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# AN APPROACH TO INTEGRAL W.R.T. MEASURE THROUGH RANDOM SUMS

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**ABSTRACT.** In this paper we construct a measure preserving transformation to the space $([0,1), \mathbf{B})$, $\mathbf{B}$ being the Borel $\sigma$-algebra, from an abstract measurable space $(\Omega, \mathcal{F})$, which yields a particular sequence of partitions, called a system of partitions. This function transforms an arbitrary measure $\mu$ on $\Omega$ to the Lebesgue measure on $[0,1)$. This transformation generalizes some results which have been obtained for $[0,1)$ to the abstract space $\Omega$. As an example, in the investigation of convergence of random Riemann sums to the Lebesgue integral, the underlying measure space is taken to be $([0,1), \mathbf{B}, m)$ where $m$ is the Lebesgue measure on $[0,1)$. In this paper we consider, instead of $[0,1)$, the abstract space $\Omega$, substitute $\mathbf{B}$ with the $\sigma$-algebra $\mathcal{F}$ generated by the given system of partitions and $m$ with an arbitrary atomless probability measure.

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## 1. PRELIMINARIES

The Idea of random Riemann sums comes from the concept of first return integral [1, 2]. The concept of random Riemann sums is considered in [7, 5, 3, 4, 6]. We present a somehow modified definition of it as follows. Denote the interval $[0,1)$ by $I_0$ and let $I_0$ be equipped with the Borel $\sigma$-algebra $\mathbf{B}$. Let $m$ be the Lebesgue measure on $\mathbf{B}$.

Let $\mathcal{P}_0$ be a finite partition of $I_0$ by intervals of positive length. We note that any collection of disjoint non-empty intervals can be ordered naturally in terms of the natural order of real numbers. Let this order too be denoted by $<$. Let $J_i$, $1 \leq i \leq n$, be the intervals constituting $\mathcal{P}_0$ s.t. $J_1 < J_2 < \cdots < J_n$. Corresponding to $\mathcal{P}_0$, there is a unique finite sequence $x_i$, $0 \leq i \leq n$, of elements of $I_0$ s.t. $0 = x_0 < x_1 < \cdots < x_n = 1$ and s.t. $x_{i-1}$ and $x_i$ are the end points of $J_i$, for $1 \leq i \leq n$. In what follows $\mathcal{P}_0$ is fixed unless otherwise stated.

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For an arbitrary nonempty set $S$ if $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions of $S$, we say $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$ and write $\mathcal{P}_1 \preceq \mathcal{P}_2$ if each element of $\mathcal{P}_1$ is a union of elements of $\mathcal{P}_2$. In this article partitions of $I_0$ in the position that $I_0$ is assumed, i.e. when it is the range space of transformations, are as defined above. Hence if $\mathcal{P}_1$ and $\mathcal{P}_2$, are partitions of $I_0$, then $\mathcal{P}_1 \preceq \mathcal{P}_2$ if and only if the sequence corresponding to $\mathcal{P}_1$ is a subsequence of that corresponding to $\mathcal{P}_2$. The norm of $\mathcal{P}_0$ w.r.t. $m$ is $\|\mathcal{P}_0\| := \max\{m(J_i) : 1 \leq i \leq n\}$. For each $i$, $1 \leq i \leq n$, let $t_i \in J_i$ be a random variable with uniform distribution in $J_i$ and let $t_i$'s be independent.

**Definition 1.1.** Let $f : I_0 \longrightarrow \mathbb{R}$ be a Lebesgue integrable function. The random Riemann sum of $f$ w.r.t. $\mathcal{P}_0$ is defined to be the r.v.

$$\mathcal{S}_{\mathcal{P}_0}(f) = \sum_{i=1}^{n} f(t_i)m(J_i).$$

In [5] is defined the random Riemann sums for a fixed and non-random sequence of partitions $\{\Delta_n\}_{n\geq 1}$ of $I_0$ such that $\Delta_n \preceq \Delta_{n+1}$, $n \geq 1$ and $\|\Delta_n\| \longrightarrow 0$ and is proved that such a sequence of random Riemann sums tends to $\int_{I_0} f dm$, a.s. There is also presented a probability space which yields the desired random elements.

In [3] some results are proved for $I_0$ in terms of arbitrary but non-random, not necessarily being refined, sequences of partitions for which again the corresponding sequence of norms w.r.t. the Lebesgue measure $m$ tends to zero.

Let, henceforth, subintervals of $I_0$ be of positive length and of the form $[x, y)$. The interested reader may easily verify that in [3], instead of taking partitions of $I_0$ to be in terms of subintervals, they can be taken to consist each of finite (or even countable) disjoint unions of subintervals. In fact if $I$ is an interval and $U$ is a union of finite disjoint intervals s.t. $m(I) = m(U)$, the measure spaces $I$ and $U$ with Borel $\sigma$-algebras and Lebesgue measures are measure theoretically isomorphic in natural ways. Thus a somewhat generalized formulation of Proposition 2.1 of [3] is the following Theorem.

**Theorem 1.2.** *For any $\epsilon > 0$, and any sequence of partitions $\mathcal{P}_n$, $n \geq 1$, of $I_0$ whose elements are finite unions of disjoint intervals, if $\lim_{n\longrightarrow\infty} \|\mathcal{P}_n\| = 0$, then*

$$P\left(|S_{\mathcal{P}_n}(f) - \int_{I_0} f dm| > \epsilon\right) \longrightarrow 0.$$

In [7] the sequence of partitions of $I_0$ based on which the random Riemann sums are defined is randomized and the results of [3] are generalized for such sequences of partitions.

In this article results of [5] and [3] are generalized, from the space $(I_0, \mathbf{B}, m)$ to a general probability space $(\Omega, \mathcal{F}, \mu)$, under some reasonable and rather weak and general conditions.

**Definition 1.3.** Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Call for a partition $\mathcal{P}$ of $\Omega$ consisting of elements of $\mathcal{F}$, $\sup_{I \in \mathcal{P}} \mu(I)$ the norm of $\mathcal{P}$, w.r.t. $\mu$ and denote it by $|\mathcal{P}|_\mu$.

**Definition 1.4.** For a probability space $(\Omega, \mathcal{F}, \mu)$ a sequence $\{\Delta_n\}_{n\geq 1}$ of partitions of $\Omega$ is called a system of partitions if:

1. for each $n \geq 1$, $\Delta_n$ is a countable collection of elements of $\mathcal{F}$;
2. the collection $\bigcup_{n\geq 1} \Delta_n$ of subsets of $\Omega$ generates $\mathcal{F}$;
3. $\lim_{n\longrightarrow\infty} |\Delta_n|_\mu = 0$.

Call a system of partitions decreasing if for each $n \geq 1$, $\Delta_{n+1}$ is a refinement of $\Delta_n$.

Henceforth $\Delta_n$, $n \geq 1$, denotes a system of partitions of $\Omega$.

**Definition 1.5.** For $\omega \in \Omega$, $n \geq 1$, let $I_n(\omega)$ be the unique element of $\Delta_n$ containing $\omega$. Call the sequence $I_n(\omega)$, $n \geq 1$, the $\omega$-tower in the system.

**Proposition 1.6.** *$\{\omega\} \in \mathcal{F}$ if and only if $\bigcap_{n\geq 1} I_n(\omega) = \{\omega\}$.*

**Definition 1.7.** If $\mathcal{P}_1$ and $\mathcal{P}_2$ are arbitrary partitions of a nonempty set $S$, call the partition

$$\left\{ I_1 \bigcap I_2 : I_1 \in \mathcal{P}_1,\ I_2 \in \mathcal{P}_2 \right\},$$

the summation of $\mathcal{P}_1$ and $\mathcal{P}_2$ and denote it by $\mathcal{P}_1 \curlyvee \mathcal{P}_2$.

**Proposition 1.8.** *Let $\Delta_1^{(c)} = \Delta_1$, ${\Delta_{n+1}}^{(c)} = {\Delta_n}^{(c)} \curlyvee \Delta_{n+1}$, $n \geq 1$. The sequence ${\Delta_n}^{(c)}$, $n \geq 1$, of partitions constitutes a decreasing system of partitions for $(\Omega, \mathcal{F}, \mu)$. For $\omega \in \Omega$, $n \geq 1$, let ${I_n}^{(c)}$ be the unique element of ${\Delta_n}^{(c)}$ containing $\omega$. For $\omega \in \Omega, \bigcap_{n\geq 1} I_n(\omega) = \bigcap_{n\geq 1} I_n^{(c)}(\omega)$.*

**Definition 1.9.** Call ${\Delta_n}^{(c)}$, $n \geq 1$, the decreasing system corresponding to $\Delta_n$, $n \geq 1$.

**Proposition 1.10.** *Let $\Delta_n$, $n \geq 1$, be a system of partitions for $(\Omega, \mathcal{F}, \mu)$. For each $\omega \in \Omega$, let $\{\omega\} \in \mathcal{F}$. The condition $|\Delta_n|_\mu \longrightarrow 0$ is equivalent to $\mu$ being diffuse, i.e. having no atoms.*

**Remark 1.11.** Euclidean spaces and more generally, locally compact second countable Hausdorff topological spaces and hence complete separable, i.e. Polish, metric spaces, with Borel $\sigma$-algebras and diffuse probability measures, when they admit such measures, yield decreasing systems of partitions which generate the Borel $\sigma$-algebra.

In the sequel we assume $(\Omega, \mathcal{F}, \mu)$ is a fixed probability space for which $\mu$ is atomless and there exists a decreasing system $\Delta_n$, $n \geq 1$ of partitions. Although to some stage we can proceed on a more general basis as described above, for the sake of simplicity and clarity we assume, in what follows, finite partitions for $\Omega$ instead of countable ones. Further we assume partitions have elements with positive, instead of nonnegative measures.

Let $\mathcal{P}$ be a (finite) partition of $\Omega$ consisting of measurable sets $A_1, A_2, \ldots, A_k$ (such that for each $i$, $1 \leq i \leq k$, $\mu(A_i) > 0$). For each $i$, $1 \leq i \leq k$, let $z_i$ be a random element of $A_i \in \mathcal{P}$, chosen according to the probability law $\mu_i(\cdot) = \frac{\mu(\cdot)}{\mu(A_i)}$ and let $z_i$'s be independent.

There are randomization mechanisms, i.e. probability spaces which yield the required random elements. In all cases in this article, appropriate randomization mechanisms exist[7, 5]. Suppose $f : (\Omega, \mathcal{F}, \mu) \longrightarrow (\mathbb{R}, \mathbf{B}_{\mathbb{R}})$ is an integrable function where $\mathbf{B}_{\mathbb{R}}$ is the Borel $\sigma$-algebra in $\mathbb{R}$.

**Definition 1.12.** The random Riemann-Stieltjes sum of $f$ w.r.t. $\mathcal{P}$ is defined to be

$$\mathcal{S}'_{\mathcal{P}}(f) = \sum_{1 \leq i \leq k} f(z_i)\mu(A_i).$$

Note that when $\Omega = I_0,\ \mathcal{F} = \mathbf{B}$, and $\mu = mA$, then $\mathcal{S}'_{\mathcal{P}}(f) = \mathcal{S}_{\mathcal{P}}(f)$.

## 2. MAIN RESULTS

We prove the following theorem which generalizes the main result of [5].

**Theorem 2.1.** *There exists a probability space which yields the corresponding random Riemann-Stieltjes sums $\mathcal{S}'_{\Delta_n}(f)$, and based on which such a sequence converges almost surly to $\int_\Omega f d\mu$.*

The proof as will be seen provides us with tools to establish further results, in particular to extend results in [3], such as Proposition2.1., whose generalized formulation is the following theorem.

**Theorem 2.2.** *Let $\{\overline{\Delta}_n\}_{n\geq 1}$ be a system of partitions of $(\Omega, \mathcal{F}, \mu)$. Then for each $\epsilon > 0$,*

$$P(|\mathcal{S}'_{\overline{\Delta}_n}(f) - \int_\Omega f d\mu| > \epsilon) \longrightarrow 0.$$

We are seeking to construct a transformation, which will be denoted by $X_0$, from the space $(\Omega, \mathcal{F}, \mu)$ to the space $(I_0, \mathbf{B}, m)$ which enable us to deduce natural analogs of the results already obtained concerning the convergence of the sequence of random Riemann sums for the case where the space is $(I_0, \mathbf{B}, m)$ for the sequence of random Riemann-Stieltjes sums for the general space $(\Omega, \mathcal{F}, \mu)$ with the assumed properties.

## 3. CHARACTERIZATION

We proceed by presenting some preliminary results. Recall that $I_0 = [0, 1)$, all subintervals of $I_0$ are assumed to be with possitive length and of the form $[x, y)$. $\Delta_n$, $n \geq 1$, is a decreasing system of partitions for $(\Omega, \mathcal{F}, \mu)$.

**Proposition 3.1.** *There is a sequence $\{\Delta'_n\}_{n\geq 1}$ of partitions of $I_0$, consisting of intervals, for which letting $\bigcup_{n\geq 1}\Delta_n = \Delta$ and $\bigcup_{n\geq 1}\Delta'_n = \Delta'$, we have a one to one correspondence $\sigma : \Delta \longrightarrow \Delta'$ s.t. for any $I \in \Delta$, we have $\mu(I) = m(\sigma(I))$ and for any $I, J \in \Delta$, if $J \subseteq I$, then $\sigma(J) \subseteq \sigma(I)$.*

**Corollary 3.2.** *In terms of the notations of Proposition* 3.1*, it follows that $\lim_{n\longrightarrow\infty} \|\Delta'_n\| = 0$. Hence $\Delta'$ generates $\mathbf{B}$ and $\{\Delta'_n\}_{n\geq 1}$ is a decreasing system of partitions for $(I_0, \mathbf{B}, m)$.*

**Definition 3.3.** Call $\sigma$ as given in Proposition 3.1, a canonical mapping from $\Delta$ to $\Delta'$. A sequence $I_n \in \Delta_n$, $n \geq 1$, is called a tower in $\Delta$, if $I_{n+1} \subseteq I_n$, $n \geq 1$. A tower in $\Delta'$ and in general is defined similarly.

**Remark 3.4.** If $I'_n$, $n \geq 1$ is a tower in $\Delta'$, then $\bigcap_{n\geq 1} I'_n$ is either empty or a singleton. It is easy to see that the collection of towers in $\Delta'$ which reduce to the empty set is countable, and hence so is the collection of corresponding towers in $\Delta$ under the correspondence $\sigma$. Let $N$ consist of the intersection of such towers of $\Delta$. It follows that $N$ is measurable and $\mu(N) = 0$.

**Proposition 3.5.** *Let $\widehat{\Omega} = \Omega - N$, and denote for $n \geq 1$, the restriction of $\mathcal{F}, \mu$, and $\Delta_n$, to $\widehat{\Omega}$, respectively, by $\widehat{\mathcal{F}}, \widehat{\mu}$, and $\widehat{\Delta}_n$. The collection $\widehat{\Delta} = \bigcup_{n\geq 1}\widehat{\Delta}_n$ generates $\widehat{\mathcal{F}}$ and the sequence $\widehat{\Delta}_n$, $n \geq 1$, constitutes a decreasing system of partitions for the space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mu})$.*

*Proof.* The truth of the first part follows in view of Definition 1.4(2). The truth of the second part is easy to see. □

**Remark 3.6.** In view of Propositions 3.1 and 3.5, let $\tau'$ be the tower in $\Delta'$ corresponding to the tower $\tau$ in $\Delta$ under $\sigma$ and $\widehat{\tau}$ be the tower in $\widehat{\Delta}$ corresponding to the tower $\tau$ in $\Delta$ when restricted to $\widehat{\Omega}$. It is now clear that if $\bigcap_{I\in\tau'} I = \emptyset$, then $\bigcap_{I\in\widehat{\tau}} I = \emptyset$, whereas $\bigcap_{I\in\tau} I$ may be empty or a non-empty subset of $N$. We denote by $\widehat{\sigma}$ the one to one correspondence from $\widehat{\Delta}$ to $\Delta'$ which is naturally defined based on $\sigma$.

Let $\omega \in \widehat{\Omega}$, and $\widehat{\tau}(\omega)$ be the $\omega$-tower in $\widehat{\Delta}$. The tower in $\Delta'$ corresponding to $\widehat{\tau}(\omega)$ under $\widehat{\sigma}$, reduces to a singleton, which we denote by $\widehat{X}(\omega)$. In this way a transformation $\widehat{X} : \widehat{\Omega} \longrightarrow I_0$ is defined.

**Proposition 3.7.** *The transformation $\widehat{X} : \widehat{\Omega} \longrightarrow I_0$ as defined above has the following properties:*

1. *$\widehat{X}$ yields a natural one to one correspondence between $\widehat{\Delta}_n$ and $\Delta'_n$ for each $n$ and between the collection of towers of $\widehat{\Delta}$ and those of $\Delta'$;*
2. *$\widehat{X}$ is $\widehat{\mathcal{F}}$-$\mathbf{B}$ measurable and in fact $\widehat{X}^{-1}(\mathbf{B}) = \widehat{\mathcal{F}}$;*
3. *$\widehat{X}$ transforms the measure $\widehat{\mu}$ on $\widehat{\Omega}$ to the Lebesgue measure on $I_0$.*

*Proof.* The truth of (1) is obvious. For (2) we note that $\widehat{X}^{-1}(I) = \widehat{\sigma}^{-1}(I)$ for $I \in \Delta'$. $\widehat{\Delta}$ generates $\widehat{\mathcal{F}}$ and $\Delta'$ generates $\mathbf{B}$. To see the truth of (3) we observe that

$$m(I) = \widehat{\mu}(\widehat{\sigma}^{-1}(I)) = \widehat{\mu}(\widehat{X}^{-1}(I)),$$

for $I \in \Delta'$. Since $\widehat{\sigma}^{-1}(\Delta') = \widehat{\Delta}$ and $\Delta'$ and $\widehat{\Delta}$ are semi-rings generating $\mathbf{B}$ and $\widehat{\mathcal{F}}$, respectively, it follows that $m(B) = \widehat{\mu}(\widehat{X}^{-1}(B))$ for $B \in \mathbf{B}$. □

**Remark 3.8.** It is worth noting that $\widehat{X}$ as defined above may not be onto. In fact the case may be such that direct images of elements of $\widehat{\mathcal{F}}$ under $\widehat{X}$ be non-Borel subsets of $I_0$.

**Proposition 3.9.** *Define $X_0 : \Omega \longrightarrow I_0$ to be*

$$X_0(\omega) = \begin{cases} \widehat{X}(\omega), & \text{if } \omega \in \widehat{\Omega}, \\ 0. & \text{if } \omega \in N. \end{cases}$$

*Then $X_0$ as a function from $(\Omega, \mathcal{F}, \mu)$ to $(I_0, \mathbf{B}, m)$ is measurable and measure preserving.*

**Corollary 3.10.** *Let $I \in \Delta$, and $I'$ be the element of $\Delta'$ corresponding to $I$ under $\sigma$. The restriction of $X_0$ to $I$ as a function from $(I, \mathcal{F}\mid_I, \frac{\mu(\cdot)}{\mu(I)})$ to $I' \bigcup \{0\}$, equipped with the Borel $\sigma$-algebra, induces the measure $\frac{m(\cdot)}{m(I')}$, i.e. the normalization of the Lebesgue measure, on $I' \bigcup \{0\}$.*

**Proposition 3.11.** *Recall that $f$ is a real integrable function on $(\Omega, \mathcal{F}, \mu)$. Let $\widehat{f}$ be the restriction of $f$ to $\widehat{\Omega}$. There is a real integrable function $g$,*

$$g : (I_0, \mathbf{B}, m) \longrightarrow (\mathbb{R}, \mathbf{B}_{\mathbb{R}}),$$

*s.t. $\widehat{f} = go\widehat{X}$ and hence*

$$f(\omega) = (goX_0)(\omega), \ \mu - a.e. \omega \in \Omega,$$

*and so*

$$\int_\Omega f d\mu = \int_{I_0} gdm.$$

*Proof.* Consider the function $\widehat{f} : (\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mu}) \longrightarrow (I_0, \mathbf{B}, m)$. The existence of the function $g$ s.t. $\widehat{f} = go\widehat{X}$ follows from a well known result. The truth of the rest of the assertion is clear. □

**Proposition 3.12.** *For each $n \geq 1$, let $I_i^{(n)}$, $1 \leq i \leq k_n$, be a sequence listing all distinct elements of $\Delta_n$. There is a probability space based on which, for $n \geq 1$, $1 \leq i \leq k_n$, an element $z_i^{(n)}$ of $I_i^{(n)}$ can be chosen according to the probability law $\frac{\mu(\cdot)}{\mu(I_i^{(n)})}$, s.t. for each fixed $n$, $z_i^{(n)}$'s, $1 \leq i \leq k_n$, are independent. It then follows that for each $n$ and $i$, $n \geq 1$, $1 \leq i \leq k_n$, the random variable $x_i^{(n)} = X_0(z_i^{(n)})$, $n \geq 1$, is* *chosen according to the law* $\frac{m(\cdot)}{m(I')}$*, i.e. has uniform distribution in* $I'^{(n)}_i$*, and for each* $n$*,* $x_i^{(n)}$*'s* $1 \le i \le k_n$*, are independent.*

*Proof.* See [5, 7]. □

## 4. **PROOFS Of THE MAIN RESULTS**

4.1. **Proof of Theorem 2.1.** In terms of the notations of Proposition 3.12, w.r.t. the basic probability space, a.s.

$$\begin{aligned}\mathcal{S}'_{\Delta_n}(f) = \sum_{i=1}^{k_n} f(z_i^{(n)})\mu(I_i^{(n)}) &= \sum_{i=1}^{k_n} g(X_0(z_i^{(n)}))\mu(I_i^{(n)}) \\ &= \sum_{i=1}^{k_n} g(x_i^{(n)})m(I'^{(n)}_i) = \mathcal{S}_{\Delta'_n}(g),\end{aligned}$$

for any $n \ge 1$. In other words, for any $n \ge 1$, the probabilistic behaviours of the random sums $\sum_{i=1}^{k_n} f(z_i^{(n)})\mu(I_i^{(n)})$ and $\sum_{i=1}^{k_n} g(x_i^{(n)})m(I'^{(n)}_i)$, both of which are generated by the same randomization mechanism, are the same. This in particular proves the assertion of Theorem 2.1.

4.2. **Proof of Theorem 2.2.** In [3], in order to investigate the behaviour of the sequence of random sums the sequence $\mathcal{P}_n$, $n \ge 1$, of partitions of $I_0$ is taken to consist each (expressing in a slightly modified manner) of disjoint subintervals of $I_0$ (of positive length and of the form $[x, y)$). The author works with the normalization of the Lebesgue measure on each such subinterval. As we stated earlier, partitions can be taken to consist of finite (or even countable) disjoint unions of subintervals, The results of [3] can be generalized to the case where we take $(\Omega, \mathcal{F}, \mu)$ instead of $(I_0, \mathbf{B}, m)$.

Let $\overline{\Delta}_n$, $n \ge 1$, be a system of partitions of $\Omega$. Corresponding to $\overline{\Delta}_n$, $n \ge 1$, define the sequence $\Delta_n$, $n \ge 1$, as $\Delta_1 = \overline{\Delta}_1$, $\Delta_2 = \overline{\Delta}_2 \curlyvee \Delta_1$, $\Delta_n = \overline{\Delta}_n \curlyvee \Delta_{n-1}$, $n \ge 2$. Then the sequence $\Delta_n$, $n \ge 1$, is a decreasing system of partitions of $\Omega$. If $\Delta'_n$, $n \ge 1$, is the sequence of partitions of $I_0$ corresponding to $\Delta_n$, $n \ge 1$, under $\sigma$, then in natural ways, there is a sequence $\overline{\Delta}'_n$, $n \ge 1$, of partitions of $I_0$ corresponding to $\overline{\Delta}_n$, $n \ge 1$, s.t. for each $n \ge 1$, each element of $\overline{\Delta}'_n$, $n \ge 1$, is a finite disjoint union of elements of $\Delta'_n$, $n \ge 1$. In this way it is clear that natural analogs of results of [3], in particular Theorem 2.2, and other similar results can be obtained for the general case $\Omega$ instead of $I_0$.

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