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OSCILLATION RESULTS FOR ODD-ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS OF MIXED TYPE

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ABSTRACT. In this paper the authors establish some new comparison theorems and Philos-type criteria for oscillation of solutions to the odd order neutral mixed type differential equation

$$(x(t) + ax(t - \tau_1) + bx(t + \tau_2))^{(n)} + p(t)x^{\alpha}(t - \sigma_1) + q(t)x^{\beta}(t + \sigma_2) = 0, \quad t \geq t_0,$$

where $n \geq 3$ is an odd integer, $\alpha \geq 1$ and $\beta \geq 1$, are ratio of odd positive integers. Examples are provided to illustrate the main results.

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1. PRELIMINARIES

This paper is concerned with the oscillation and asymptotic behavior of solutions of odd order nonlinear neutral mixed type differential equation of the form

$$(1.1)\quad (x(t) + ax(t - \tau_1) + bx(t + \tau_2))^{(n)} + p(t)x^{\alpha}(t - \sigma_1) + q(t)x^{\beta}(t + \sigma_2) = 0,\ t \geq t_0,$$

where $n \geq 3$ is an odd integer, $\alpha \geq 1$ and $\beta \geq 1$ are the ratios of odd positive integers, $p(t)$ and $q(t)$ are continuous and positive functions for all $t \geq t_0$, and $a, b, \tau_1, \tau_2, \sigma_1, \sigma_2$ are non-negative constants. We set $z(t) = x(t) + ax(t - \tau_1) + bx(t + \tau_2)$. By a solution of equation (1.1), we mean a function $x(t) \in C([T_x, \infty), \mathbb{R})$, $T_x \geq t_0$, which has the property $z(t) \in C^n([T_x, \infty), \mathbb{R})$ and satisfies equation (1.1) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of equation (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that equation (1.1) possesses such a solution. A solution of equation (1.1) is called oscillatory if it has infinitely large zeros in $[T_x, \infty)$ and otherwise, it is said to be nonoscillatory. Equation (1.1) is said to be almost oscillatory if all its solutions are either oscillatory or convergent to zero asymptotically.

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Recently, there have been a lot of interest in studying the oscillatory and asymptotic behavior of solutions of neutral type differential equations, see for example [2, 3, 4, 5, 6, 11, 13, 18, 21, 22, 23, 24].

Very recently, there are some results regarding the oscillatory properties of neutral differential equations with mixed arguments, see the papers [7, 8, 10, 14, 26, 27]. In [8], the author has obtained some oscillation theorems for the odd order neutral differential equation

$$\big(x(t) + p_1 x(t-\tau_1) + p_2 x(t+\tau_2)\big)^{(n)} = q_1 x(t-\sigma_1) + q_2 x(t+\sigma_2), \quad t \geq t_0, \tag{1.2}$$

where $n \geq 1$ is odd.

Grace [10] and Yan [27] obtained several sufficient conditions for the oscillation of all solutions of higher order neutral functional differential equation of the form

$$\big(x(t) + cx(t-h) + Cx(t+H)\big)^{(n)} + qx(t-g) + Qx(t+G) = 0, \quad t \geq t_0, \tag{1.3}$$

where q and Q are nonnegative real constants.

In [20], Li and Thandapani considered the following odd order neutral differential equation

$$(x(t) + p(t)x(\tau(t)))^{(n)} + q(t)x(\sigma(t)) + v(t)x(\eta(t)) = 0 \tag{1.4}$$

and obtained some oscillation results for the equation (1.4). Clearly the equations (1.3) and (1.4) are special cases of equation (1.1).

The purpose of this paper is to study the oscillatory and asymptotic behavior of solutions of equation (1.1) so our results generalize and extends some of the results obtained in [8, 10, 20, 27].

2. OSCILLATION RESULTS

In this section, we present some new oscillation criteria for the equation (1.1). We begin with the following definition.

Definition 2.1. Consider the sets $\mathbb{D}_0 = \{(t,s) : t > s \geq t_0\}$ and $\mathbb{D} = \{(t,s) : t \geq s \geq t_0\}$. Assume that $H \in C(\mathbb{D}, \mathbb{R})$ satisfies the following assumptions:

(A_1) $H(t,t) = 0$, $t \geq t_0$; $H(t,s) > 0$, $(t,s) \in \mathbb{D}_0$;
(A_2) H has a non-positive continuous partial derivative with respect to the second variable in $\mathbb{D}_0$.

Then the function H has the property P.

Lemma 2.2 ([16, 17], Kiguradze's lemma)**.** *Let $f \in C^n([t_0,\infty), \mathbb{R})$ and its derivatives up to order $(n-1)$ are of constant sign in $[t_0,\infty)$. If $f^{(n)}$ is of constant sign and not* *identically zero on a sub-ray of* $[t_0, \infty)$*, then there exist* $m \in \mathbb{Z}$ *and* $t_1 \in [t_0, \infty)$ *such that* $0 \le m \le n-1$*, and* $(-1)^{n+m} f f^{(n)} \ge 0$*,*

$$ff^{(j)} > 0 \text{ for } j = 0, 1, \ldots, m-1 \text{ when } m \ge 1$$

and

$$(-1)^{m+j} f f^{(j)} > 0 \text{ for } j = m, m+1, \ldots, n-1 \text{ when } m \le n-1$$

hold on $[t_1, \infty)$*.*

Lemma 2.3 ([1, Lemma 2.2.3])**.** *Let* f *be a function as in Lemma* 2.1*. If* $\lim_{t\to\infty} f(t) \neq 0$*, then for every* $\lambda \in (0,1)$*, there exists* $t_\lambda \in [t_1, \infty)$ *such that*

$$|f| \ge \frac{\lambda}{(n-1)!} t^{n-1} |f^{(n-1)}|$$

holds on $[t_\lambda, \infty)$*.*

Lemma 2.4 ([21])**.** *Let* f *be a function as in Lemma* 2.1*. If*

$$f^{(n-1)}(t) f^{(n)}(t) \le 0,$$

then for any constant $\theta \in (0,1)$ *and sufficiently large* t*, there exists a constant* $M > 0$*, satisfying*

$$|f'(\theta t)| \ge M t^{n-2} |f^{(n-1)}(t)|.$$

Lemma 2.5. *If* x *is a positive solution of* (1.1)*, then the corresponding function* $z(t) = x(t) + ax(t-\tau_1) + bx(t+\tau_2)$ *satisfies*

$$z(t) > 0, \quad z^{(n-1)}(t) > 0, \quad z^{(n)}(t) \le 0 \tag{2.1}$$

eventually.

Due to Lemma 2.1, the proof of the above lemma is simple and so is omitted.

Lemma 2.6 ([19, Lemma 2.6])**.** *Assume that* $\alpha \in (0, \infty)$ *and* $c \ge 0$ *and* $d \ge 0$*. Then*

$$c^\alpha + d^\alpha \ge (c+d)^\alpha \text{ if } 0 < \alpha < 1$$

and

$$c^\alpha + d^\alpha \ge \frac{1}{2^{\alpha-1}} (c+d)^\alpha \text{ if } \alpha \ge 1. \tag{2.2}$$

Lemma 2.7 ([25])**.** *Assume that for large* t

$$q(s) \neq 0 \text{ for all } s \in [t, t^*],$$

where t^* *satisfies* $\sigma(t^*) = t$*. Then*

$$x'(t) + q(t) [x(\sigma(t))]^\alpha = 0, \quad t \ge t_0,$$

has an eventually positive solution if and only if the corresponding inequality

$$x'(t) + q(t) [x(\sigma(t))]^\alpha \le 0, \quad t \ge t_0,$$

has an eventually positive solution.

In [9, 12, 18, 25], the authors investigated the oscillatory behavior of the following equation

$$x'(t) + q(t)[x(\sigma(t))]^{\alpha} = 0, \quad t \geq t_0, \tag{2.3}$$

where $q \in C([t_0, \infty), \mathbb{R}^+)$, $\sigma \in C([t_0, \infty), \mathbb{R})$, $\sigma(t) < t$, $\lim_{t\to\infty} \sigma(t) = \infty$ and $\alpha \in (0, \infty)$ is a ratio of odd positive integers.

Let $\alpha \in (0, 1)$. Then it is shown that every solution of the sublinear equation (2.3) oscillates if and only if

$$\int_{t_0}^{\infty} q(s)ds = \infty. \tag{2.4}$$

Let $\alpha = 1$. Then equation (2.3) reduces to the linear delay differential equation

$$x'(t) + q(t)x(\sigma(t)) = 0, \quad t \geq t_0, \tag{2.5}$$

and it is shown that every solution of equation (2.5) oscillates if

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} q(s)ds > \frac{1}{e}. \tag{2.6}$$

Let $\alpha \in (1, \infty)$ and $\sigma(t) = t - \sigma$. Then equation (2.3) reduces to

$$x'(t) + q(t)x^{\alpha}(t - \sigma) = 0, \quad t \geq t_0, \tag{2.7}$$

for which the following results was obtained: If there exists $\lambda \in (\sigma^{-1} \ln \alpha, \infty)$ such that

$$\liminf_{t\to\infty} q(t) \exp(-e^{\lambda t}) > 0, \tag{2.8}$$

then every solution of equation (2.7) oscillates.

Next, we state and prove our main results. For the sake of convenience, let

$$\begin{aligned} R(t) &= P(t) + Q(t), \\ P(t) &= \min\{p(t), p(t - \tau_1), p(t + \tau_2)\}, \\ Q(t) &= \min\{q(t), q(t - \tau_1), q(t + \tau_2)\}. \end{aligned} \tag{2.9}$$

Theorem 2.8. *Assume that*

(H_1) $\int_{t_0}^{\infty} t^{n-1}R(t)dt = \infty$;
(H_2) $1 \leq \alpha \leq \beta$.

If the first order differential inequality

$$w'(t)+\frac{R(t)}{4^{\alpha-1}(1+a^{\alpha}+b^{\alpha})^{\alpha}}\left(\frac{\lambda}{(n-1)!}(t-\sigma_1)^{n-1}\right)^{\alpha}w^{\alpha}(t-(\sigma_1-\tau_1))\le 0, \tag{2.10}$$

has no positive solution for some $0<\lambda<1$ *and* $t\ge t_0$*. Then equation* (1.1) *is almost oscillatory.*

Proof. Assume that x is a nonoscillatory solution of equation (1.1), which does not tend to zero asymptotically. Without loss of generality we may assume that x is a positive solution of equation (1.1), which does not tend to zero asymptotically. Let

$$z(t)=x(t)+ax(t-\tau_1)+bx(t+\tau_2),\quad t\ge t_1\ge t_0. \tag{2.11}$$

Then $z(t)>0$ and it follows from equation (1.1) that

$$z^{(n)}(t)=-p(t)x^{\alpha}(t-\sigma_1)-q(t)x^{\beta}(t+\sigma_2)\le 0,\quad t\ge t_1. \tag{2.12}$$

Moreover

$$a^{\alpha}z^{(n)}(t-\tau_1)+a^{\alpha}p(t-\tau_1)x^{\alpha}(t-\tau_1-\sigma_1)+a^{\alpha}q(t-\tau_1)x^{\beta}(t-\tau_1+\sigma_2)=0, \tag{2.13}$$

and

$$b^{\alpha}z^{(n)}(t+\tau_2)+b^{\alpha}p(t+\tau_2)x^{\alpha}(t+\tau_2-\sigma_1)+b^{\alpha}q(t+\tau_2)x^{\beta}(t+\tau_2+\sigma_2)=0. \tag{2.14}$$

Combining (2.12), (2.13), (2.14) and using Lemma 2.5, (2.9) and (H_2), we obtain for $t\ge t_1$,

$$\begin{aligned}&\left(z^{(n-1)}(t)+a^{\alpha}z^{(n-1)}(t-\tau_1)+b^{\alpha}z^{(n-1)}(t+\tau_2)\right)'\\ &+P(t)\frac{1}{4^{\alpha-1}}z^{\alpha}(t-\sigma_1)+Q(t)\frac{1}{4^{\alpha-1}}z^{\alpha}(t+\sigma_2)\le 0,\quad t\ge t_1.\end{aligned} \tag{2.15}$$

Next, we claim that $z'(t)>0$ eventually. If not, then we have $z(t)>0$ and $z'(t)\le 0$ for all $t\ge t_1$. Thus $\lim_{t\to\infty}z(t)=M\ge 0$, and then $\lim_{t\to\infty}z^{(k)}(t)=0$ for $k=1,2,x,\ldots,n-1$. Integrating (2.15) from t to ∞ for a total of $(n-1)$ times and integrating the resulting inequality from $t_1(t_1$ is large enough) to ∞, we obtain

$$\int_{t_1}^{\infty}\frac{(s-t_1)^{n-1}}{(n-1)!4^{\alpha-1}}\left(P(s)z^{\alpha}(s-\sigma_1)+Q(s)z^{\alpha}(s+\sigma_2)\right)ds<\infty.$$

Then for $s\ge T\ge 2t_1$, and since $z(t)$ is bounded, we have from the last inequality

$$\int_{T}^{\infty}s^{n-1}(P(s)+Q(s))ds<\infty.$$

This contradicts (H_1). Hence we have $z'(t)>0$ and $z(t-\sigma_1)\le z(t+\sigma_2)$. Then, from (2.9) and (2.15), we obtain

$$z^{(n)}(t)+a^{\alpha}z^{(n)}(t-\tau_1)+b^{\alpha}z^{(n)}(t+\tau_2)+\frac{R(t)}{4^{\alpha-1}}z^{\alpha}(t-\sigma_1)\le 0,\quad t\ge t_1. \tag{2.16}$$

By the Lemma 2.2 and Lemma 2.4, we obtain

$$z(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} z^{(n-1)}(t) \text{ for every } \lambda \in (0,1), \quad t \geq t_1.$$

Thus, it follows from (2.16) that

$$\begin{aligned} &\left(z^{(n-1)}(t) + a^\alpha z^{(n-1)}(t-\tau_1) + b^\alpha z^{(n-1)}(t+\tau_2)\right)' \\ &+ \frac{R(t)}{4^{\alpha-1}} \left(\frac{\lambda}{(n-1)!}(t-\sigma_1)^{n-1} z^{(n-1)}(t-\sigma_1)\right)^\alpha \leq 0, \quad t \geq t_1. \end{aligned} \tag{2.17}$$

Then, setting $z^{(n-1)}(t) = y(t) > 0$ is a decreasing solution of equation

$$\begin{aligned} &(y(t) + a^\alpha y(t-\tau_1) + b^\alpha y(t+\tau_2))' \\ &+ \frac{R(t)}{4^{\alpha-1}} \left(\frac{\lambda}{(n-1)!}(t-\sigma_1)^{n-1}\right)^\alpha y^\alpha(t-\sigma_1) \leq 0, \quad t \geq t_1. \end{aligned} \tag{2.18}$$

We denote

$$w(t) = y(t) + a^\alpha y(t-\tau_1) + b^\alpha y(t+\tau_2) \leq (1 + a^\alpha + b^\alpha) y(t-\tau_1), \quad t \geq t_1.$$

Substituting this into (2.18), we obtain that w is a positive solution of (2.10), a contradiction. This completes the proof. □

Theorem 2.9. *Assume that* (H$_1$) *holds and*

(H$_3$) $1 \leq \beta \leq \alpha$.

If the first order differential inequality

$$w'(t) + \frac{R(t)}{4^{\beta-1}(1 + a^\beta + b^\beta)^\beta} \left(\frac{\lambda}{(n-1)!}(t-\sigma_1)^{n-1}\right)^\beta w^\beta(t - (\sigma_1 - \tau_1)) \leq 0, \tag{2.19}$$

has no positive solution for some $\lambda \in (0,1)$ and $t \geq t_0$. Then equation (1.1) *is almost oscillatory.*

Proof. The proof is similar to that of Theorem 2.1 and hence the details are omitted. □

Corollary 2.10. *Assume that condition* (H$_1$) *holds, $\alpha = 1$ and $\sigma_1 - \tau_1 > 0$. If*

$$\liminf_{t\to\infty} \int_{t-(\sigma_1-\tau_1)}^{t} R(s)(s-\sigma_1)^{n-1} ds > \frac{(1+a+b)(n-1)!}{\lambda e}, \tag{2.20}$$

then equation (1.1) *is almost oscillatory.*

Proof. According to Lemma 2.6 and condition (2.6), the condition (2.20) guarantees that (2.10) with $\alpha = 1$ has no positive solution. Hence by Theorem 2.1, equation (1.1) is almost oscillatory. This completes the proof. □

Corollary 2.11. *Assume that condition* (H_1) *holds,* $\sigma_1 - \tau_1 > 0$ *and* $\alpha \in (0,1)$. *If*

$$\int_{t_0}^{\infty} R(s)(s-\sigma_1)^{\alpha(n-1)} ds = \infty, \quad t \geq t_0, \tag{2.21}$$

then equation (1.1) *is almost oscillatory.*

Proof. According to Lemma 2.6 and condition (2.4), the condition (2.21) guarantees that (2.10) with $\alpha < 1$ has no positive solution. Hence by Theorem 2.1, equation (1.1) is almost oscillatory. This completes the proof. □

Corollary 2.12. *Assume that condition* (H_1) *holds,* $\sigma_1 - \tau_1 > 0$ *and* $\alpha \in (1,\infty)$. *If there exists* $\mu \in ((\sigma_1 - \tau_1)^{-1} \ln \alpha, \infty)$ *such that*

$$\liminf_{t\to\infty} R(t) \left(\frac{(t-\sigma_1)^{n-1}}{(n-1)!} \right)^{\alpha} \exp(-e^{\mu t}) > 0, \tag{2.22}$$

then equation (1.1) *is almost oscillatory.*

Proof. According to Lemma 2.6 and condition (2.8), the condition (2.22) guarantees that (2.10) with $\alpha > 1$ has no positive solution. Hence by Theorem 2.1, equation (1.1) is almost oscillatory. This completes the proof. □

Corollary 2.13. *Assume that condition* (H_1) *holds,* $\sigma_1 - \tau_1 > 0$ *and* $\beta = 1$. *If*

$$\liminf_{t\to\infty} \int_{t-(\sigma_1-\tau_1)}^{t} R(s)(s-\sigma_1)^{n-1} ds > \frac{(1+a+b)(n-1)!}{\lambda e}, \tag{2.23}$$

then equation (1.1) *is almost oscillatory.*

Proof. According to Lemma 2.6 and condition (2.6), the condition (2.23) guarantees that (2.19) with $\beta = 1$ has no positive solution. Hence by Theorem 2.2, equation (1.1) is almost oscillatory. This completes the proof. □

Corollary 2.14. *Assume that condition* (H_1) *holds,* $\sigma_1 - \tau_1 > 0$ *and* $\beta \in (0,1)$. *If*

$$\int_{t_0}^{\infty} R(s)(s-\sigma_1)^{\beta(n-1)} ds = \infty, \quad t \geq t_0, \tag{2.24}$$

then equation (1.1) *is almost oscillatory.*

Proof. According to Lemma 2.6 and condition (2.4), the condition (2.24) guarantees that (2.19) with $\beta < 1$ has no positive solution. Hence by Theorem 2.2, equation (1.1) is almost oscillatory. This completes the proof. □

Corollary 2.15. *Assume that condition* (H_1) *holds,* $\sigma_1 - \tau_1 > 0$ *and* $\beta \in (1,\infty)$. *If there exists* $\nu \in ((\sigma_1 - \tau_1)^{-1} \ln \beta, \infty)$ *such that*

$$\liminf_{t\to\infty} R(t) \left(\frac{(t-\sigma_1)^{n-1}}{(n-1)!} \right)^{\beta} \exp(-e^{\nu t}) > 0, \tag{2.25}$$

then equation (1.1) *is almost oscillatory.*

Proof. According to Lemma 2.6 and condition (2.8), the condition (2.25) guarantees that (2.19) with $\beta > 1$ has no positive solution. Hence by Theorem 2.2, equation (1.1) is almost oscillatory. This completes the proof. □

Next, we shall establish some Philos-type oscillation criteria for the oscillation of equation (1.1).

Theorem 2.16. *Assume that* (H_1), (H_2) *and* $\sigma_1 \leq \tau_1$ *hold. Further, assume that the function* $H \in C(\mathbb{D}, \mathbb{R})$ *has the property* P *and there exist functions* $h \in C(\mathbb{D}_0, \mathbb{R})$ *and* $\rho \in C^1([t_0, \infty), (0, \infty))$ *such that*

$$-\frac{\partial}{\partial s} H(t,s) - H(t,s)\frac{\rho'(s)}{\rho(s)} = h(t,s), \quad (t,s) \in \mathbb{D}_0. \tag{2.26}$$

If

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} K_1(t,s)ds = \infty, \tag{2.27}$$

for all constants $M > 0$ *and* $L > 0$, *where*

$$K_1(t,s) := \left(\frac{L}{4}\right)^{\alpha-1} H(t,s)\rho(s)R(s) - (1 + a^\alpha + b^\alpha)\frac{\rho(s)h^2(t,s)}{4MH(t,s)(s-\tau_1)^{n-2}},$$

then equation (1.1) *is almost oscillatory.*

Proof. Assume that x is a nonoscillatory solution of equation (1.1), which does not tend to zero asymptotically. Without loss of generality we may assume that x is a positive solution of equation (1.1), which does not tend to zero asymptotically. Proceeding as in the proof of Theorem 2.1, we obtain (2.15) and $z'(t) > 0$ for all $t \geq t_1$. Define

$$w(t) = \rho(t)\frac{z^{(n-1)}(t)}{z(t-\tau_1)}, \quad t \geq t_1. \tag{2.28}$$

then $w(t) > 0$ and

$$w'(t) = \rho'(t)\frac{z^{(n-1)}(t)}{z(t-\tau_1)} + \rho(t)\frac{z^{(n)}(t)z(t-\tau_1) - z^{(n-1)}(t)z'(t-\tau_1)}{z^2(t-\tau_1)}, \quad t \geq t_1. \tag{2.29}$$

It follows from Lemma 2.3 and Lemma 2.4 that there exists a constant $M > 0$, such that

$$z'(t-\tau_1) \geq M(t-\tau_1)^{n-2} z^{(n-1)}(t-\tau_1), \tag{2.30}$$

which in view of (2.28) and (2.29) yields

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)}w(t) + \rho(t)\frac{z^{(n)}(t)}{z(t-\tau_1)} - \frac{M(t-\tau_1)^{n-2}}{\rho(t)}w^2(t). \tag{2.31}$$

Define

$$v(t) = \rho(t)\frac{z^{(n-1)}(t-\tau_1)}{z(t-\tau_1)}, \quad t \geq t_1, \tag{2.32}$$

then $v(t) > 0$ and

$$v'(t) \le \frac{\rho'(t)}{\rho(t)} v(t) + \rho(t) \frac{z^{(n)}(t-\tau_1)}{z(t-\tau_1)} - \frac{M(t-\tau_1)^{n-2}}{\rho(t)} v^2(t). \tag{2.33}$$

Define

$$u(t) = \rho(t) \frac{z^{(n-1)}(t+\tau_2)}{z(t-\tau_1)}, \quad t \ge t_1, \tag{2.34}$$

then $u(t) > 0$ and

$$u'(t) \le \frac{\rho'(t)}{\rho(t)} u(t) + \rho(t) \frac{z^{(n)}(t+\tau_2)}{z(t-\tau_1)} - \frac{M(t-\tau_1)^{n-2}}{\rho(t)} u^2(t). \tag{2.35}$$

In the view of (2.31), (2.33) and (2.35), we obtain

$$\begin{aligned} w'(t) + a^\alpha v'(t) + b^\alpha u'(t) \le{}& \frac{\rho'(t)}{\rho(t)} \left(w(t) + a^\alpha v(t) + b^\alpha u(t)\right) \\ &+ \frac{\rho(t)}{z(t-\tau_1)} \left(z^{(n)}(t) + a^\alpha z^{(n)}(t-\tau_1) + b^\alpha z^{(n)}(t+\tau_2)\right) \\ &- \frac{M(t-\tau_1)^{n-2}}{\rho(t)} \left(w^2(t) + a^\alpha v^2(t) + b^\alpha u^2(t)\right). \end{aligned} \tag{2.36}$$

From (2.16), (2.36) and $z'(t) > 0$, we obtain

$$\begin{aligned} w'(t) + a^\alpha v'(t) + b^\alpha u'(t) \le{}& \frac{\rho'(t)}{\rho(t)} \left(w(t) + a^\alpha v(t) + b^\alpha u(t)\right) \\ &- \frac{\rho(t)}{4^{\alpha-1} z(t-\tau_1)} R(t) z^\alpha(t-\sigma_1) \\ &- \frac{M(t-\tau_1)^{n-2}}{\rho(t)} \left(w^2(t) + a^\alpha v^2(t) + b^\alpha u^2(t)\right). \end{aligned}$$

Since $z(t) \ge L > 0$, we have from the last inequality

$$\begin{aligned} w'(t) + a^\alpha v'(t) + b^\alpha u'(t) \le{}& \frac{\rho'(t)}{\rho(t)} \left(w(t) + a^\alpha v(t) + b^\alpha u(t)\right) \\ &- \left(\frac{L}{4}\right)^{\alpha-1} \rho(t) R(t) - \frac{M(t-\tau_1)^{n-2}}{\rho(t)} \left(w^2(t) + a^\alpha v^2(t) + b^\alpha u^2(t)\right). \end{aligned} \tag{2.37}$$

Multiplying (2.37), with t replaced by s, by $H(t,s)$ and integrating from T to t with $T \ge t_1$, we have

$$\begin{aligned} \int_T^t \left(\frac{L}{4}\right)^{\alpha-1} H(t,s)\rho(s)R(s)ds \le{}& -\int_T^t H(t,s) \left(w'(s) + a^\alpha v'(s) + b^\alpha u'(s)\right) ds \\ &+ \int_T^t H(t,s) \frac{\rho'(s)}{\rho(s)} \left(w(s) + a^\alpha v(s) + b^\alpha u(s)\right) ds \\ &- M \int_T^t H(t,s) \frac{(s-\tau_1)^{n-2}}{\rho(s)} (w^2(s) + a^\alpha v^2(s) + b^\alpha u^2(s)) ds. \end{aligned}$$

It follows from the above inequality and (2.26) that

$$\left(\frac{L}{4}\right)^{\alpha-1}\int_T^t H(t,s)\rho(s)R(s)ds \le H(t,T)\left(w(T)+a^\alpha v(T)+b^\alpha u(T)\right)$$
$$-M\int_T^t H(t,s)\frac{(s-\tau_1)^{n-2}}{\rho(s)}\left(\left(w^2(s)+\frac{h(t,s)\rho(s)w(s)}{MH(t,s)(s-\tau_1)^{n-2}}\right)\right.$$
$$\left.+a^\alpha(v^2(s)+\frac{h(t,s)\rho(s)v(s)}{MH(t,s)(s-\tau_1)^{n-2}})+b^\alpha\left(u^2(s)+\frac{h(t,s)\rho(s)w(s)}{MH(t,s)(s-\tau_1)^{n-2}}\right)\right)ds.$$

Now using completing the square, we obtain

$$\int_T^t\left[\left(\frac{L}{4}\right)^{\alpha-1}H(t,s)\rho(s)R(s)-(1+a^\alpha+b^\alpha)\frac{\rho(s)h^2(t,s)}{4M(s-\tau_1)^{n-2}H(t,s)}\right]ds$$
$$\le H(t,T)(w(T)+a^\alpha v(T)+b^\alpha u(T))\le H(t,t_0)(w(T)+a^\alpha v(T)+b^\alpha u(T)),$$

which yields

$$\frac{1}{H(t,t_0)}\int_T^t\left[\left(\frac{L}{4}\right)^{\alpha-1}H(t,s)\rho(s)R(s)-(1+a^\alpha+b^\alpha)\frac{\rho(s)h^2(t,s)}{4M(s-\tau_1)^{n-2}H(t,s)}\right]ds<\infty.$$

This contradicts condition (2.27). The proof is complete. □

Theorem 2.17. *Assume that* (H$_1$), (H$_3$), (2.26) *and* $\sigma_1 \le \tau_1$ *hold. If*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t K_2(t,s)ds=\infty, \tag{2.38}$$

for all constants $M>0$ *and* $L>0$*, where*

$$K_2(t,s):=\left(\frac{L}{4}\right)^{\beta-1}H(t,s)\rho(s)R(s)-(1+a^\beta+b^\beta)\frac{\rho(s)h^2(t,s)}{4MH(t,s)(s-\tau_1)^{n-2}},$$

then equation (1.1) *is almost oscillatory.*

Proof. The proof is similar to that of Theorem 2.3 and hence the details are omitted. □

Theorem 2.18. *Assume that* (H$_1$), (H$_2$) *and* $\tau_1\le\sigma_1$ *hold. Further, assume that the function* $H\in C(\mathbb{D},\mathbb{R})$ *has the property* P *and there exist functions* $h\in C(\mathbb{D}_0,\mathbb{R})$ *and* $\rho\in C^1([t_0,\infty),(0,\infty))$ *such that* (2.26) *holds. If*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t K_3(t,s)ds=\infty, \tag{2.39}$$

for all constants $M>0$ *and* $L>0$*, where*

$$K_3(t,s):=\left(\frac{L}{4}\right)^{\alpha-1}H(t,s)\rho(s)R(s)-(1+a^\alpha+b^\alpha)\frac{\rho(s)h^2(t,s)}{4MH(t,s)(s-\sigma_1)^{n-2}},$$

then equation (1.1) *is almost oscillatory.*

Proof. The proof is similar to that of Theorem 2.3 by taking $w(t) = \rho(t)\frac{z^{(n-1)}(t)}{z(t-\sigma_1)}$, $v(t) = \rho(t)\frac{z^{(n-1)}(t-\tau_1)}{z(t-\sigma_1)}$ and $u(t) = \rho(t)\frac{z^{(n-1)}(t+\tau_2)}{z(t-\sigma_1)}$, for $t \geq t_1$. Therefore the details are omitted. □

Theorem 2.19. *Assume that* (H_1), (H_3) *and* $\tau_1 \leq \sigma_1$ *hold. Further, assume that the function* $H \in C(\mathbb{D}, \mathbb{R})$ *has the property* P *and there exist functions* $h \in C(\mathbb{D}_0, \mathbb{R})$ *and* $\rho \in C^1([t_0, \infty), (0, \infty))$ *such that* (2.26) *holds. If*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} K_4(t,s)ds = \infty, \tag{2.40}$$

for all constants $M > 0$ *and* $L > 0$*, where*

$$K_4(t,s) := \left(\frac{L}{4}\right)^{\beta-1} H(t,s)\rho(s)R(s) - (1 + a^\beta + b^\beta)\frac{\rho(s)h^2(t,s)}{4MH(t,s)(s-\sigma_1)^{n-2}},$$

then equation (1.1) *is almost oscillatory.*

Proof. The proof is similar to that of Theorem 2.4 by taking $w(t) = \rho(t)\frac{z^{(n-1)}(t)}{z(t-\sigma_1)}$, $v(t) = \rho(t)\frac{z^{(n-1)}(t-\tau_1)}{z(t-\sigma_1)}$, and $u(t) = \rho(t)\frac{z^{(n-1)}(t+\tau_2)}{z(t-\sigma_1)}$, for $t \geq t_1$. Therefore the details are omitted. □

Corollary 2.20. *Let condition* (2.27) *in Theorem* 2.3 *be replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)R(s)ds = \infty,$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\rho(s)h^2(t,s)}{H(t,s)(s-\tau_1)^{n-2}}ds < \infty.$$

Then equation (1.1) *is almost oscillatory.*

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