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QUENCHING RATES FOR PARABOLIC PROBLEMS DUE TO A CONCENTRATED NONLINEAR SOURCE

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ABSTRACT. Let q, a, b, p and T be real numbers such that $q \geq 0$, $a > 0$, $0 < b < a$, $p > 0$, and $0 < T$, $D = (0, a)$, $\Omega = D \times (0, T]$. This article studies the following degenerate parabolic initial boundary value problem:

$$x^q u_t - u_{xx} = \delta(x-b)(1-u(x,t))^{-p} \text{ in } \Omega,$$

$$u(x,0) = 0 \text{ on } \bar{D}, u(0,t) = 0 = u(a,t) \text{ for } 0 < t \leq T,$$

where $\delta(x)$ is the Dirac delta function. The growth rate of the solution u as $u \to 1^-$ is established.

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1. INTRODUCTION

Let q, a, b, p and T be real numbers such that $q \geq 0$, $a > 0$, $0 < b < a$, $p > 0$, and $0 < T$, $D = (0, a)$, $\Omega = D \times (0, T]$. Let $L_q u = x^q u_t - u_{xx}$. This article studies the following degenerate parabolic initial boundary value problem:

$$L_q u = \delta(x-b)(1-u(x,t))^{-p} \text{ in } \Omega, \tag{1.1}$$

$$u(x,0) = 0 \text{ on } \bar{D}, u(0,t) = 0 = u(a,t) \text{ for } 0 < t \leq T, \tag{1.2}$$

where $\delta(x)$ is the Dirac delta function. These types of problems are motivated by applications in which the ignition of a combustible medium is accomplished through the use of either a heated wire or a pair of small electrodes to supply a large amount of energy to a very confined area. In particularly, when $q = 0$, it can be used to describe the temperature distribution on a rod with a concentrated nonlinear source at point b (cf. [6]). When $q = 1$, it may be used to describe the temperature u of the channel flow of a fluid with temperature dependent viscosity in the boundary layer

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(cf. [4, 5]). For the case $q = 0$, Deng and Roberts [12] studied the corresponding nonlinear Volterra equation at the site b in a finite domain $(0, a)$:

$$u(b,t) = a^2 \int_0^t G(b,t;b,\tau)(1-u(b,\tau))^{-p}\,d\tau,$$

where $G(x,t;\xi,\tau)$ denotes the Green's function corresponding to the problem (1.1)–(1.2). They showed that there is a a^* such that for $a \leq a^*$, the solution $u(b,t)$ of the integral equation exists for all time and is uniformly bounded away from 1. When $a > a^*$, there exists a finite time T such that $\lim_{t\to T} u(b,t) = 1$, and $\lim_{t\to T} u_t(b,t) = \infty$.

Chan and Jiang [6] investigated the solution $u(x,t)$ of the problem (1.1)–(1.2). They showed that the problem has a unique continuous solution which satisfies (1.1)–(1.2), and $u_{xx}(x,t) \geq 0$ for $x \in (0,b)$ and $x \in (b,a)$. Also, $u_t(b,t) = \infty$ for any $t > 0$.

For $q = 0$, the study of the problems when the singularity of the right-hand side of the equation (1.1) is replaced by $(1 - u(x,t))^{-p}$ was initiated in 1975 by Kawarada [17], and since then, it has attracted much attention (cf. [5, 11]). Chan and Tragoonsirisak [9], and Chan and Treeyaprasert [10] studied the existence and quenching of the solution of a parabolic problem with a concentrated nonlinear source in an infinite strip and on a semi-infinite interval respectively. Chan [2], and Chan and Tragoonsirisak [8] investigated the quenching behavior of the solution in the multi-dimensional cases.

The rate of change of the solution when $\max\{u(x,t) : x \in \bar{D}\} \to 1^-$ as $t \to T$ were studies by Deng and Levine [13]. For $q \geq 0$, Yuen [20] studied the rate of change of the solution. When the right-hand side of the equation (1.1) is replaced by $u^p(x,t)$, the rate of change of the solution as $u(x,t) \to \infty$ was studied by Fila and Hulshof [14], Guo [15], and Guo, Sasayama and Wang [16].

In this paper, a solution of the problem (1.1)–(1.2) is a continuous function which satisfies (1.1)–(1.2). The solution is said to quench if there is a T such that $\max\{u(x,t) : x \in \bar{D}\} \to 1^-$ as $t \to T$.

2. MAIN RESULTS

In this section, we consider the problem (1.1)–(1.2) for the case when $q > 0$. The Green's function $G(x,t;\xi,\tau)$ corresponding to the problem (1.1) is determined by the following system:

$$L_q G = \delta(x-\xi)\delta(t-\tau),$$

with $G(x,t;\xi,\tau)=0$ for $t<\tau$, and $G(0,t;\xi,\tau)=0=G(a,t;\xi,\tau)$. By Chan and Chan [3], we have

$$G(x,t;\xi,\tau)=\sum_{n=1}^{\infty}\phi_n(x)\phi_n(\xi)e^{-\lambda_n(t-\tau)},$$

where λ_n $(n=1,2,\dots)$ are the eigenvalues of the problem

$$\phi''+\lambda x^q\phi=0,\quad \phi(0)=0=\phi(a),$$

and their corresponding eigenfunctions are given by

$$\phi_n(x)=(q+2)^{1/2}x^{1/2}\frac{J_{\frac{1}{q+2}}\left(\frac{2\lambda_n^{1/2}}{q+2}x^{(q+2)/2}\right)}{\left|J_{1+\frac{1}{q+2}}\left(\frac{2\lambda_2^{1/2}}{q+2}\right)\right|}$$

where $J_{1/(q+2)}$ is the Bessel function of the first kind of order $1/(q+2)$. The eigenvalues satisfies $0<\lambda_1<\lambda_2<\cdots<\lambda_n<\lambda_{n+1}<\cdots$, $\lambda_n\approx O(n^2)$, and the set $\{\phi_n(x)\}$ is a maximal orthonormal set with the weight function x^q (cf. [18, p. 506]).

By using the Green's function $G(x,t;\xi,\tau)$, the solution $u(x,t)$ of the problem (1.1)–(1.2) is given as

$$u(x,t)=\frac{2}{a}\int_0^t\sum_{n=1}^{\infty}\phi_n(x)\phi_n(b)e^{-\lambda_n(t-\tau)}(1-u(b,\tau))^{-p}d\tau,$$

for $0<t$. The solution $u(x,t)$ can be shown to be unique, continuous, increasing with respect to t in Ω. Furthermore, $u(x,t)$ satisfies the problem (1.1)–(1.2). Also there is a positive real number a^* such that for $a>a^*$, the solution $u(x,t)$ quenches in a finite T, and b is the only quenching point (cf. [6]). Chan and Tian [7] shows that there is a positive constant k such that $|\phi_n(x)|\le kx^{-q/4}$ for $x\in D$.

Theorem 2.1. *If the solution $u(x,t)$ of the problem* (1.1) *quenches in a finite time T at $x=b\in D$, then there exist two positive numbers C_1 and C_2 such that* $\varliminf_{t\to T}(1-u(b,t))(T-t)^{-1/(p+1)}=C_1$, *and* $\varlimsup_{t\to T}(1-u(b,t))(T-t)^{-1/(p+1)}=C_2$.

Proof. Firstly, let us determine a lower bound of $(1-u(x,t))$ when t is close to T.

Since $u(b,t)\to 1$ as $t\to T$ and $u(x,t)$ is continuous, we have

$$\frac{2}{a}\int_0^T\sum_{n=1}^{\infty}\left(\phi_n(b)\right)^2e^{-\lambda_n(T-\tau)}(1-u(b,\tau))^{-p}d\tau=1.$$

By a direct computation, we have

$$\begin{aligned}
1-u(b,t) &= \frac{2}{a}\sum_{n=1}^{\infty}(\phi_n(b))^2\left(\int_0^T e^{-\lambda_n(T-\tau)}(1-u(b,\tau))^{-p}d\tau\right.\\
&\quad\left.-\int_0^t e^{-\lambda_n(t-\tau)}(1-u(b,\tau))^{-p}d\tau\right)\\
&= \frac{2}{a}\sum_{n=1}^{\infty}(\phi_n(b))^2\left[\int_0^t\left(e^{-\lambda_n(T-\tau)}-e^{-\lambda_n(t-\tau)}\right)(1-u(b,\tau))^{-p}d\tau\right.\\
&\quad\left.+\int_t^T e^{-\lambda_n(T-\tau)}(1-u(b,\tau))^{-p}d\tau\right]\\
&\geq \frac{2}{a}\sum_{n=1}^{\infty}(\phi_n(b))^2(1-u(b,t))^{-p}\left[\int_0^t\left(e^{-\lambda_n(T-\tau)}-e^{-\lambda_n(t-\tau)}\right)d\tau\right.\\
&\quad\left.+\int_t^T e^{-\lambda_n(T-\tau)}d\tau\right]\\
&= \frac{2}{a}\sum_{n=1}^{\infty}(\phi_n(b))^2(1-u(b,t))^{-p}\left[\frac{1}{\lambda_n}\left(e^{-\lambda_n t}-e^{-\lambda_n T}\right)\right].
\end{aligned}$$

It follows from the Mean Value Theorem that there is a η satisfying $t<\eta<T$, such that $e^{-\lambda_n t}-e^{-\lambda_n T}=\lambda_n e^{-\lambda_n \eta}(T-t)$. Then we obtain

$$1-u(b,t)\geq\left(\frac{2}{a}\sum_{n=1}^{\infty}(\phi_n(b))^2 e^{-\lambda_n T}\right)(T-t)(1-u(b,t))^{-p}.$$

Since $|\phi_n(b)|\leq kb^{-q/4}$ for some constant $k>0$, the series $\sum_{n=1}^{\infty}(\phi_n(b))^2 e^{-\lambda_n T}$ converges. Thus, there is $K_1>0$ such that

$$(1-u(b,t))^{p+1}\geq K_1(T-t) \tag{2.1}$$

for any $0<t<T$. This gives $\underline{\lim}_{t\to T}(1-u(b,t))(T-t)^{-1/(p+1)}=C_1$, for some positive constant C_1.

For the upper bound, we consider

$$\begin{aligned}
1-u(b,t) &= \frac{2}{a}\sum_{n=1}^{\infty}(\phi_n(b))^2\left(\int_0^T e^{-\lambda_n(T-\tau)}(1-u(b,\tau))^{-p}d\tau\right.\\
&\quad\left.-\int_0^t e^{-\lambda_n(t-\tau)}(1-u(b,\tau))^{-p}d\tau\right)\\
&= \frac{2}{a}\sum_{n=1}^{\infty}(\phi_n(b))^2\left[\int_0^t\left(e^{-\lambda_n(T-\tau)}-e^{-\lambda_n(t-\tau)}\right)(1-u(b,\tau))^{-p}d\tau\right.\\
&\quad\left.+\int_t^T e^{-\lambda_n(T-\tau)}(1-u(b,\tau))^{-p}d\tau\right].
\end{aligned}$$

It follows from the Mean Value Theorem that there is a ρ satisfying $t < \rho < T$ such that

$$\begin{aligned}\int_0^t & \left(e^{-\lambda_n(T-\tau)} - e^{-\lambda_n(t-\tau)}\right)(1-u(b,\tau))^{-p}d\tau \\ &= -\lambda_n e^{-\lambda_n \rho}(T-t)\int_0^t (1-u(b,\tau))^{-p}d\tau.\end{aligned}$$

On the other hand, by use of the inequality (2.1), we have

$$\int_t^T e^{-\lambda_n(T-\tau)}(1-u(b,\tau))^{-p}d\tau \le K_1^{-p/(p+1)}\int_t^T e^{-\lambda_n(T-\tau)}(T-\tau)^{-p/(p+1)}d\tau. \tag{2.2}$$

By using the substitution $s = T-\tau$, the integral on the right-hand side of the above inequality becomes

$$\begin{aligned}\int_0^{T-t} & e^{-\lambda_n s}s^{-p/(p+1)}ds = (p+1)(T-t)^{-p/[2(p+1)]}\left(\lambda_n\right)^{-1+p/[2(p+1)]} \\ & \times e^{-(1/2)\lambda_n(T-t)}M\left(-\tfrac{p}{2(p+1)}, -\tfrac{p}{2(p+1)}+\tfrac{1}{2}, \lambda_n(T-t)\right),\end{aligned}$$

where $M(k,m,z)$ is the Whittaker function (cf. [19]). The Whittaker function can be rewritten in terms of confluent hypergeometric function as

$$M(k,m,z) = z^{m+\frac{1}{2}}e^{-\frac{z}{2}}\Phi\left(m-k+\frac{1}{2}, 2m+1, z\right),$$

where $\Phi(\alpha,\gamma,z)$ is the confluent hypergeometric function. Note that the hypergeometric function Φ has a Kummer series expansion

$$\Phi(\alpha,\gamma,z) = 1 + \frac{\alpha}{\gamma}z + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)2!}z^2 + \cdots,$$

which shows that this function is entire for any z. Then, we get

$$\begin{aligned}& M\left(-\frac{p}{2(p+1)}, -\frac{p}{2(p+1)}+\frac{1}{2}, \lambda_n(T-t)\right) \\ &= \left[\lambda_n(T-t)\right]^{1-\frac{p}{2(p+1)}} e^{-(1/2)\lambda_n(T-t)}\Phi\left(1, 1+\frac{1}{p+1}, \lambda_n(T-t)\right).\end{aligned}$$

By putting the Whittaker function back into the estimation (2.2), we obtain

$$\begin{aligned}\int_t^T & e^{-\lambda_n(T-\tau)}(1-u(b,\tau))^{-p}d\tau \\ \le\ & K_1^{-p/(p+1)}(p+1)(T-t)^{1/(p+1)}e^{-\lambda_n(T-t)}\Phi\left(1, 1+\frac{1}{p+1}, \lambda_n(T-t)\right).\end{aligned}$$

For $0 < t_0 < T$, since $e^{-\lambda_n(T-t)}\Phi(1, 1+1/(p+1), \lambda_n(T-t))$ is continuous and is decreasing on $[0,T]$, there is $M > 0$ such that for any $t_0 \le t_1 \le t_2 \le T$, we have

$$\begin{aligned}&\left|e^{-\lambda_n(T-t_1)}\Phi\left(1, 1+\frac{1}{p+1}, \lambda_n(T-t_1)\right) - e^{-\lambda_n(T-t_2)}\Phi\left(1, 1+\frac{1}{p+1}, \lambda_n(T-t_2)\right)\right| \\ &\le M\left|e^{-\lambda_n(T-t_0)}\Phi\left(1, 1+\frac{1}{p+1}, \lambda_n(T-t_0)\right)\right|.\end{aligned}$$

Thus, for any $t_0 \le t \le T$, we get

$$\begin{aligned}&\left| e^{-\lambda_n(T-t)}\Phi\left(1, 1+\frac{1}{p+1}, \lambda_n(T-t)\right)\right| \\ &\quad \le (M+1)\left| e^{-\lambda_n(T-t_0)}\Phi\left(1, 1+\frac{1}{p+1}, \lambda_n(T-t_0)\right)\right|.\end{aligned}$$

Since $\lambda_n e^{-\lambda_n(T-t_0)}\Phi(1, 1+1/(p+1), \lambda_n(T-t_0)) \to 0$ as $n\to\infty$, there exists $K_2 > 0$ such that

$$K_1^{-p/(p+1)}(p+1)\lambda_n e^{-\lambda_n(T-t)}\Phi\left(1, 1+\frac{1}{p+1}, \lambda_n(T-t)\right) \le K_2.$$

This gives

$$\begin{aligned}&\int_0^t \left(e^{-\lambda_n(T-\tau)} - e^{-\lambda_n(t-\tau)}\right)(1-u(b,\tau))^{-p}d\tau \\ &\quad + \int_t^T e^{-\lambda_n(T-\tau)}(1-u(b,\tau))^{-p}d\tau \\ &\le -\lambda_n e^{-\lambda_n \rho}(T-t)\int_0^t (1-u(b,\tau))^{-p}d\tau \\ &\quad + K_1^{-p/(p+1)}(p+1)(T-t)^{1/(p+1)}e^{-\lambda_n(T-t)}\Phi\left(1, 1+\frac{1}{p+1}, \lambda_n(T-t)\right) \\ &\le \frac{K_2}{\lambda_n}(T-t)^{1/p+1}.\end{aligned}$$

Thus,

$$1-u(b,t) \le \frac{2K_2}{a}(T-t)^{1/p+1}\sum_{n=1}^{\infty}(\phi_n(b))^2\frac{1}{\lambda_n}.$$

Since $\sum_{n=1}^{\infty}(\phi_n(b))^2\frac{1}{\lambda_n}$ converges, we have $1-u(b,t) \le K_3(T-t)^{1/p+1}$ for some $K_3 > 0$. This shows that $\overline{\lim_{t\to T}}(1-u(b,t))(T-t)^{-1/(p+1)} = C_2$. □

When $q = 0$, the operator L_0 in the problem (1.1) becomes $L_0 u = u_t - u_{xx}$ which is the heat operator, and its corresponding Green's function on Ω is given as (cf. [1])

$$G(x,t;\xi,\tau) = \frac{2H(t-\tau)}{a}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{a}\sin\frac{n\pi\xi}{a}e^{-\left(\frac{n\pi}{a}\right)^2(t-\tau)},$$

where $H(t-\tau)$ is the Heaviside function. The representation form of the solution $u(x,t)$ of the problem (1.1)–(1.2) is given as

$$u(x,t) = \frac{2}{a}\int_0^t\sum_{n=1}^{\infty}\sin\frac{n\pi x}{a}\sin\frac{n\pi b}{a}e^{-\left(\frac{n\pi}{a}\right)^2(t-\tau)}(1-u(b,\tau))^{-p}d\tau,$$

for $0 < t$. The solution $u(x,t)$ can be shown to be unique, continuous, increasing with respect to time t in Ω. Furthermore, $u(x,t)$ satisfies the problem (1.1)–(1.2). Also there is a positive real number a^*, for $a > a^*$, the solution $u(x,t)$ quenches in a finite T, and b is the only quenching point. Since $|\sin\frac{n\pi b}{a}| \le 1$, the series $\sum_{n=1}^{\infty}\left(\sin\frac{n\pi b}{a}\right)^2\frac{1}{\left(\frac{n\pi}{a}\right)^2}$ converges. It follows from a similar argument as in the proof of the Theorem 2.1 that the quenching rate can be estimated to obtain the following result.

Corollary 2.2. *For $q = 0$, if the solution $u(x,t)$ of the problem (1.1) quenches in a finite time T at $x = b$, then there exist two positive numbers C_3 and C_4 such that* $\underline{\lim}_{t\to T}(1-u(b,t))(T-t)^{-1/(p+1)} = C_3$, *and* $\overline{\lim}_{t\to T}(1-u(b,t))(T-t)^{-1/(p+1)} = C_4$.

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