Dynamic Systems and Applications 23 (2014) 39-62

ASSET ALLOCATION FOR REGIME-SWITCHING MARKET MODELS UNDER PARTIAL OBSERVATION

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ABSTRACT. As researchers and practitioners continuously search for good models to capture price movements of financial assets, regime-Switching models have received increasing attention. Using a regime-switching model, we study asset allocation problems with one risk-free asset and one risky asset. One of the main features of the paper is that the switching process is not observable. Thus we are in the framework of asset allocation under partial observation. We resort to Wonham filters to recover necessary information required for optimal control of the problems under consideration. After converting the partial observable controls to completely observable controls, we characterize the associated value function in terms of solutions of a partial differential equation, the Hamilton-Jacobi-Bellman (HJB) equation. Owing to its nonlinearity, it is difficult to obtain the close-form solution of the HJB equation. Markov chain approximation methods are used to find solutions numerically.

AMS (MOS) Subject Classification. 91G80, 93E11, 93E20

1. INTRODUCTION

When an investor invests in various assets, she/he has numerous choices. All these choices can be categorized into two types: risky investments or risk-free investments. Risky investments such as stocks, bring the possibility for higher profits but exposes more risk. Risk-free investments such as T-bond or money market, secure predictable amount of profit but at a lower return rate. When making investment discussions, the problem of balancing wealth between risk-free and risky investments such as bonds and stocks, constantly comes up. The term "asset allocation" refers to the process of spreading wealth across different types of financial asset classes. The modern financial market offers unprecedented opportunity of moving money from one class to another. Between the two types of investment, the price of a risk-free asset is easy to model, which is often assumed to satisfy an ordinary differential equation for a continuous compounding bank account. The modeling of the price for a risky asset such as a stock is considerably more complex and difficult. This paper aims at deriving optimal

This research was supported in part by the Army Research Office under W911NF-12-1-0223.
Received April 21, 2013

strategies to dividing the proportion of these two different assets. To capture trending markets, we adopt a regime-switching formulation.

As researchers and practitioners continuously search for better models to capture price movements of financial assets, regime-switching models have received increasing attention. Traditionally, a geometric Brownian motion (GBM) formulation has been widely used in finance to capture the movement of stock prices because of its simple structure. However, practitioners and researchers noted its short comings. One of its main drawbacks is the constant return rate and volatility. To capture both longer term market trends and shorter-term market uncertainty, an effective model is the regime-switching diffusion, in which both the return rates and the volatility depend on a Markov switching process. Regime-switching models assume that the market has finitely many modes. Under different market mode, there are different sets of market parameters such as the rate of returns, volatility or the risk-free interest rates. The market movement from one mode to another mode over time is often assumed to follow a Markov switching process. A simple case is a two-state Markov chain with one state representing the bull market and the other bear market. Justification of regime-switching models in marketplaces can be found in [21] and [19]; see also [6] and [9] among others for empirical studies in connection with regime-switching models.

It is only until very recently, researchers and practitioners have recognized the importance of regime-switching modeling in asset allocation. To name a few, we mention the work [1] and [5]. When the underlying Markov process is observable. In [22], a closed-form portfolio selection is developed using a mean-variance technique. For a hidden Markov model of a special structure, an optimal trading strategy has been presented in [10]. In [19], nearly-optimal asset allocation strategies were developed to maximize the expected returns.

In a regime-switching model, the market mode is typically not observable. In this case, its states need to be estimated using filtering techniques. Related work can be found in [15], [10], and [14]. In [15], they considered a simple case of two-state Markov chain with an absorbing state. Wonham filter is used to estimate the conditional probability of the market mode. A selling rule was studied under this framework. In [10], an unnormalized filter is used to derive an optimal investment strategy in an asset allocation problem. In [14], similar asset allocation problem was considered and classical probabilistic solutions are derived with a logarithmic utility and a power utility function.

In this paper, we consider a regime-switching asset model that is modulated by a continuous-time Markov chain. We assume that the modulating force, the underlying Markov chain, can only be observed indirectly through stock prices. To deal with incomplete observation, we adopt the approach of [15] and [14] and use the Wonham filtering techniques [17]. We consider general setting of the asset allocation and characterize the value function in terms of solutions of Hamilton-Jacobi-Bellman (HJB) equations. We show that the value function is the only viscosity solution to the associated HJB equations. It is difficult to obtain the close-form solution of the HJB equation. For computational methods, a finite difference method is often used to compute the solution. However, in our problem, due to the specific degenerate feature of the stochastic differential equation, finite difference method is not directly applicable. Instead of using the finite different methods, we use Kushner's Markov chain approximation methods to find numerical solutions of the problem; A Markov approximation is based on a probabilistic interpretation of the stochastic differential equations and leads to an approximation to the solution; see [12] and [13] for complete treatment of the methods.

To summarize, our contributions in this paper include:

1. We use a regime-switching model for optimal asset allocation with partial observation. We use the Wonham filtering techniques to convert the partially observable control problem into a completely observable one which allows to treat the general asset allocation problem with terminal wealth.
2. We derive the associated HJB equation for the converted completely observable control problem and demonstrate that there is a unique solution.
3. Owing to the lack of close-form solution of the HJB equation in the general case, we resort to numerical methods and use Markov chain approximation techniques to obtain the convergence of the numerical algorithm.
4. We study a separable case which allows us separate the time variable and the state variables so as to achieve the reduction of the overall computation.

The rest of the paper is arranged as follows. Section 2 begins with the problem formulation. Section 3 recalls the notion of Wonham filters, and derive such filter for our problem. After converting the partially observed control to that of complete-observation control, we characterize the value function in Section 4. Section 5 device numerical approximation based on Markov chain approximation techniques. Section 6 considers a separable case. Section 7 provides a numerical example. Finally, Section 8 concludes the paper.

2. PROBLEM SETUP

In this paper, we consider a continuous-time market setting with one risk-free investment and one risky investment. Their prices are denoted by $P_1(t)$ and $P_2(t)$, respectively. The risk-free investment $P_1(t)$ pays a constant interest rate of $r > 0$. The risky investment $P_2(t)$ is modeled according to a regime-switching model, namely,

$P_1(t)$ and $P_2(t)$ satisfies

$$\begin{aligned} dP_1(t) &= P_1(t)rdt, \\ \frac{dP_2(t)}{P_2(t)} &= \mu(\alpha(t))dt + \sigma dW(t), \end{aligned} \tag{2.1}$$

where $\alpha(t) \in \mathcal{M} = \{1, 2, \ldots, m\}$ is a continuous-time Markov chain that governs the market mode, and $\mu(i)$ is the return rate of the stock when the market mode is i. Note that this is a model where the rate of return follows the market trend, but the volatility is assumed to be constant. This is mainly needed to meet the Wonham filter conditions. Such assumption is acceptable in asset allocation problems because the dependence on volatility is not as crucial as in derivative pricing.

Let $\xi(t)$ denote the wealth of the investor at time t and $u(t)$ to portion allocated to risky investment. That is, at each time t, $u(t)\xi(t)$ is put into risky investment and $(1-u(t))\xi(t)$ is put in risk-free investment. We will assume self-financing, which means $\xi(t)$ equals to the sum of the values of the above investments and no external funds are transferred to it or from it. There is no cash inflow or outflow and no short sell. Therefore, we must have

$$\frac{d\xi(t)}{\xi(t)} = (1-u(t))rdt + u(t)(\mu(\alpha(t))dt + \sigma dW(t)). \tag{2.2}$$

Suppose the initial time is s, and the initial wealth is $\xi(s) = y$. We assume the investor did not consume any amount of the investment. The investor's objective is to dynamically adjust $u(t)$ over time to maximize the expected a utility function $\Phi(\xi(T))$.

This type of asset allocation with regime switching have been studied extensively in the literature. Nevertheless, most of the research is concerned with completely observable models, which is far from reality. In this paper, we aim at the partially observed cases and develop a method for this type of asset allocation problems.

Note that $u(t)$ is a feedback control. Denote the filtration generated by $P_2(t)$ as $\mathcal{F}_t$. A control u is *admissible* if u is progressively measurable with respect to $\{\mathcal{F}_t\}$ and $u(t) \in [0, 1]$ for all $t \in [0, T]$. Denote the set of admissible control by $\mathcal{A}$.

In this paper, the price of the stock is observable but the market mode $\alpha(t)$ cannot be directly observed. For example, in a two-state Markov chain case, it is not possible to label the market mode to be either 'bull' or 'bear.' In view of this, $\alpha(t)$ is often considered as a "hidden Markov chain." In this case, one viable way of solving the problem is to come up with some form of estimation of $\alpha(t)$ based on the observation $P_2(t)$ so as to extract needed information.

Note that $P_2(t)$ is a function of $\alpha(t)$. To estimate the state of $\alpha(t)$, we use the Wonham filter; see [17]. The Wonham filter enables us to compute the conditional probability of $\alpha(t)$ given the past observation $\mathcal{F}_t = \{P_2(r), s < r < t\}$.

3. WONHAM FILTER

Let $\alpha(t)$ be a continuous-time Markov chain having finite state space $\mathcal{M} = \{1, \ldots, m\}$, and generator $Q = (q_{ij}) \in \mathbb{R}^{m \times m}$. Consider a function $y(t)$ of the Markov chain that is observable with additive Gaussian noise. Let $y(t)$ be the observation measurement given by

$$dy(t) = f(\alpha(t))dt + \sigma dW(t), \quad y(0) = 0, \tag{3.1}$$

where σ is a positive constant and $W(t)$ is a standard Brownian motion. Let $p_i(t)$ denote the conditional probability of $\alpha(t) = i$ given the observations up to time t, i.e.,

$$p_i(t) = P(\alpha(t) = i \mid y(s) : s \leq t);$$

for $i = 1, \ldots, m$. Let $p(t) = (p_1(t), \ldots, p_m(t)) \in \mathbb{R}^{1 \times m}$. The *Wonham filter* is given by

$$dp(t) = p(t)Qdt - \frac{1}{\sigma^2}\left(\sum_{i=1}^{m} f(i)p_i(t)\right)p(t)A(t)dt + \frac{1}{\sigma^2}p(t)A(t)dy(t), \tag{3.2}$$

$p(0) = p_0$, being the initial probability, where

$$A(t) = \operatorname{diag}(f(1), \ldots, f(m)) - \sum_{i=1}^{m} f(i)p_i(t)I,$$

and I is the $m \times m$ identity matrix.

Define $y(t) = \log(P_2(t))$. Then it is easy to see that

$$dy(t) = \left[\mu(\alpha(t)) - \frac{\sigma^2}{2}\right]dt + \sigma dW(t).$$

The corresponding Wonham filter for $\alpha(t)$ is given by the following SDE:

$$dp(t) = p(t)Qdt - \frac{1}{\sigma^2}\left(\sum_{i=1}^{m}\left[\mu(i) - \frac{\sigma^2}{2}\right]p_i(t)\right)p(t)A(t)dt + \frac{1}{\sigma^2}p(t)A(t)dy(t), \tag{3.3}$$

where $p(0) = p_0$ is the initial probability, and

$$\begin{aligned} A(t) &= \operatorname{diag}\left(\mu(1) - \frac{\sigma^2}{2}, \ldots, \mu(m) - \frac{\sigma^2}{2}\right) - \sum_{i=1}^{m}\left[\mu(i) - \frac{\sigma^2}{2}\right]p_i(t)I. \\ &= \operatorname{diag}(\mu(1), \ldots, \mu(m)) - \sum_{i=1}^{m}\mu(i)p_i(t)I. \end{aligned}$$

Denote $\tilde{\alpha}(t) = \sum_{i=1}^{m}\left[\mu(i) - \frac{\sigma^2}{2}\right]p_i(t)$. We have

$$\begin{aligned} dp(t) &= p(t)Qdt - \frac{1}{\sigma^2}\tilde{\alpha}(t)p(t)A(t)dt + \frac{1}{\sigma^2}p(t)A(t)dy(t) \\ &= p(t)Qdt + \frac{p(t)A(t)}{\sigma}d\hat{v}(t), \end{aligned} \tag{3.4}$$

where

$$d\hat{v} = \frac{d\log(P_2) - \tilde{\alpha}dt}{\sigma}, \quad \hat{v}(0) = 0$$

is the innovation process.

We may rewrite the stock price equation of P_2 in terms of the innovation process as follows:

$$\frac{dP_2}{P_2} = \left(\widetilde{\alpha}(t) + \frac{\sigma^2}{2}\right) dt + \sigma d\hat{v}.$$

Note that both $\tilde{\alpha}(t)$ and $d\hat{v}$ are observable.

Because

$$\widetilde{\alpha}(t) = \sum_{i=1}^{m}\left[\mu(i) - \frac{\sigma^2}{2}\right] p_i(t) = \sum_{i=1}^{m} \mu(i)p_i(t) - \frac{\sigma^2}{2},$$

we can simplify our notation and obtain

$$\frac{dP_2(t)}{P_2(t)} = \widehat{\alpha}(t)dt + \sigma d\hat{v}(t),$$

by letting

$$\widehat{\alpha}(t) = \sum_{i=1}^{m} \mu(i)p_i(t).$$

Now the dynamic of the wealth function $\xi(t)$ can be reformulated by

$$\begin{aligned} \frac{d\xi(t)}{\xi(t)} &= [1 - u(t)]rdt + u(t)(\widehat{\alpha}dt + \sigma d\hat{v}(t)), \\ &= [(1 - u(t))r + u(t)\widehat{\alpha}(t)]dt + u(t)\sigma d\hat{v}(t). \end{aligned} \tag{3.5}$$

The objective is to find an optimal control u to maximizes

$$J = E_{sy}(\Phi(\xi(T))),$$

where T is a finite time and Φ is a utility function.

Denote $Z(t) = \log \xi(t)$ and $z = \log y$. Then we have

$$\begin{aligned} dZ(t) &= [(1 - u(t))r + u(t)\hat{\alpha}(t) - (1/2)(u(t)\sigma)^2]dt + u(t)\sigma d\hat{v}(t), \\ Z(s) &= z, \\ dp(t) &= p(t)Qdt + \frac{p(t)A(t)}{\sigma}d\hat{v}(t), \\ p(s) &= p. \end{aligned}$$

Then, we have

$$J(s, z, p, u(\cdot)) = E_{sz}(\Phi(\exp(Z(T)))).$$

The value function is given by

$$v(s, z, p) = \sup_{u(\cdot)\in\mathcal{A}} J(s, z, p, u(\cdot)).$$

Let $Y(t) = (Z(t) \vdots p(t))'$, where A' denote the transpose of the matrix (or vector) A. Then

$$dY(t) = \begin{pmatrix} (1-u(t))r + u(t)\hat{\alpha}(t) - (1/2)(u(t)\sigma)^2 \\ Q'p(t)' \end{pmatrix} dt + \begin{pmatrix} u(t)\sigma \\ \dfrac{A(t)p(t)'}{\sigma} \end{pmatrix} d\hat{v}(t).$$

Let

$$f(t, Y, u) = \begin{pmatrix} (1-u)r + u\hat{\alpha} - (1/2)(u\sigma)^2 \\ Q'p(t)' \end{pmatrix}$$

and

$$\Sigma(t, Y, u) = \begin{pmatrix} u(t)\sigma \\ \dfrac{A(t)p(t)'}{\sigma} \end{pmatrix}.$$

Then

$$dY(t) = f(t, Y(t), u(t))dt + \Sigma(t, Y(t), u(t))d\hat{v}(t).$$

Define

$$H(Y, P, G) = fP + \frac{1}{2}\mathrm{tr}\{(\Sigma\Sigma')G\} \tag{3.6}$$

where P is an $1 \times (m+1)$ vector and G is an $(m+1) \times (m+1)$ matrix. Here, fP should be understood as the inner product of two vectors.

Formally, the associated HJB equation is given as follows:

$$\frac{\partial v}{\partial s} + \sup_u H\left(Y, \frac{\partial v}{\partial Y}, \frac{\partial^2 v}{\partial Y^2}\right) = 0 \tag{3.7}$$

with the boundary condition $v(T, z, p) = \Phi(\mathrm{e}^z)$, where $z = \log y$, y is the initial wealth, and p is the initial probability vector.

4. PROPERTIES OF THE VALUE FUNCTIONS

An analytical solution to equation (3.7) is difficult to obtain (if not impossible). It is not even clear if equation (3.7) has a classical solution. In this paper, We use viscosity solution to characterize the dynamics of the system.

The theory of viscosity solutions applies to partial differential equations of the form $F(x, u, Du, D^2u) = 0$ where $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S(N) \to \mathbb{R}$ and $S(N)$ is the set of symmetric $N \times N$ matrices. The notion of viscosity solutions was first introduced by Crandall and Lion for solving first-order Hamilton-Jacobi equations. The user's guide by Crandall, Ishii and Lion [7] offers a complete treatment of this topic. Readers are referred to [8] for applications to deterministic and stochastic control theory. Viscosity solution is also useful for characterizing numerical solutions of partial differential equations of the form $F(x, u, Du, D^2u) = 0$ where Du is the gradient vector of u, D^2u is its Hessian matrix. The condition on F is that it has to be *proper* defined as follows.

Definition 4.1. Function F is *proper* if it satisfies

$$F(x,r,p,X) \leq F(x,s,p,Y) \text{ whenever } r \leq s \text{ and } Y \leq X.$$

Definition 4.2. Let Ω be an open subset of $\mathbb{R}^N$, F be proper and $u : \Omega \to \mathbb{R}$.
(a) u is a viscosity subsolution of $F(x,u,Du,D^2u) = 0$ in Ω if it is upper semicontinuous and for each $\phi \in C^2(\Omega)$ and local maximum point x_0 of $u - \phi$ we have

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0.$$

(b) u is a viscosity supersolution of $F(x,u,Du,D^2u) = 0$ in Ω if it is lower semicontinuous and for each $\phi \in C^2(\Omega)$ and local minimum point x_0 of $u - \phi$ we have

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0.$$

(c) u is a viscosity solution of $F(x,u,Du,D^2u) = 0$ in Ω if it is both viscosity subsolution and supersolution of $F(x,u,Du,D^2u) = 0$.

Let

$$F(Y,v,Dv,D^2v) = \sup_u H\left(Y, \frac{\partial v}{\partial Y}, \frac{\partial^2 v}{\partial Y^2}\right).$$

We consider

$$v_t + F(Y,v,Dv,D^2v) = 0,$$

which is a classical parabolic equation. Here $Dv = \frac{\partial v}{\partial Y}$ and $D^2 v = \frac{\partial^2 v}{\partial Y^2}$.

We need the following condition for the utility function Φ.

$$|\Phi(y)| \leq K(1 + |\log y|^{k_1} + y^{k_2} + y^{-k_3}),$$

for $y \in (0,\infty)$, for some nonnegative constants K, k_i, $i = 1,2,3$.

Moreover, for any $y_1, y_2 \in (0,\infty)$ and some $K_1 > 0$, either

$$|\Phi(y_1) - \Phi(y_2)| \leq K_1 |\log y_1 - \log y_2|$$

or, for some $\gamma < 1$ and $\gamma \neq 0$,

$$|\Phi(y_1) - \Phi(y_2)| \leq K_1 |y_1^\gamma - y_2^\gamma|.$$

These conditions are needed so that the value functions have certain growth rate; see [19] for related estimates.

Lemma 4.3. *$v(s,z,p)$ is continuous with respect to s, z, and p.*

Proof. (1) $v(s,\cdot,p)$ is continuous with respect to z.

Fix (s,p). For given z_1, z_2 and u define

$$R(t) = \int_s^t \left[(1-u(x))r + u(x)\hat{\alpha}(x) - \frac{1}{2}(u(x)\sigma)^2\right] dx + \int_s^t u(x)\sigma d\hat{v}(x).$$

Let $Z_i(t)$, $i = 1, 2$, be defined as $Z_i(t) = z_i + R(t)$ then $Z_1(t) - Z_2(t) = z_1 - z_2$ and

$$\begin{aligned}|v(s, z_1, p) - v(s, z_2, p)| &\leq \sup_u |E\Phi(e^{z_1} e^{R(T)}) - E\Phi(e^{z_2} e^{R(T)})| \\ &\leq \sup_u |E\{[\Phi(e^{z_1}) - \Phi(e^{z_2})] e^{R(T)}\}| \\ &\leq K_1 |z_1 - z_2| E(e^{R(T)}).\end{aligned}$$

Or

$$\begin{aligned}|v(s, z_1, p) - v(s, z_2, p)| &\leq \sup_u |E\Phi(e^{z_1} e^{R(T)}) - E\Phi(e^{z_2} e^{R(T)})| \\ &\leq \sup_u |E\{[\Phi(e^{z_1}) - \Phi(e^{z_2})] e^{R(T)}\}| \\ &\leq EK_1 |(e^{z_1 R(T)})^\gamma - (e^{z_2 R(T)})^\gamma| \\ &= K_1 |e^{\gamma z_1} - e^{\gamma z_2}| E(e^{\gamma R(T)}).\end{aligned}$$

It suffices to show the boundedness of $E(e^{\gamma R(T)})$.

By Ito's differential rule, we have

$$de^{\gamma R(t)} = e^{\gamma R(T)} \left[\gamma dR(t) + \frac{\gamma^2}{2} (u(t)\sigma)^2 dt \right].$$

Taking expectation on both sides yields

$$Ee^{\gamma R(t)} \leq 1 + C \int_s^t Ee^{\gamma R(x)} dx,$$

for some constant $C > 0$. Then the Gronwall inequality implies

$$Ee^{\gamma R(t)} \leq e^{C(t-s)}, \quad s \leq t \leq T.$$

(2) $v(\cdot, z, p)$ is continuous with respect to s.

Fix (z, p). For a given $s, \Delta s > 0$, and u define

$$\begin{aligned}\hat{Z}(t) &= Z(t - \Delta s), \\ \tilde{u}(t) &= u(t - \Delta s), \\ \tilde{p}(t) &= p(t - \Delta s), \\ \tilde{\alpha}(t) &= \sum_{i=1}^m \mu(i) p(t - \Delta s).\end{aligned}$$

Write $Z(\cdot)$ in terms of $\tilde{u}(t)$:

$$Z(t) = z + \int_{s+\Delta s}^{t+\Delta s} (1 - \tilde{u}(x))r + \tilde{u}(x)\tilde{\alpha}(x) - \frac{1}{2}(\tilde{u}(x)\sigma)^2 dx + \int_{s+\Delta s}^{t+\Delta s} \tilde{u}(x)\sigma d\hat{v}(x).$$

Let

$$\hat{Z}(t) = z + \int_{s+\Delta s}^{t} (1 - \tilde{u}(x))r + \tilde{u}(x)\tilde{\alpha}(x) - \frac{1}{2}(\tilde{u}(x)\sigma)^2 dx + \int_{s+\Delta s}^{t} \tilde{u}(x)\sigma d\hat{v}(x).$$

Then

$$J(s + \Delta s, z, \tilde{p}, \tilde{u}) = E\Phi(e^{\hat{Z}(t)}).$$

Moreover,

$$Z(T) - \hat{Z}(t) = z + \int_t^{t+\Delta s} (1 - \tilde{u}(x))r + \tilde{u}(x)\tilde{\alpha}(x) - \frac{1}{2}(\tilde{u}(x)\sigma)^2 dx + \int_t^{t+\Delta s} \tilde{u}(x)\sigma d\hat{v}(x),$$

$$\begin{aligned}|J(s,z,p,u) - J(s+\Delta s, z, \tilde{p}, \tilde{u})| &= |E\Phi(e^{Z(T)}) - E\Phi(e^{\hat{Z}(t)})| \\ \text{either } &\leq K_1 E|Z(T) - \hat{Z}(T) \leq K_1\sqrt{\Delta s} \\ \text{or} &\leq K_1|E(e^{\gamma Z(T)} - e^{\gamma \hat{Z}(T)})|.\end{aligned}$$

Note that $E(e^{\gamma Z(T)}) \leq K$.

By the Cauchy-Schwarz inequality, we have

$$|E(e^{\gamma Z(T)} - e^{\gamma \hat{Z}(T)})|^2 \leq E(e^{\gamma(Z(T) - \hat{Z}(T))} - 1)^2 \leq K\Delta s.$$

The last inequality follows by Ito's rule.

(3) $v(s,z,\cdot)$ is continuous with respect to p.

Fix (s,z). For given p_1, p_2 and u, define

$$R_i(t) = \int_s^t \left[(1-u(x))r + u(x)\hat{\alpha}_i(x) - \frac{1}{2}u^2(x)\sigma^2\right] dx + \int_s^t u(x)\sigma d\hat{v}(x),$$

where $\hat{\alpha}_i(x) = \sum_{k=1}^m \mu(k)p_k^i(x)$, $i = 1, 2$.

Since $Z_i(t) = z + R_i(t), i = 1, 2$, then $Z_1(t) - Z_2(t) = R_1(t) - R_2(t)$. Either

$$\begin{aligned}|J(s,z,p_1,u) - J(s,z,p_2,u)| &= |E\Phi(e^{Z_1(T)}) - E\Phi(e^{Z_2(T)})| \\ &= |E\Phi(e^{z+R_1(T)}) - E\Phi(e^{z+R_2(T)})| \\ &\leq K_1 E|Z_1(T) - Z_2(T)| = K_1 E|R_1(T) - R_2(T)|,\end{aligned}$$

or

$$\begin{aligned}|J(s,z,p_1,u) - J(s,z,p_2,u)| &= |E\Phi(e^{Z_1(T)}) - E\Phi(e^{Z_2(T)})| \\ &= |E\Phi(e^{z+R_1(T)}) - E\Phi(e^{z+R_2(T)})| \\ &\leq K_1 E|(e^{z+R_1(T)})^\gamma - (e^{z+R_2(T)})^\gamma| \\ &= K_1 e^{\gamma z} E|e^{\gamma R_1(T)} - e^{\gamma R_2(T)}|.\end{aligned}$$

$$\begin{aligned}E|R_1(T) - R_2(T)| &= E\int_s^T u(x)\left[\hat{\alpha}_1(x) - \hat{\alpha}_2(x)\right] dx \\ &= E\int_s^T u(x)\sum_{k=1}^m \mu_k \left[p_k^1(x) - p_k^2(x)\right] dx \\ &\leq \|p_1 - p_2\|.\end{aligned}$$

□

Theorem 4.4. *The value function $v(s, z, p)$ is the unique viscosity solution of HJB equation* (3.7)*.*

Proof. The proof can be found in [18]. □

5. MARKOV CHAIN APPROXIMATION

In the previous section, we have shown that the value function v is the unique viscosity solution of the HJB equations

$$\frac{\partial v}{\partial s} + H\left(Y, \frac{\partial v}{\partial Y}, \frac{\partial^2 v}{\partial Y^2}\right) = 0 \tag{5.1}$$

with the boundary condition $v(T, z, p) = \Phi(\mathrm{e}^z)$.

A common choice to compute the value function is to use finite difference approximation. But finite difference method is not applicable in this problem. The obstacle is due to the fact that the matrix $a(u, p) = \Sigma\Sigma'$ is not diagonally dominant. To apply finite difference method, one needs

$$a_{ii}(u, p) - \sum_{j:j\neq i} |a_{ij}(u, p)| \geq 0.$$

We can see that this is not the case in this paper. For example, if $p(t)$ is 2-dimensional,

$$A(t)P(t)' = \begin{pmatrix} \mu_1 - \hat{\alpha} & 0 \\ 0 & \mu_2 - \hat{\alpha} \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} = \begin{pmatrix} (\mu_1 - \hat{\alpha})p_1 \\ (\mu_2 - \hat{\alpha})p_2 \end{pmatrix},$$

it follows that

$$\Sigma(t, Y, u) = \begin{pmatrix} u\sigma \\ \dfrac{A(t)p(t)'}{\sigma} \end{pmatrix} = \begin{pmatrix} u\sigma \\ \frac{1}{\sigma}(\mu_1 - \hat{\alpha})p_1 \\ \frac{1}{\sigma}(\mu_2 - \hat{\alpha})p_2 \end{pmatrix}.$$

Thus

$$\Sigma\Sigma' = \begin{pmatrix} u\sigma \\ \frac{1}{\sigma}(\mu_1 - \hat{\alpha})p_1 \\ \frac{1}{\sigma}(\mu_2 - \hat{\alpha})p_2 \end{pmatrix} \left(u\sigma \; \frac{1}{\sigma}(\mu_1 - \hat{\alpha})p_1 \; \frac{1}{\sigma}(\mu_2 - \hat{\alpha})p_2 \right),$$

$$\Sigma\Sigma' = \begin{pmatrix} u^2\sigma^2 & u(\mu_1 - \hat{\alpha})p_1 & u(\mu_2 - \hat{\alpha})p_2 \\ u(\mu_1 - \hat{\alpha})p_1 & \frac{1}{\sigma^2}(\mu_1 - \hat{\alpha})^2 p_1^2 & \frac{1}{\sigma^2}(\mu_1 - \hat{\alpha})p_1(\mu_2 - \hat{\alpha})p_2 \\ u(\mu_2 - \hat{\alpha})p_2 & \frac{1}{\sigma^2}(\mu_1 - \hat{\alpha})p_1(\mu_2 - \hat{\alpha})p_2 & \frac{1}{\sigma^2}(\mu_2 - \hat{\alpha})^2 p_2^2 \end{pmatrix}.$$

In order to have

$$a_{ii}(u, p) - \sum_{j:j\neq i} |a_{ij}(u, p)| \geq 0,$$

we must have

$$
\begin{aligned}
u\sigma^2 &\geq |\mu_1 - \hat{\alpha}|p_1 + |\mu_2 - \hat{\alpha}|p_2 \\
\frac{1}{\sigma^2}(\mu_1 - \hat{\alpha})^2 p_1^2 &\geq u|\mu_1 - \hat{\alpha}|p_1 + \frac{1}{\sigma^2}(\mu_1 - \hat{\alpha})p_1(\mu_2 - \hat{\alpha})p_2 \\
\frac{1}{\sigma^2}(\mu_2 - \hat{\alpha})^2 p_2^2 &\geq u|\mu_2 - \hat{\alpha}|p_2 + \frac{1}{\sigma^2}(\mu_1 - \hat{\alpha})p_1(\mu_2 - \hat{\alpha})p_2.
\end{aligned}
$$

Suppose $\mu_1 > \mu_2$, hence $\mu_1 - \hat{\alpha} > 0, \mu_2 - \hat{\alpha} < 0$. We then hope to have

$$
\begin{aligned}
u\sigma^2 &\geq (\mu_1 - \hat{\alpha})p_1 + (\hat{\alpha} - \mu_2)p_2, \\
(\mu_1 - \hat{\alpha})p_1 &\geq u\sigma^2 + (\mu_2 - \hat{\alpha})p_2, \\
(\mu_2 - \hat{\alpha})p_2 &\leq u\sigma^2 + (\mu_1 - \hat{\alpha})p_1.
\end{aligned}
$$

The third condition holds. To satisfy the first two condition, we must have

$$u\sigma^2 = (\mu_1 - \hat{\alpha})p_1 + (\hat{\alpha} - \mu_2)p_2.$$

However, u is the control function that is allowed to take value 0.

To find numerical solutions, we apply Kushner's Markov chain approximation method see [12]. The main idea is: Based on probabilistic methods, we construct a Markov chain with specified transition probabilities leading to the approximation to the cost function, and the value functions etc. We refer the reader to [16] for Markov chain approximation to regime-switching diffusions using relaxed control setup and weak convergence approach.

One of the key requirements in finding the proper Markov chain approximation is to verify the "local consistency conditions," which basically means that the approximating chain should have local properties that are consistent with that of the original chain. Recall that we denote $Y(t) = (Z(t) \vdots p(t))'$, and $Y(t)$ evolves according to the stochastic process

$$dY(t) = f(t, Y(t), u(t))dt + \Sigma(t, Y(t), u(t))d\hat{v}(t).$$

So the approximating chain $Y^h(t)$ should satisfy the following local consistency conditions:

$$
\begin{aligned}
E^{h,u}_{z,p,n}\Delta Y^h_n &= f(t, Y(t), u(t))\Delta t^h(Y, u) + o(\Delta t^h(Y, u)) \\
\mathrm{cov}^{h,u}_{z,p,n}\Delta Y^h_n &= \Sigma(t, Y(t), u(t))\Sigma(t, Y(t), u(t))'\Delta t^h(Y, u) + o(\Delta t^h(Y, u))
\end{aligned}
$$

If we can find approximating chain $Y^h(t)$ whose transition probability $P^h(Y, Z \mid u)$ and time step functions $\Delta t^h(Y, u)$ satisfy the "local consistency conditions" then we can use it to compute the value function $v^h(s, Y^h)$ for the approximating chain. For a detailed discussion of this method, see [12].

The value function is

$$v(s, Y_0) = \sup_u E[\Phi(\exp(Z(T))) \mid Y(s) = Y_0],$$

where $Y_0 = (z, p)$ is the initial condition. By the principle of dynamic programming,

$$v(s - \Delta t^h, Y_0) = \sup_u E[v(s, Y(s)) \mid Y(s - \Delta t^h) = Y_0].$$

The approximation function v^h should have the same property

$$v^h(s - \Delta t^h, Y_0) = \sup_u E[v^h(s, Y(s)) \mid Y(s - \Delta t^h) = Y_0].$$

The degenerate structure of the noise covariance matrix suggests that the part of the transitions of any approximating Markov chain which approximates the effects of the "noise" would move the chain in the directions $\pm\Sigma(s, Y, u)$. Let the state space S_h be such that

$$Y \pm h\Sigma(s, Y, u) \in S_h, \quad \text{for } Y \in S_h,$$

and

$$Y \pm e_i h \in S_h, \quad \text{for } Y \in S_h,$$

We use the following steps to choose a set of transition probabilities $P^h(Y, Z \mid u)$ and time step functions $\Delta t^h(Y, u)$ to satisfy the "local consistency conditions." First we consider the stochastic process

$$dY(t) = f(t, Y(t), u(t))dt + \Sigma(t, Y(t), u(t))d\hat{v}(t)$$

as having two different components, represented respectively by

$$dY(t) = \Sigma(t, Y(t), u(t))d\hat{v}(t)$$

and

$$dY(t) = f(t, Y(t), u(t))dt.$$

We choose two different sets of transition probability and time step functions, so these two SDE's individual "local consistency conditions" can be satisfied. Then we combine them to obtain a choice that can satisfy the "local consistency conditions" for the original state $Y(t)$. The idea of this construction can be found in [13, p. 118].

(1) One set of transition probabilities for a locally consistent chain for the component represented by

$$dY(t) = \Sigma(t, Y(t), u(t))d\hat{v}(t)$$

is $P_1^h(Y, Y \pm h\Sigma(s, Y, u) \mid u) = 1/2$. With these transition probabilities, the covariance of the state transition can be written as

$$\sum_Z (Z - Y)(Z - Y)' P_1^h(Y, Z \mid u) = \Sigma\Sigma' h^2.$$

Then, if we define the interpolation interval $\Delta t_1^h(Y, u) = h^2$, $P_1^h(Y, Y \pm h\Sigma(s, Y, u) \mid u)$ is locally consistent.

(2) One possibility for the transition probability of the approximation to

$$dY(t) = f(t, Y(t), u(t))dt$$

is

$$P_2^h(Y, Y \pm e_i h \mid u) = f_i^{\pm}(t, Y, u) \times \text{ normalization},$$

where the normalization is

$$\frac{1}{Q_2^h(Y,u)} = \frac{1}{\sum_{i=1}^{m+1} f_i(t, Y, u)},$$

$f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$. Define

$$\Delta t_2^h(Y,u) = \frac{h}{\sum_{i=1}^{m+1} f_i(t, Y, u)}.$$

The local consistency can be shown by the calculations

$$\sum_Z (Z - Y) P_2^h(Y, Z \mid u) = f(t, Y, u) \times \Delta t_2^h(Y, u),$$

where $Z \in \{Y \pm e_i h, i = 1, \ldots, m\}$, and

$$\sum_Z (Z - Y)(Z - Y)' P_2^h(Y, Z \mid u) = o(\Delta t_2^h(Y, u)).$$

(3) Combine the above "partial" transition probabilities from the diffusion and drift component to get

$$P^h(Y, Y \pm h\Sigma(s, Y, u) \mid u) = \frac{1}{2Q^h(Y,u)},$$

$$P^h(Y, Y \pm e_i h \mid u) = f_i^{\pm}(t, Y, u) \frac{h}{Q^h(Y,u)}$$

where

$$Q^h(Y,u) = 1 + h \sum_{i=1}^{m+1} |f_i(Y,u)|,$$

and

$$\Delta t^h(Y,u) = \frac{h^2}{Q^h(Y,u)}.$$

To show that the local consistency is satisfied, we see that

$$\sum_{Z \in S_h} (Z - Y) P^h(Y, Z \mid u) = f(t, Y, u) \frac{h^2}{Q^h(Y,u)},$$

where $Z \in \{Y \pm e_i h, i = 1, Y \pm h\Sigma(s, Y, u), \ldots, m\}$, and

$$\sum_{Z \in S_h} (Z - Y)(Z - Y)' P^h(Y, Z \mid u) = \Sigma\Sigma' \frac{h^2}{Q^h(Y,u)} + o(\Delta t_2^h(Y, u)).$$

The numerical scheme for the value function is

$$v^h(s - \Delta t^h(Y,u), Y) = \sup_u \left[\sum_{Z \in S_h} P^h(Y, Z \mid u) v^h(s, Z) \right] \tag{5.2}$$

with

$$v^h(T,z,p) = \Phi(e^z), (z \vdots p) \in S_h. \tag{5.3}$$

For calculation purposes, it would be better if we can find a constant interpolation intervals Δt^h. This can be done by defining

$$\overline{Q}^h = \sup_{u,p} Q^h(Y,u).$$

Then the following are locally consistent:

$$\begin{aligned}
\Delta t^h &= h^2/\overline{Q}^h, \\
P^h(Y, Y \pm h\Sigma(s,Y,u) \mid u) &= 1/2\overline{Q}^h, \\
P^h(Y, Y \pm e_i h \mid u) &= f_i^{\pm}(t,Y,u)h/\overline{Q}^h, \\
P^h(Y, Y \mid u) &= (\overline{Q}^h - Q^h(Y,u))/\overline{Q}^h.
\end{aligned}$$

Let

$$\mathcal{F}_h(\phi)(Y) = \sup_u \left[\sum_{Z \in S_h} P^h(Y, Z \mid u)\phi(Z) \right].$$

Then the scheme for computing the value function approximation can be rewritten as

$$\begin{aligned}
v^h(s,Y) &= \mathcal{F}_h(v^h(s + \Delta t^h(Y,u), \cdot))(Y), Y \in S_h, \\
v^h(T,z,p) &= \Phi(e^z), (z \vdots p) \in S_h.
\end{aligned}$$

In order to use the Barles-Souganidis method [2] to prove the desired convergence, we need to check the following condition:

$$\mathcal{F}_h(\phi_1) \le \mathcal{F}_h(\phi_2) \text{ if } \phi_1 \le \phi_2 \text{ (monotonicity)}.$$

For $0 < h < 1$, there exists a solution v^h to the computation scheme and a constant K such that $\|v^h\| \le K$(stability).

For every "test function" $w \in C^{1,2}(\mathbf{R}^{m+1})$,

$$\lim_{(t,q) \underset{h \downarrow 0}{\rightarrow} (s,p)} h^{-1}[\mathcal{F}_h(w(t+h,\cdot))(q) - w(t,q)] = \frac{\partial w}{\partial s} + H\left(Y, \frac{\partial w}{\partial Y}, \frac{\partial^2 w}{\partial Y^2}\right) \text{ (consistency)}.$$

We have the consistency because

$$\begin{aligned}
&\lim_{(t,q) \underset{h \downarrow 0}{\rightarrow} (s,p)} h^{-1}[\mathcal{F}_h(w(t+h,\cdot))(q) - w(t,q)] \\
&= \lim_{(t,q) \underset{h \downarrow 0}{\rightarrow} (s,p)} \frac{\sup_u [\sum_Z P^h(q,Z \mid u) w(t+h,Z)] - w(t,q)}{h} \\
&= \lim_{(t,q) \underset{h \downarrow 0}{\rightarrow} (s,p)} \frac{\sup_u \left[\sum_Z P^h(q,Z \mid u)[w(t+h,Z) - w(t+h,q)]\right] + w(t+h,q) - w(t,q)}{h} \\
&= \frac{\partial w}{\partial s} + H(Y, \frac{\partial w}{\partial Y}, \frac{\partial^2 w}{\partial Y^2}).
\end{aligned}$$

Since $P^h(Y,Z\mid u)\geq 0$, the monotonicity is immediate.

$$\begin{aligned}\|\mathcal{F}_h(\phi_1)(Y)-\mathcal{F}_h(\phi_2)(Y)\| &= \left\|\sup_u\left[\sum_{p\in S_h}P^h(Y,Z\mid u)[\phi_1(Z)-\phi_2(Z)]\right]\right\|\\ &\leq \sup_u\left[\sum_{p\in\Sigma_0^h}P^w(p,q)\right]\|\phi_1-\phi_2\|\\ &= \sup_u\|\phi_1-\phi_2\|.\end{aligned}$$

Therefore $\mathcal{F}_h$ is a contraction mapping. The fixed point v^h of this contraction mapping is the solution of (5.2). This proves the stability.

Define

$$v^*(s,Y)=\limsup_{(t,Z)\underset{h\downarrow 0}{\to}(s,Y)} v^h(t,Z)$$

$$v_*(s,Y)=\liminf_{(t,Z)\underset{h\downarrow 0}{\to}(s,Y)} v^h(t,Z).$$

Lemma 5.1. *v^* is a viscosity subsolution of equation* (5.1)*, and v_* is a viscosity supersolution.*

Proof. In order to prove that v^* is a viscosity subsolution, we suppose that ϕ is a test function such that $v^*-\phi$ has a strict local maximum at (s,Y). Then there is a sequence converging to zero denoted by h, such that $v^h-\phi$ has a local maximum at (t_h,Y_h) which converges to (s,Y) as $h\downarrow 0$.

$$v^h(t_h,Y_h)-\phi(t_h,Y_h)\geq v^h(t_h+h,Y_h)-\phi(t_h+h,Y_h),$$

$$\phi(t_h+h,Y_h)-\phi(t_h,Y_h)\geq v^h(t_h+h,Y_h)-v^h(t_h,Y_h).$$

By the monotonicity we proved above,

$$\mathcal{F}_h(\phi(t_h+h,\cdot))(Y_h)-\phi(t_h,Y_h)\geq\mathcal{F}_h(v^h(t_h+h,\cdot))(Y_h)-v^h(t_h,Y_h).$$

Since v^h is the solution of (5.2), the right side is 0. We divide by h and let $h\downarrow 0$. By the consistency, we have

$$\frac{\partial\phi}{\partial s}+H\left(Y,\frac{\partial\phi}{\partial Y},\frac{\partial^2\phi}{\partial Y^2}\right)\geq 0$$

Therefore, v^* is a viscosity subsolution.

Similarly, suppose that $\phi\in C^{1,2}$ is a test function such that $v_*-\phi$ has a strict local minimum at (s,Y). Then there is a sequence converging to zero denoted by h, such that $v^h-\phi$ has a local minimum at (t_h,Y_h) which converges to (s,Y) as $h\downarrow 0$.

$$v^h(t_h,Y_h)-\phi(t_h,Y_h)\leq v^h(t_h+h,Y_h)-\phi(t_h+h,Y_h),$$

$$\phi(t_h+h,Y_h)-\phi(t_h,Y_h)\leq v^h(t_h+h,Y_h)-v^h(t_h,Y_h).$$

By the monotonicity we proved above,

$$\mathcal{F}_h(\phi(t_h+h,\cdot))(Y_h)-\phi(t_h,Y_h)\le \mathcal{F}_h(v^h(t_h+h,\cdot))(Y_h)-v^h(t_h,Y_h).$$

Since v^h is the solution of (5.2), the right side is 0. We divide both sides by h and let $h\downarrow 0$. By the consistency, we have

$$\frac{\partial \phi}{\partial s}+H\left(Y,\frac{\partial \phi}{\partial Y},\frac{\partial^2\phi}{\partial Y^2}\right)\le 0$$

Therefore, v_* is a viscosity supersolution. □

Theorem 5.2. *As $h\to 0$ the solution v^h of* (5.2) *converges uniformly on any compact subset of $[0,T]\times R^1\times[0,1]^m$ to the unique continuous viscosity v of* (5.1).

Proof. By Lemma 5.1, v^* is a viscosity subsolution of equation (5.1). By comparison result for viscosity solutions, $v^*\le v$. Similarly, $v_*\ge v$. Since $v_*\le v^*$, we have proved

$$\lim_{(t,Z)\underset{h\downarrow 0}{\rightarrow}(s,Y)} v^h(t,Z)=v(s,Y).$$

□

6. SEPARABLE CASE

In this section, we consider a special case in which the value function can be written as the product of a function of (s,p) and that of z. Such a separation allows us to reduce the complexity of the overall problem and leads to simpler HJB equations for numerical solutions. We have proved that v is the unique viscosity solution of the PDE

$$\frac{\partial v}{\partial s}+H(Y,\frac{\partial v}{\partial Y},\frac{\partial^2 v}{\partial Y^2})=0 \tag{6.1}$$

with the boundary condition $v(T,z,p)=\Phi(\mathrm{e}^z)$, where $z=\ln y$ and y is the initial wealth, and p is the initial probability vector. When the utility function is of the form $\Phi(x)=x^k$, we can simplify the numerical solution even further by separation variables.

Recall that

$$H\left(Y,\frac{\partial v}{\partial Y},\frac{\partial^2 v}{\partial Y^2}\right)=\sup_u\left\{f\frac{\partial v}{\partial Y}+\frac{1}{2}\mathrm{tr}\left\{(\Sigma\Sigma')\frac{\partial^2 v}{\partial Y^2}\right\}\right\}.$$

Let

$$f(t,Y,u)=\begin{pmatrix}(1-u)r+u\hat{\alpha}-(1/2)(u\sigma)^2\\ Q'p(t)'\end{pmatrix}.$$

If we denote $f(t,Y,u)=(f_u,f_p)'$, then

$$f\frac{\partial v}{\partial Y}=f_u\frac{\partial v}{\partial z}+f_p\frac{\partial v}{\partial p},$$

where $f_p\frac{\partial v}{\partial p}$ is the inner product of the two vectors.

Recall that

$$\Sigma(t,Y,u)=\begin{pmatrix} u\sigma \\ \dfrac{A(t)p(t)'}{\sigma}\end{pmatrix}.$$

We denote $\Sigma(t,Y,u)=(c_u,c_p)'$. Then

$$(\Sigma\Sigma')\frac{\partial^2 v}{\partial Y^2}=\begin{pmatrix} c_u^2 & c_u c_p' \\ c_p c_u & c_p c_p'\end{pmatrix}\begin{pmatrix} \dfrac{\partial^2 v}{\partial z^2} & \dfrac{\partial^2 v}{\partial z\partial p} \\ \dfrac{\partial^2 v}{\partial p\partial z} & \dfrac{\partial^2 v}{\partial p^2}\end{pmatrix}.$$

So,

$$\frac{1}{2}\mathrm{tr}\left\{(\Sigma\Sigma')\frac{\partial^2 v}{\partial Y^2}\right\}=\frac{1}{2}\left(c_u^2\frac{\partial^2 v}{\partial z^2}+c_u c_p'\frac{\partial^2 v}{\partial p\partial z}+c_u' c_p\frac{\partial^2 v}{\partial z\partial p}\right)+\frac{1}{2}\mathrm{tr}\left(c_p c_p'\frac{\partial^2 v}{\partial p^2}\right).$$

The PDE becomes

$$\begin{aligned}0&=\frac{\partial v}{\partial s}+H\left(Y,\frac{\partial v}{\partial Y},\frac{\partial^2 v}{\partial Y^2}\right)\\ &=\frac{\partial v}{\partial s}+\sup_u\left\{f_u\frac{\partial v}{\partial z}+f_p\frac{\partial v}{\partial p}+\frac{1}{2}\left(c_u^2\frac{\partial^2 v}{\partial z^2}+c_u c_p'\frac{\partial^2 v}{\partial p\partial z}+c_u' c_p\frac{\partial^2 v}{\partial z\partial p}\right)\right.\\ &\qquad\left.+\frac{1}{2}\mathrm{tr}\left(c_p c_p'\frac{\partial^2 v}{\partial p^2}\right)\right\}\\ &=\frac{\partial v}{\partial s}+\sup_u\left\{f_u\frac{\partial v}{\partial z}+\frac{1}{2}c_u^2\frac{\partial^2 v}{\partial z^2}+f_p\frac{\partial v}{\partial p}+c_u' c_p\frac{\partial^2 v}{\partial z\partial p}\right\}+\frac{1}{2}\mathrm{tr}\left(c_p c_p'\frac{\partial^2 v}{\partial p^2}\right).\end{aligned}$$

Suppose that the value function has the form

$$v(s,z,p)=y^k w(s,p)=e^{kz}w(s,p).$$

Then

$$\begin{aligned}\frac{\partial v}{\partial s}&=e^{kz}\frac{\partial w}{\partial s},\\ \frac{\partial v}{\partial z}&=ke^{kz}w(s,p),\\ \frac{\partial v}{\partial p}&=e^{kz}\frac{\partial w}{\partial p},\\ \frac{\partial^2 v}{\partial z^2}&=k^2e^{kz}w(s,p),\\ \frac{\partial^2 v}{\partial z\partial p}&=\left(\frac{\partial^2 v}{\partial p\partial z}\right)'=ke^{kz}\frac{\partial w}{\partial p},\\ \frac{\partial^2 v}{\partial p^2}&=e^{kz}\frac{\partial^2 w}{\partial p^2}.\end{aligned}$$

It follows that

$$\begin{aligned} 0 = &\ \frac{\partial v}{\partial s} + H\left(Y, \frac{\partial v}{\partial Y}, \frac{\partial^2 v}{\partial Y^2}\right) \\ = &\ e^{kz}\frac{\partial w}{\partial s} + \sup_u \left\{ f_u k e^{kz} w(s,p) + \frac{1}{2}c_u^2 k^2 e^{kz} w(s,p) \right. \\ &\left. + f_p e^{kz}\frac{\partial w}{\partial p} + c_u c_p k e^{kz}\frac{\partial w}{\partial p}\right\} + \frac{1}{2}\mathrm{tr}\left(c_p c_p' e^{kz}\frac{\partial^2 w}{\partial p^2}\right). \end{aligned}$$

Therefore, if the value function has the form $v(s,z,p) = y^k w(s,p) = e^{kz}w(s,p)$, then

$$\begin{aligned} \frac{\partial w}{\partial s} &+ \sup_u \left\{ f_u k w(s,p) + \frac{1}{2}c_u^2 k^2 w(s,p) \right. \\ &+ \left. f_p \frac{\partial w}{\partial p} + c_u c_p k \frac{\partial w}{\partial p}\right\} + \frac{1}{2}\mathrm{tr}\left(c_p c_p' \frac{\partial^2 w}{\partial p^2}\right) = 0. \end{aligned} \tag{6.2}$$

This is a reduced PDE that only contains the variables s and p.

Theorem 6.1. *If $w(s,p)$ is the viscosity solution of the PDE* (6.2)*, then $v(s,z,p) = e^{kz}w(s,p)$ is the viscosity solution of the HJB equation* (3.7)*.*

Proof. Suppose $w(s,p)$ is the viscosity solution of the PDE (6.2), then

$$\begin{aligned} &\frac{\partial \phi}{\partial s} + \sup_u \left\{ f_u k w(s,p) + \frac{1}{2}c_u^2 k^2 w(s,p) \right. \\ &\quad \left. + f_p \frac{\partial \phi}{\partial p} + c_u c_p k \frac{\partial \phi}{\partial p}\right\} + \frac{1}{2}\mathrm{tr}\left(c_p c_p' \frac{\partial^2 \phi}{\partial p^2}\right) \le 0. \end{aligned}$$

for all $\phi \in C^2$ such that $w - \phi$ has a local minimum at (s,p). Then $w(s,p) - \phi(s,p) \le w(t,q) - \phi(t,q)$.

Suppose $v(s,z,p) = e^{kz}w(s,p)$ and $\psi \in C^2$ such that $v - \psi$ has a local minimum at (s,z,p). That is,

$$e^{kz}w(s,p) - \psi(s,z,p) \le e^{kx}w(t,q) - \psi(t,x,q) \tag{6.3}$$

for all (t,x,q)in a neighborhood $N(s,z,p)$.

(1) Let $t = s, q = p, x = z + \Delta z$ in (6.3) we have

$$e^{kz}w(s,p) - \psi(s,z,p) \le e^{k(z+\Delta z)}w(s,p) - \psi(s,z+\Delta z,p),$$

or,

$$\psi(s,z+\Delta z,p) - \psi(s,z,p) \le e^{k(z+\Delta z)}w(s,p) - e^{kz}w(s,p). \tag{6.4}$$

Therefore,

$$\frac{\psi(s,z+\Delta z,p) - \psi(s,z,p)}{\Delta z} \le \frac{e^{k(z+\Delta z)} - e^{kz}}{\Delta z}w(s,p).$$

Letting $\Delta z \to 0$, we have

$$\frac{\partial \psi}{\partial z} \le \frac{\partial v}{\partial z} \text{ at } (s,z,p). \tag{6.5}$$

Similarly, we have

$$\psi(s, z - \Delta z, p) - \psi(s, z, p) \le e^{k(z-\Delta z)} w(s, p) - e^{kz} w(s, p). \tag{6.6}$$

Add (6.4) and (6.6). We have

$$\psi(s, z+\Delta z, p) - 2\psi(s, z, p) + \psi(s, z-\Delta z, p) \le e^{k(z+\Delta z)} w(s, p) + e^{k(z-\Delta z)} w(s, p) - 2e^{kz} w(s, p).$$

Hence

$$\frac{\psi(s, z + \Delta z, p) - 2\psi(s, z, p) + \psi(s, z - \Delta z, p)}{(\Delta z)^2} \le \frac{e^{k(z+\Delta z)} - 2e^{kz} + +e^{k(z-\Delta z)}}{(\Delta z)^2} w(s, p).$$

Letting $\Delta z \to 0$, we have

$$\frac{\partial^2 \psi}{\partial z^2} \le \frac{\partial^2 v}{\partial z^2} \text{ at } (s, z, p). \tag{6.7}$$

(2) Let $x = z$ in (6.3). We also have

$$e^{kz} w(s, p) - \psi(s, z, p) \le e^{kz} w(t, q) - \psi(t, z, q).$$

Fix z and divide both sides by e^{kz}, we have

$$w(s, p) - \frac{\psi(s, z, p)}{e^{kz}} \le w(t, q) - \frac{\psi(t, z, q)}{e^{kz}},$$

for all (t, q) in the neighborhood $N(s, p)$. Because w is the viscosity solution of (6.2), we must have

$$\begin{aligned} \frac{1}{e^{kz}} \frac{\partial \psi}{\partial s} &+ \sup_{u(\cdot)} \Bigg\{ f_u k w(s, z, p) + \frac{1}{2} c_u^2 k^2 w(s, z, p) \\ &+ f_p \frac{1}{e^{kz}} \frac{\partial \psi}{\partial p} + c_u c_p k \frac{1}{e^{kz}} \frac{\partial \psi}{\partial p} Bigg\} + \frac{1}{2} \frac{1}{e^{kz}} \mathrm{tr} \left(c_p c_p' \frac{\partial^2 \psi}{\partial p^2} \right) \le 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial \psi}{\partial s} &+ \sup_{u(\cdot)} \Bigg\{ f_u k e^{kz} w(s, z, p) + \frac{1}{2} c_u^2 k^2 e^{kz} w(s, z, p) \\ &+ f_p \frac{\partial \psi}{\partial p} + c_u c_p k \frac{\partial \psi}{\partial p} \Bigg\} + \frac{1}{2} \mathrm{tr} \left(c_p c_p' \frac{\partial^2 \psi}{\partial p^2} \right) \le 0. \end{aligned}$$

This is true for all value of z. So

$$\begin{aligned} \frac{\partial \psi}{\partial s} &+ \sup_{u(\cdot)} \Bigg\{ f_u \frac{\partial v}{\partial z} + \frac{1}{2} c_u^2 \frac{\partial^2 v}{\partial z^2} \\ &+ f_p \frac{\partial \psi}{\partial p} + c_u c_p k \frac{\partial \psi}{\partial p} \Bigg\} + \frac{1}{2} \mathrm{tr} \left(c_p c_p' \frac{\partial^2 \psi}{\partial p^2} \right) \le 0. \end{aligned}$$

TABLE 1. Different initial time

s	$v(s, \log(1000), 0.8)$	MC
0	33.5278	33.8943
0.1	33.2449	33.4353
0.2	32.8735	33.2977
0.3	32.4707	32.4048
0.4	32.0480	32.1506

Consider (6.5) and (6.7). We have

$$\begin{aligned}\frac{\partial \psi}{\partial s} + \sup_{u(\cdot)} \Bigg\{ f_u \frac{\partial \psi}{\partial z} + \frac{1}{2} c_u^2 \frac{\partial^2 \psi}{\partial z^2} \\ + f_p \frac{\partial \psi}{\partial p} + c_u c_p k \frac{\partial \psi}{\partial p} \Bigg\} + \frac{1}{2} \mathrm{tr} \left(c_p c_p' \frac{\partial^2 \psi}{\partial p^2} \right) \le 0.\end{aligned}$$

This proves that $v(s, z, p) = e^{kz} w(s, p)$ is a viscosity subsolution of (3.7). The proof for supersolution is similar. □

Recall that the numerical scheme for the value function is

$$v^h(s - \Delta t, Y) = \sup_u \left[\sum_Z P^h(Y, Z \mid u) v^h(s, Z) \right] \tag{6.8}$$

with

$$v^h(T, z, p) = \Phi(z), \quad z \in S_h. \tag{6.9}$$

Now with $v(s, z, p) = e^{kz} w(s, p)$ we can simplify this scheme and have

$$w^h(s - \Delta t, p) = \sup_u \left[\sum_q P^h(p, q \mid u) \delta(q) w^h(s, p) \right], \tag{6.10}$$

with

$$w^h(T, p) = 1. \tag{6.11}$$

7. A NUMERICAL EXAMPLE

In order to test the numerical scheme in this paper, we compare the value function from the Markov chain approximation and from the Monte Carlo simulation. To take advantage of the separable case, we assume the utility function is $\Phi(x) = x^{1/2}$.

Assume $T = 0.5$, a half year time frame. With initial investment of \$1000, the value function $v(s, x, p)$ for different initial time s compared with the data from Monte Carlo simulation is shown in Table 1.

TABLE 2. Different initial probability

p	$v(0.2, \log(1000), p)$	MC
0	32.5956	32.9894
0.1	32.5964	32.7831
0.2	32.5981	32.8393
0.3	32.6024	32.8257
0.4	32.6140	32.9907
0.5	32.6511	32.9886
0.6	32.6963	33.0124
0.7	32.7415	33.1261
0.8	32.7868	33.1319
0.9	32.8321	33.2506
1.0	32.8588	33.2377

With initial investment of \$1000, the value function $v(s, x, p)$ for different initial probability p compared with the data from Monte Carlo simulation is shown in Table 2.

8. CONCLUSIONS

In this paper, we considered an asset allocation problem under the formulation of geometric Brownian motion with regime switching. We focused on the problem with partial observations. Wonham filter is used to estimate the conditional probability of the market mode process. We used Kushner's Markov approximation to solve the problem. We also considered a separable case and showed that only a simpler HJB equation is needed for the corresponding numerical solutions.

It would be interesting to extend our results to incorporate the case when the volatility is Markov chain dependent. Nevertheless, this requires further study of extended Wonham filters or approximation methods for handling the associated infinite dimensional filtering equations.

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