cdot (\overline{\phi^{'}_n + \Psi_n}))(x) \right|_{x=0}^{x=1} \\ & \quad + \frac{K}{\rho_1} \left(\left\langle \phi^{'}_n + \Psi_n, \Psi''_n \right\rangle - \left\langle \Psi''_n, \phi^{'}_n + \Psi_n \right\rangle \right). \end{aligned} \tag{4.19}$$

Using (4.9), (4.10), and (4.19), we obtain

$$\begin{aligned} & - \| \alpha_n \Psi_n \|^2 - \frac{K}{\rho_1} \left[\Psi^{'}_n \overline{(\phi^{'}_n + \Psi_n)} \right]_0^1 + \frac{K}{\rho_2} \| \phi^{'}_n + \Psi_n \|^2 \\ & + \frac{\beta}{\rho_2} \left\langle w^{'}_n, \phi^{'}_n + \Psi_n \right\rangle + \frac{K}{\rho_1} \left(\left\langle \phi^{'}_n + \Psi_n, \Psi^{''}_n \right\rangle - \left\langle \Psi^{''}_n, \phi^{'}_n + \Psi_n \right\rangle \right) \longrightarrow 0, \end{aligned} \tag{4.20}$$

multiplying (4.20) with $\frac{1}{\alpha_n^2}$, we have

$$\begin{aligned} & - \| \Psi_n \|^2 - \frac{K}{\rho_1} \left[\frac{\Psi^{'}_n}{\alpha_n} \frac{\overline{(\phi^{'}_n + \Psi_n)}}{\alpha_n} \right]_0^1 + \frac{\beta}{\rho_2 \alpha_n} \left\langle w^{'}_n, \frac{\phi^{'}_n + \Psi_n}{\alpha_n} \right\rangle \\ & + \frac{K}{\rho_2} \| \frac{\phi^{'}_n + \Psi_n}{\alpha_n} \|^2 + \frac{K}{\alpha_n^2 \rho_1} \left(\left\langle \phi^{'}_n + \Psi_n, \Psi^{''}_n \right\rangle - \left\langle \Psi^{''}_n, \phi^{'}_n + \Psi_n \right\rangle \right) \longrightarrow 0. \end{aligned} \tag{4.21}$$

Now, using the fact that $\|z_n\|_{\mathcal{H}} = 1, \ \forall \ n \succeq 0$ and multiplying (4.7) by α_n, we deduce that

$$\left(\| \left(\frac{\phi^{'}_n + \Psi_n}{\alpha_n} \right)^{'} \| \right)_{n \in \mathbb{N}} \text{ is uniformly bounded.} \tag{4.22}$$

In addition, we apply the triangular inequality to estimate the boundary term in (4.21), we get

$$\left| \frac{\Psi^{'}_n}{\alpha_n} \overline{\left(\frac{\phi^{'}_n + \Psi_n}{\alpha_n} \right)} (x) \right| \longrightarrow 0. \tag{4.23}$$

Taking the real part of (4.21), using (4.13), (4.18) and (4.23), we deduce that

$$\| \frac{\phi^{'}_n + \Psi_n}{\alpha_n} \| \longrightarrow 0. \tag{4.24}$$

Step 4. Taking the inner product of $iv_n - \dfrac{1}{\alpha_n \rho_2} \left(b\Psi^{''}_n - K \left(\phi^{'}_n + \Psi_n \right) - \beta w^{'}_n \right)$ with v_n in $L^2(0.1)$, we obtain

$$i \| v_n \|^2 - \frac{b}{\rho_2} \left\langle \Psi^{'}_n, \frac{v^{'}_n}{\alpha_n} \right\rangle - \frac{K}{\rho} \left\langle \frac{\phi^{'}_n + \Psi_n}{\alpha_n}, v_n \right\rangle - \frac{\beta}{\rho_2} \left\langle w^{'}_n, \frac{v_n}{\alpha_n} \right\rangle \longrightarrow 0. \tag{4.25}$$

Using (4.10) and (4.18), we have

$$\| \frac{v^{'}_n}{\alpha_n} \| \longrightarrow 0. \tag{4.26}$$

From (4.13), (4.18), (4.24), (4.25), and (4.26), we obtain

$$\|v_n\| \longrightarrow 0., \tag{4.27}$$

applying again the triangular inequality, using (4.10), and (4.27), we deduce that

$$\|\alpha_n \Psi_n\| \longrightarrow 0. \tag{4.28}$$

Step 5. Taking the inner product of (4.7) with u_n in $L^2(0.1)$, we get

$$i\|\alpha_n^{\frac{1}{2}} u_n\|^2 + \frac{K}{\rho_1}\left\langle \phi_n^{'}, u_n^{'} \right\rangle + \frac{K}{\rho_1}\left\langle \Psi_n, u_n^{'} \right\rangle \longrightarrow 0, \tag{4.29}$$

replacing $u_n^{'}$ by $i\alpha_n \phi_n^{'}$ in (4.29), we get

$$i\|\alpha_n^{\frac{1}{2}} u_n\|^2 + i\frac{K}{\rho_1}\|\alpha_n^{\frac{1}{2}} \phi_n^{'}\|^2 - \frac{K}{\rho_1}\left\langle \Psi_n^{'}, u_n \right\rangle \longrightarrow 0. \tag{4.30}$$

From (4.18) and (4.30), we deduce that

$$\|u_n\| \longrightarrow 0, \text{ and } \|\phi_n^{'}\| \longrightarrow 0, \tag{4.31}$$

By Poincare's inequality, we get

$$\|\Psi_n\| \longrightarrow 0. \tag{4.32}$$

Using the triangular inequality and (4.31), we obtain

$$\|\phi_n^{'} + \Psi_n\| \longrightarrow 0. \tag{4.33}$$

Finally from (4.13), (4.14), (4.18), (4.27), (4.31) and (4.33), we deduce that

$$\|z_n\| \longrightarrow 0. \tag{4.34}$$

Therefore we get the contradiction with (4.4). □

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