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A NEW CLASS OF DYNAMIC INEQUALITIES OF HARDY'S TYPE ON TIME SCALES

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ABSTRACT. In this paper, the authors prove some new dynamic inequalities of Hardy's types on time scales. The proofs make use of some algebraic inequalities, the Hölder inequality, and a simple consequence of Keller's chain rule on time scales. The well-known Hardy type inequalities in differential and difference forms with a best constant 1/4 are derived as special cases.

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1. INTRODUCTION

The classical Hardy inequality states that if $f \geq 0$ is integrable over any finite interval $(0, x)$, $p > 1$, and f^p is integrable and its integral over $(0, \infty)$ converges, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{1.1}$$

and equality holds if and only if $f(x) = 0$ a.e. The constant $(p/(p-1))^p$ is the best possible. This inequality was proved by Hardy in 1925, but it appeared as the continuous version of a discrete inequality in his work in 1920 when he tried to find a new elementary proof of Hilbert's inequality for double series. Hardy [6] showed that this inequality follows from the discrete version of (1.1):

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n, \quad (a_n > 0 \;\; p > 1).$$

These two inequalities are known in the literature as Hardy-Hilbert type inequalities. Since the discovery of these inequalities, many papers containing new proofs, various generalizations, and extensions have appeared. Hardy's inequality (1.1) was generalized by Hardy himself in [7] where he showed that, for $p > 1$ and any integrable

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function $f(x) > 0$ on $(0, \infty)$,

$$\int_0^\infty \frac{1}{x^m}\left(\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{m-1}\right)^p \int_0^\infty \frac{1}{x^{m-p}} f^p(x)dx, \quad m > 1, \tag{1.2}$$

and

$$\int_0^\infty \frac{1}{x^m}\left(\int_x^\infty f(t)dt\right)^p dx \leq \left(\frac{p}{1-m}\right)^p \int_0^\infty \frac{1}{x^{m-p}} f^p(x)dx, \ m < 1, \tag{1.3}$$

The study of Hardy inequalities (continuous and discrete) or Hardy operators focused on the investigations of new inequalities or operators with weight functions. These results are of interest and important in analysis, not only because the mappings are optimal in the sense that the size of weight classes cannot be improved, but also because the weight conditions themselves are of interest. These inequalities have natural applications in the theory of differential equations (ordinary and partial) and have led to many interesting questions and connections between different areas of mathematical analysis. For example, Hardy inequalities are closely related to the quasi-additivity properties of capacitors [1] and recently have been used to find the gaps between zeros of solutions of differential equations arising in the bending of beams [22]. This intensively investigated area of mathematical analysis resulted in the publication of numerous research papers and monographs, and we refer the reader to the books [12, 13, 17] and the papers [2, 5, 10, 11, 14–16, 19, 20].

In what follows, we assume that the reader is familiar with the concepts and notation from time scale calculus as can be found in Bohner and Peterson [3, 4].

Recently, a number of dynamic inequalities of Hardy's type on time scales have been established [18, 21, 23, 25]. In [21], Řehak proved a time scale version of (1.1) showing that if $p > 1$ and g is nonnegative and such that the delta integral $\int_a^\infty g^p(t)\Delta t$ exits as a finite number, then

$$\int_a^\infty \left(\frac{1}{\sigma(x) - a}\int_a^{\sigma(x)} g(t)\Delta t\right)^p \Delta x \leq \left(\frac{p}{p-1}\right)^p \int_a^\infty g^p(x)dx. \tag{1.4}$$

If, in addition, $\mu(t)/t \to 0$ as $t \to \infty$, then the constant is the best possible. However, it is an open problem to determine whether the constant in inequality (1.4) is the best possible on all time scales or just those satisfying $\lim_{t\to\infty}(\mu(t)/t) = 0$.

Ozkan and Yildirim [18] established a new inequality with weight functions that can be considered as a time scale version of inequality (1.6) and proved that if $u \in C_{rd}([a, b], \mathbb{R})$ is a nonnegative function such that the delta integral $\int_t^b \frac{u(s)}{(s-a)(\sigma(s)-a)}\Delta s$ exists as a finite number, the function v is defined by

$$v(t) = (t - a)\int_t^b \frac{u(s)}{(s-a)(\sigma(s) - a)}\Delta s, \ \ t \in [a, b],$$

and $\Phi : (c, d) \to \mathbb{R}$, c, $d \in \mathbb{R}$, is continuous and convex, then the inequality

$$\int_a^b u(t)\Phi\left(\frac{1}{(\sigma(t)-a)}\int_a^{\sigma(t)} g(s)\Delta s\right)\frac{\Delta t}{t-a} \le \int_a^b v(t)\Phi(g(t))\frac{\Delta t}{t-a}, \tag{1.5}$$

holds for all delta integrable functions $g \in C_{rd}([a,b],\mathbb{R})$ such that $g(t) \in (c,d)$. Inequality (1.5) can be considered as the time scale version of the (Hardy-Knopp type) inequality

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{dx}{x} \le \int_0^\infty \Phi\left(f(x)\right)\frac{dx}{x}, \tag{1.6}$$

that was proved by Kaijser et al. [11], where Φ is a convex function on $(0,\infty)$.

We assume that our time scale $\mathbb{T}$ satisfies $\sup \mathbb{T} = \infty$, and we define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty) \cap \mathbb{T}$.

Our aim in this paper is to prove some new inequalities of Hardy's type on time scales by making use of the chain rule, Hölder's inequality, and some algebraic inequalities. The inequalities to be proved contain the inequalities (1.2) and (1.3) and some new discrete inequalities as special cases. As one special case of our results, we establish the well-known inequality due to Hardy

$$\int_0^1 \left(U'(t)\right)^2 dt \ge \frac{1}{4}\int_0^1 \frac{1}{t^2}U^2(t), \text{ with } U(0)=0,$$

where $1/4$ is the best possible constant.

2. MAIN RESULTS

In this section, we will prove our main results. As indicated above, concepts and results on the time scale calculus as can be found in [3, 4] will be used as needed. However, it will be convenient to have the following at our disposal. First, the expression

$$\left(x^\lambda(t)\right)^\Delta = \lambda\left\{\int_0^1 \left[hx^\sigma(t)+(1-h)x(t)\right]^{\lambda-1} dh\right\} x^\Delta(t), \tag{2.1}$$

is a well known consequence of Keller's chain rule [3, Theorem 1.90]. And using the fact that $g(\sigma(t)) = g(t) + \mu(t)g^\Delta(t)$, we have

$$\left(x^\lambda(t)\right)^\Delta = \lambda\left\{\int_0^1 \left[x(t)+h\mu(t)x^\Delta(t)\right]^{\lambda-1} dh\right\} x^\Delta(t). \tag{2.2}$$

For easy reference, we also give the time scale integration by parts formula

$$\int_a^b u(t)v^\Delta(t)\Delta t = u(t)v(t)\big|_a^b - \int_a^b u^\Delta(t)v^\sigma(t)\Delta t. \tag{2.3}$$

Finally, to prove our main results, we will use the following Hölder Inequality [3, Theorem 6.13]:

Let a, $b \in \mathbb{T}$. For u, $v \in C_{rd}(\mathbb{T}, \mathbb{R})$, we have

$$\int_a^b |u(t)v(t)|\,\Delta t \le \left[\int_a^b |u(t)|^p\,\Delta t\right]^{\frac{1}{p}} \left[\int_a^b |v(t)|^q\,\Delta t\right]^{\frac{1}{q}}, \tag{2.4}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Throughout the paper, we will assume that the functions f and g are nonnegative rd-continuous functions that are Δ-differentiable, locally delta integrable, and the left hand sides of the inequalities exits if the right hand sides exist. We will assume also, without loss of generality, that

$$\frac{s}{\sigma(s)} \ge \frac{1}{K} \quad \text{for} \quad s \ge a, \tag{2.5}$$

for some constant $K > 0$. Now, we are ready to state and prove the main results in this paper and begin with the case where $p/q \ge 2$.

Theorem 2.1. *Let $\mathbb{T}$ be a time scale, $a \in \mathbb{T}$, $\gamma > 1$, and p and q be real numbers with $p > q > 0$ and $p/q \ge 2$. Assume that f is nonincreasing on $[a, \infty)_{\mathbb{T}}$ and define*

$$\Lambda(t) := \frac{1}{f(t)} \int_a^t \frac{f(s)g(s)}{s}\Delta s, \quad \textit{for} \quad t \in [a, \infty)_{\mathbb{T}}. \tag{2.6}$$

If

$$1 + \frac{p\left(2^{p/q-2}\right)K^\gamma}{q\,(\gamma - 1)} \frac{t f^\Delta(t)}{f^\sigma(t)} \ge \frac{1}{m} > 0 \tag{2.7}$$

for some constant $m > 0$, then

$$\int_a^\infty \frac{1}{t^\gamma}\left[\Lambda^{\frac{p}{q}}(t) - \frac{t\,2^{\frac{p}{q}-2} m K^\gamma}{\gamma - 1}\mu^{\frac{p}{q}-1}(t)\left(\Lambda^\Delta(t)\right)^{\frac{p}{q}}\right]\Delta t \le \frac{2^{\frac{p}{q}-2} p m K^\gamma}{q\,(\gamma - 1)} \left[\int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)g(t)}{f^\sigma(t)}\right)^{\frac{p}{q}}\Delta t\right]^{\frac{q}{p}} \left[\int_a^\infty \frac{\Lambda^{\frac{p}{q}}(t)}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}}. \tag{2.8}$$

Proof. Integrating the term $\int_a^\infty \frac{1}{t^\gamma}\left(\Lambda^\sigma(t)\right)^{p/q}\Delta t$ by parts with

$$u^\Delta(t) = \frac{1}{t^\gamma} \quad \text{and} \quad v^\sigma(t) = \left(\Lambda^\sigma(t)\right)^{p/q},$$

we obtain

$$\int_a^\infty \frac{\left(\Lambda^\sigma(t)\right)^{p/q}}{t^\gamma}\Delta t = uv|_a^\infty + \int_a^\infty (-u(t))\left(\Lambda^{p/q}(t)\right)^\Delta \Delta t, \tag{2.9}$$

where

$$u(t) = \int_t^\infty \left(\frac{-1}{s^\gamma}\right)\Delta s. \tag{2.10}$$

Applying the chain rule (2.1), we see that

$$\begin{aligned}\left(\frac{-1}{s^{\gamma-1}}\right)^{\Delta} &= (\gamma-1)\int_0^1 \frac{1}{[h\sigma(s)+(1-h)s]^{\gamma}}dh \\ &\geq (\gamma-1)\int_0^1 \frac{1}{[h\sigma(s)+(1-h)\sigma(s)]^{\gamma}}dh \\ &= \int_0^1 \left(\frac{\gamma-1}{\sigma^{\gamma}(s)}\right)dh = \frac{(\gamma-1)}{\sigma^{\gamma}(s)}. \qquad (2.11)\end{aligned}$$

Then (2.5) and (2.11) imply

$$\left(\frac{-1}{s^{\gamma-1}}\right)^{\Delta} \geq \frac{(\gamma-1)}{K^{\gamma}s^{\gamma}}.$$

Hence,

$$\begin{aligned}\int_t^{\infty} \frac{-1}{s^{\gamma}}\Delta s &\geq \frac{-K^{\gamma}}{(\gamma-1)}\int_t^{\infty}\left(\frac{-1}{s^{\gamma-1}}\right)^{\Delta}\Delta s = \frac{K^{\gamma}}{(\gamma-1)}\left(\frac{1}{s^{\gamma-1}}\right)\Big|_t^{\infty} \\ &= \frac{-K^{\gamma}}{(\gamma-1)}\left(\frac{1}{t^{\gamma-1}}\right). \qquad (2.12)\end{aligned}$$

Therefore,

$$-u(t) = -\int_t^{\infty}\left(\frac{-1}{s^{\gamma}}\right)\Delta s \leq \frac{K^{\gamma}}{\gamma-1}\left(\frac{1}{t^{\gamma-1}}\right). \qquad (2.13)$$

Using (2.2), we have

$$\left(\Lambda^{p/q}(t)\right)^{\Delta} = \frac{p}{q}\left\{\int_0^1 \left[\Lambda+\mu h\Lambda^{\Delta}\right]^{\frac{p}{q}-1}dh\right\}\Lambda^{\Delta}(t). \qquad (2.14)$$

From (2.9), (2.13), and (2.14), and the fact that $u(\infty)=0$ and $\Lambda(a)=0$, we have

$$\int_a^{\infty}\frac{1}{t^{\gamma}}\left(\Lambda^{\sigma}(t)\right)^{p/q}\Delta t \leq \frac{pK^{\gamma}}{q(\gamma-1)}\left\{\int_a^{\infty}\frac{1}{t^{\gamma-1}}\int_0^1\left[\Lambda+\mu h\Lambda^{\Delta}\right]^{\frac{p}{q}-1}dh\right\}\Lambda^{\Delta}(t)\Delta t. \qquad (2.15)$$

From the definition of $\Lambda(t)$ and the fact that $f^{\Delta}(t)\leq 0$ we see that

$$\begin{aligned}\Lambda^{\Delta}(t) &= \left(\frac{1}{f(t)}\int_a^t \frac{f(s)g(s)}{s}\Delta s\right)^{\Delta} = \frac{f(t)g(t)}{tf^{\sigma}(t)} - \frac{f^{\Delta}(t)\int_a^t \frac{f(s)g(s)}{s}\Delta s}{f(t)f^{\sigma}(t)} \\ &= \frac{f(t)g(t)}{tf^{\sigma}(t)} - \frac{f^{\Delta}(t)}{f^{\sigma}(t)}\Lambda(t) > 0. \qquad (2.16)\end{aligned}$$

Recalling that $p/q\geq 2$ and applying the inequality

$$a^{\lambda}+b^{\lambda} \leq (a+b)^{\lambda} \leq 2^{\lambda-1}(a^{\lambda}+b^{\lambda}) \quad \text{if} \quad a,b\geq 0 \quad \text{and} \quad \lambda\geq 1, \qquad (2.17)$$

to the expression $\left[\Lambda+h\mu\Lambda^{\Delta}\right]^{(p/q)-1}$ gives

$$\frac{p}{q}\int_0^1\left[\Lambda+h\mu\Lambda^{\Delta}\right]^{\frac{p}{q}-1}dh \leq \frac{p}{q}2^{(p/q)-2}\Lambda^{\frac{p}{q}-1}(t) + 2^{(p/q)-2}(\mu\Lambda^{\Delta})^{\frac{p}{q}-1}. \qquad (2.18)$$

Substituting (2.16) and (2.18) into (2.15), we get

$$\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma}\Delta t \;\leq\; \frac{p\left(2^{p/q-2}\right)K^\gamma}{q(\gamma-1)}\int_a^\infty \frac{(\Lambda(t))^{p/q-1}}{t^{\gamma-1}}\left[\frac{f(t)g(t)}{tf^\sigma(t)}-\frac{f^\Delta(t)}{f^\sigma(t)}\Lambda(t)\right]\Delta t + \frac{2^{(p/q)-2}K^\gamma}{\gamma-1}\int_a^\infty \frac{(\mu(t))^{p/q-1}\left(\Lambda^\Delta(t)\right)^{p/q}}{t^{\gamma-1}}\Delta t.$$

Now $(\Lambda^\sigma(t))^{p/q} > (\Lambda(t))^{p/q}$ since $\Lambda^\Delta(t) > 0$, so

$$\begin{aligned}\int_a^\infty \frac{1}{t^\gamma}(\Lambda(t))^{p/q}&\left[1+\frac{p\left(2^{(p/q-2)}\right)K^\gamma}{q(\gamma-1)}\frac{tf^\Delta(t)}{f^\sigma(t)}\right]\Delta t\\ &\leq \frac{2^{p/q-2}pK^\gamma}{q(\gamma-1)}\int_a^\infty\left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{f^\sigma(t)}\right]\left[(t^\gamma)^{-(p-q)/p}\Lambda^{(p-q)/q}(t)\right]\Delta t\\ &+\frac{2^{p/q-2}K^\gamma}{\gamma-1}\int_a^\infty \frac{\mu^{\frac{p}{q}-1}(t)}{t^{\gamma-1}}(\Lambda^\Delta(t))^{p/q}\Delta t. \end{aligned}\tag{2.19}$$

Applying the Hölder inequality (2.4) with indices p/q and $p/(p-q)$ to the expression

$$\int_a^\infty\left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{f^\sigma(t)}\right]\left[(t^\gamma)^{-(p-q)/p}\Lambda^{(p-q/q)}(t)\right]\Delta t$$

gives

$$\begin{aligned}\int_a^\infty&\left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{f^\sigma(t)}\right]\left[(t^\gamma)^{-(p-q)/p}\Lambda^{(p-q/q)}(t)\right]\Delta t\\ &\leq\left[\int_a^\infty\left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{tf^\sigma(t)}\right]^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty\frac{\Lambda^{p/q}(t)}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}}.\end{aligned}\tag{2.20}$$

Substituting (2.20) into (2.19), we have

$$\begin{aligned}\int_a^\infty \frac{1}{t^\gamma}(\Lambda(t))^{\frac{p}{q}}&\left[1+\frac{p\left(2^{p/q-2}\right)K^\gamma}{q(\gamma-1)}\frac{tf^\Delta(t)}{f^\sigma(t)}\right]\Delta t\\ &\leq\frac{2^{\frac{p}{q}-2}(p/q)K^\gamma}{\gamma-1}\left[\int_a^\infty\left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{f^\sigma(t)}\right]^{\frac{p}{q}}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty\frac{\Lambda^{\frac{p}{q}}(t)}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}}\\ &+\frac{2^{\frac{p}{q}-2}K^\gamma}{\gamma-1}\int_a^\infty\frac{\mu^{p/q-1}(t)}{t^{\gamma-1}}(\Lambda^\Delta(t))^{p/q}\Delta t.\end{aligned}$$

From condition (2.7), we obtain

$$\begin{aligned}\frac{1}{m}\int_a^\infty &\frac{1}{t^\gamma}(\Lambda(t))^{p/q}\Delta t\\ &\leq\frac{2^{\frac{p}{q}-2}pK^\gamma}{q(\gamma-1)}\left[\int_a^\infty\left[\frac{(t^\gamma)^{(p-q)/p}}{t^{\gamma-1}}\frac{f(t)g(t)}{tf^\sigma(t)}\right]^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty\frac{\Lambda^{p/q}(t)}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}}\\ &+\frac{2^{\frac{p}{q}-2}K^\gamma}{\gamma-1}\int_a^\infty\frac{\mu^{\frac{p}{q}-1}(t)}{t^{\gamma-1}}(\Lambda^\Delta(t))^{\frac{p}{q}}\Delta t.\end{aligned}$$

This implies

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda(t))^{p/q}\Delta t - \frac{2^{p/q-2}mK^\gamma}{\gamma-1}\int_a^\infty \frac{\mu^{p/q-1}(t)}{t^{\gamma-1}}\left(\Lambda^\Delta(t)\right)^{p/q}\Delta t$$

$$\leq \frac{2^{p/q-2}pmK^\gamma}{q(\gamma-1)}\left[\int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)}{f^\sigma(t)}\right)^{p/q} g^{p/q}(t)\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty \frac{\Lambda^{p/q}(t)}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}},$$

and upon simplification, we get the desired inequality (2.8). This completes the proof of the theorem. □

Theorem 2.2. *Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$, $\gamma > 1$, and p and q be real numbers with $p > q > 0$ and $p/q \geq 2$, and f be a nonincreasing function. If*

$$1 + \frac{pK^\gamma}{q(\gamma-1)}\left(\frac{\Lambda(t)}{\Lambda^\sigma(t)}\right)^{\frac{p}{q}}\frac{tf^\Delta(t)}{f^\sigma(t)} \geq \frac{1}{m} > 0 \tag{2.21}$$

for some constant $m > 0$, then

$$\int_a^\infty \frac{1}{t^\gamma}\left(\frac{1}{f^\sigma(t)}\int_a^{\sigma(t)}\frac{f(s)g(s)}{s}\Delta s\right)^{\frac{p}{q}}\Delta t \leq \left(\frac{pmK^\gamma}{q(\gamma-1)}\right)^{\frac{p}{q}}\int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)g(t)}{f^\sigma(t)}\right)^{\frac{p}{q}}\Delta t. \tag{2.22}$$

Proof. We proceed as in the proof of Theorem 2.1 to obtain

$$\int_a^\infty \frac{1}{t^\gamma}\left(\Lambda^\sigma(t)\right)^{p/q}\Delta t \leq \frac{K^\gamma}{(\gamma-1)}\int_a^\infty \frac{1}{t^{\gamma-1}}(\Lambda^{p/q}(t))^\Delta\Delta t. \tag{2.23}$$

Applying the chain rule [3, Theorem 1.87]

$$F^\Delta(g(t)) = F'(g(c))g^\Delta(t) \quad \text{where} \quad c \in [t, \sigma(t)],$$

to the term $\left(\Lambda^{p/q}(t)\right)^\Delta$, we see that

$$\left(\Lambda^{p/q}(t)\right)^\Delta = \frac{p}{q}\Lambda^{\frac{p}{q}-1}(c)\Lambda^\Delta(t) \quad \text{for} \quad c \in [t, \sigma(t)]. \tag{2.24}$$

From (2.16), we see that $\Lambda^\sigma(t) \geq \Lambda(c)$ since $\sigma(t) \geq c$. Substituting this into (2.23) and using (2.16), we have

$$\begin{aligned}
\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma}\Delta t &\leq \frac{pK^\gamma}{q(\gamma-1)}\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q-1}}{t^{\gamma-1}}\left[\frac{f(t)g(t)}{tf^\sigma(t)} - \frac{f^\Delta(t)}{f^\sigma(t)}\Lambda(t)\right]\Delta t \\
&= \frac{pK^\gamma}{q(\gamma-1)}\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q-1}}{t^{\gamma-1}}\frac{f(t)g(t)}{tf^\sigma(t)}\Delta t \\
&\quad - \frac{pK^\gamma}{q(\gamma-1)}\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q-1}\Lambda(t)}{t^{\gamma-1}}\frac{f^\Delta(t)}{f^\sigma(t)}\Delta t \\
&\leq \frac{pK^\gamma}{q(\gamma-1)}\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q-1}}{t^{\gamma-1}}\frac{f(t)g(t)}{tf^\sigma(t)}\Delta t \\
&\quad - \frac{pK^\gamma}{q(\gamma-1)}\int_a^\infty \frac{(\Lambda(t))^{p/q}}{t^{\gamma-1}}\frac{f^\Delta(t)}{f^\sigma(t)}\Delta t.
\end{aligned}$$

Hence,

$$\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma}\left[1+\frac{pK^\gamma}{q(\gamma-1)}\left(\frac{\Lambda(t)}{\Lambda^\sigma(t)}\right)^{p/q}\frac{tf^\Delta(t)}{f^\sigma(t)}\right]\Delta t$$

$$\leq \frac{pK^\gamma}{q(\gamma-1)}\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q-1}}{t^{\gamma-1}}\frac{f(t)g(t)}{tf^\sigma(t)}\Delta t. \tag{2.25}$$

Using condition (2.21) and rewriting the right hand side, we have

$$\begin{aligned}\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma}\Delta t &\leq \frac{pmK^\gamma}{q(\gamma-1)}\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q-1}}{t^{\gamma-1}}\frac{f(t)g(t)}{tf^\sigma(t)}\Delta t\\ &= \frac{pmK^\gamma}{q(\gamma-1)}\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{f^\sigma(t)}\right]\left[(t^\gamma)^{-(p-q)/p}(\Lambda^\sigma(t))^{(p-q)/q}\right]\Delta t.\end{aligned}$$

We now apply Hölder's inequality (2.4) with indices p/q and $p/(p-q)$ to the right hand side to obtain

$$\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma}\Delta t \leq \left(\frac{pmK^\gamma}{q(\gamma-1)}\right)\left[\int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)g(t)}{f^\sigma(t)}\right)^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}},$$

which gives the desired inequality (2.22). This completes the proof of the theorem. □

From (2.16), we have

$$\Lambda^\Delta(t) = \frac{f(t)g(t)}{tf^\sigma(t)} - \frac{f^\Delta(t)}{f^\sigma(t)}\Lambda(t). \tag{2.26}$$

Notice that the condition $f^\Delta(t) \leq 0$ in Theorems 2.1 and 2.2 can be replaced by $f^\Delta(t) \geq 0$ provided we require the additional condition

$$\frac{f(t)g(t)}{tf^\sigma(t)} \geq \frac{f^\Delta(t)}{f^\sigma(t)}\Lambda(t).$$

Using this inequality, we see that $\Lambda^\Delta(t) \geq 0$ and we would have

$$\Lambda^\Delta(t) \leq \frac{f(t)g(t)}{tf^\sigma(t)}. \tag{2.27}$$

Proceeding as in the proof of Theorem 2.2 and using (2.27) in (2.24), we can obtain the following result.

Theorem 2.3. *Let $\mathbb{T}$ be a time scale, $a \in \mathbb{T}$, $\gamma > 1$, p and q be real numbers with $p > q > 0$ and $p/q \geq 2$, and let f be nondecreasing. If (2.27) holds, then*

$$\int_a^\infty \frac{1}{t^\gamma}\left(\frac{1}{f^\sigma(t)}\int_a^{\sigma(t)} \frac{f(s)g(s)}{s}\Delta s\right)^{\frac{p}{q}}\Delta t \leq \left(\frac{pK^\gamma}{q(\gamma-1)}\right)^{\frac{p}{q}}\int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)g(t)}{f^\sigma(t)}\right)^{\frac{p}{q}}\Delta t.$$

In the following, we will use the form of the chain rule given in (2.2) to obtain

$$(\Lambda^{p/q}(t))^\Delta = \frac{p}{q}\left\{\int_0^1 [h\Lambda^\sigma + (1-h)\Lambda]^{\frac{p}{q}-1}\,dh\right\}\Lambda^\Delta(t), \tag{2.28}$$

and use this in place of (2.14). We can then prove the following result.

Theorem 2.4. *Let* $\mathbb{T}$ *be a time scale,* $a \in \mathbb{T}$, $\gamma > 1$, p *and* q *be real numbers with* $p > q > 0$ *and* $p/q \geq 2$, *and let* f *be nonincreasing. If*

$$1 + \frac{2^{p/q-1}K^\gamma}{\gamma - 1}\frac{tf^\Delta(t)\Lambda(t)}{f^\sigma(t)\Lambda^\sigma(t)} \geq \frac{1}{m} > 0 \tag{2.29}$$

for some constant $m > 0$, *then*

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t \leq \left(\frac{2^{\frac{p}{q}-1}mK^\gamma}{\gamma - 1}\right)^{\frac{p}{q}} \int_a^\infty \left(\frac{f(t)}{f^\sigma(t)}\right)^{p/q} \frac{1}{t^\gamma} g^{p/q}(t)\Delta t. \tag{2.30}$$

Proof. Proceed as in the proof of Theorem 2.1 to obtain

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t \leq \frac{K^\gamma}{\gamma - 1}\int_a^\infty \frac{(\Lambda^{p/q}(t))^\Delta}{t^{\gamma-1}}\Delta t \tag{2.31}$$

after combining (2.9) and (2.13). From (2.28) and (2.17), we see that

$$\begin{aligned} (\Lambda^{p/q}(t))^\Delta &\leq 2^{\frac{p}{q}-1}\frac{p}{q}\int_0^1 \left[(h\Lambda^\sigma)^{\frac{p}{q}-1} + (1-h)^{\frac{p}{q}-1}\Lambda^{\frac{p}{q}-1}\right] dh\Lambda^\Delta(t) \\ &= 2^{\frac{p}{q}-2}\left[(\Lambda^\sigma)^{\frac{p}{q}-1} + \Lambda^{\frac{p}{q}-1}\right]\Lambda^\Delta(t) \end{aligned} \tag{2.32}$$

Now $\Lambda^\Delta(t) \geq 0$, so

$$(\Lambda^{p/q}(t))^\Delta \leq 2^{\frac{p}{q}-1}(\Lambda^\sigma(t))^{\frac{p}{q}-1}\Lambda^\Delta(t). \tag{2.33}$$

This, (2.16), and (2.31) imply

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t \leq \frac{2^{\frac{p}{q}-1}K^\gamma}{(\gamma-1)}\int_a^\infty \frac{1}{t^{\gamma-1}}(\Lambda^\sigma(t))^{\frac{p}{q}-1}\left[\frac{f(t)g(t)}{tf^\sigma(t)} - \frac{f^\Delta(t)}{f^\sigma(t)}\Lambda(t)\right]\Delta t. \tag{2.34}$$

Thus,

$$\begin{aligned} &\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t \left[1 + \frac{\left(2^{(p/q)-1}\right)K^\gamma}{(\gamma-1)}\frac{tf^\Delta(t)\Lambda(t)}{f^\sigma(t)\Lambda^\sigma(t)}\right]\Delta t \\ &\qquad \leq \frac{2^{\frac{p}{q}-1}K^\gamma}{(\gamma-1)}\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{f^\sigma(t)}\right]\left[(t^\gamma)^{-(p-q)/p}(\Lambda^\sigma(t))^{(p-q)/q}\right]\Delta t. \end{aligned} \tag{2.35}$$

Applying the Hölder inequality (2.4) with indices p/q and $p/(p-q)$ to the right hand side, we see that

$$\begin{aligned} &\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{f^\sigma(t)}\right]\left[(t^\gamma)^{-(p-q)/p}(\Lambda^\sigma(t))^{(p-q)/q}\right]\Delta t \\ &\qquad \leq \left[\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{f^\sigma(t)}\right]^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}}. \end{aligned} \tag{2.36}$$

Substituting (2.36) into (2.35) and using (2.29), we have

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t \le \frac{2^{\frac{p}{q}-1}mK^\gamma}{\gamma-1}\left[\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{f^\sigma(t)}\right]^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty \frac{(\Lambda^\sigma(t))^{p/q}}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}},$$

so

$$\left[\int_a^\infty \frac{1}{t^\gamma(t)}(\Lambda^\sigma(t))^{p/q}\Delta t\right]^{1-\frac{p-q}{p}} \le \frac{2^{\frac{p}{q}-1}mK^\gamma}{\gamma-1}\left[\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^{\gamma-1}}\frac{f(t)g(t)}{tf^\sigma(t)}\right]^{p/q}\Delta t\right]^{\frac{q}{p}}.$$

Hence,

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t \le \left(\frac{2^{\frac{p}{q}-1}mK^\gamma}{\gamma-1}\right)^{\frac{p}{q}}\int_a^\infty \left(\frac{f(t)}{f^\sigma(t)}\right)^{p/q}\frac{1}{t^\gamma}g^{p/q}(t)\Delta t,$$

which is the desired inequality (2.8) and completes the proof of the theorem. □

Remark 2.5. In Theorems 2.1, 2.2, and 2.4, we assumed that the function f is nonincreasing. We note that this may be replaced by the condition

$$g(t) \ge \frac{tf^\Delta(t)}{f^2(t)}\int_a^t \frac{g(s)f(s)}{s}\Delta s \tag{2.37}$$

(see (2.16)).

As special cases when $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{N}$, we can derive some new differential and discrete inequalities from Theorems 2.1–2.4. We begin with Theorem 2.1 for $\mathbb{T}=\mathbb{R}$. In this case, $\mu(t)=0$, $\sigma(t)=t$, and $K=1$, so Theorem 2.1 reduces to the following corollary after p/q is replaced by $\lambda\ge 2$.

Corollary 2.6. *Let $a\in\mathbb{R}^+$, $\lambda\ge 2$ and $\gamma>1$ be real numbers, and f be nonincreasing. If*

$$1+\frac{2^{\lambda-2}\lambda}{\gamma-1}\frac{tf'(t)}{f(t)} \ge \frac{1}{m} > 0 \tag{2.38}$$

for some constant $m>0$, then

$$\int_a^\infty \frac{1}{t^\gamma}\left[\frac{1}{f(t)}\int_a^t \frac{f(s)g(s)}{s}ds\right]^\lambda dt \le \left(\frac{2^{\lambda-2}\lambda m}{\gamma-1}\right)^\lambda \int_a^\infty \frac{1}{t^\gamma}g^\lambda(t)dt. \tag{2.39}$$

Remark 2.7. From Corollary 2.6, if $f(t)=1/\alpha t$ for $\alpha>1$, we see that condition (2.38) becomes

$$1-\frac{\lambda 2^{\lambda-2}}{\gamma-1} \ge \frac{1}{m} > 0, \tag{2.40}$$

and the inequality (2.39) reduces to

$$\int_a^\infty \frac{1}{t^{\gamma-\lambda}}\left(\int_a^t \frac{g(s)}{s^2}ds\right)^\lambda dt \le \left(\frac{2^{\lambda-1}\lambda m}{\gamma-1}\right)^\lambda \int_a^\infty \frac{1}{t^\gamma}g^\lambda(t)dt, \quad \lambda\ge 2. \tag{2.41}$$

With $\mathbb{T} = \mathbb{R}$ and p/q replaced by λ, Theorem 2.2 yields the following corollary.

Corollary 2.8. *Let $a \in \mathbb{R}^+$, $\lambda \geq 2$ and $\gamma > 1$ be real numbers, and let f be nonincreasing. If*

$$1 + \frac{\lambda}{\gamma - 1}\frac{tf'(t)}{f(t)} \geq \frac{1}{m} > 0$$

for some constant $m > 0$, then

$$\int_a^\infty \frac{1}{t^\gamma}\left(\frac{1}{f(t)}\int_a^t \frac{f(s)g(s)}{s}ds\right)^\lambda dt \leq \left(\frac{\lambda m}{\gamma - 1}\right)^\lambda \int_a^\infty \frac{1}{t^\gamma}g^\lambda(t)dt. \tag{2.42}$$

Theorem 2.4 with $\mathbb{T} = \mathbb{R}$ and p/q replaced by λ gives the following corollary.

Corollary 2.9. *Let $a \in \mathbb{R}^+$, $\lambda \geq 2$ and $\gamma > 1$ be real numbers, and f be nonincreasing. If*

$$1 + \frac{2^{\lambda-1}}{\gamma - 1}\frac{tf'(t)}{f(t)} \geq \frac{1}{m} > 0$$

for some constant $m > 0$, then

$$\int_a^\infty \frac{1}{t^\gamma}\left[\frac{1}{f(t)}\int_a^t \frac{f(s)g(s)}{s}ds\right]^\lambda dt \leq \left(\frac{2^{\lambda-1}m}{\gamma - 1}\right)^\lambda \int_a^\infty \frac{1}{t^\gamma}g^\lambda(t)dt.$$

As the special case with $f(t) = 1$, we see that $m = 1$, and inequality (2.42) reduces to

$$\int_a^\infty \frac{1}{t^\gamma}\left(\int_a^t \frac{g(s)}{s}ds\right)^\lambda dt \leq \left(\frac{\lambda}{\gamma - 1}\right)^\lambda \int_a^\infty \frac{1}{t^\gamma}g^\lambda(t)dt. \tag{2.43}$$

Setting $G(t) = g(t)/t$ in (2.43), we have the Hardy inequality (1.2) in the form

$$\int_a^\infty \frac{1}{t^\gamma}\left(\int_a^t G(s)ds\right)^\lambda dt \leq \left(\frac{\lambda}{\gamma - 1}\right)^\lambda \int_a^\infty \frac{1}{t^{\gamma-\lambda}}G^\lambda(t)dt. \tag{2.44}$$

As a special case with $\gamma = 2$ and $\lambda = 2$, (2.43) becomes

$$\int_a^\infty \frac{1}{t^2}\left[\int_a^t \frac{g(s)}{s}ds\right]^2 dt \leq 4\int_a^\infty \left(\frac{g(t)}{t}\right)^2 dt.$$

With $u(t) = g(t)/t$, this takes the form

$$\int_a^\infty \frac{1}{t^2}\left[\int_a^t u(s)ds\right]^2 dt \leq 4\int_a^\infty u^2(t)dt,$$

which is equivalent to

$$\int_a^\infty \left(U'(t)\right)^2 dt \geq \frac{1}{4}\int_a^\infty \frac{1}{t^2}U^2(t), \quad \text{with } \ U(a) = 0.$$

By choosing $a = 0$ and replacing ∞ by 1, we obtain the well-known inequality due to Hardy [8, page 330]

$$\int_0^1 \left(U'(t)\right)^2 dt \geq \frac{1}{4}\int_0^1 \frac{1}{t^2}U^2(t), \text{ with } \ U(0) = 0, \tag{2.45}$$

with the best constant 1/4.

If $\mathbb{T}=\mathbb{N}$, we have the following result as a special case of Theorem 2.1. Notice that for $\mathbb{T}=\mathbb{N}$, the K in (2.5) can be chosen to be $\frac{a+1}{a}$.

Corollary 2.10. *Let $a\in\mathbb{N}$, $\lambda\geq 2$, and $\gamma>1$. Let $f(n)$ and $g(n)$ be nonnegative sequences such that $\Delta f(n)\leq 0$ and define*

$$\Lambda(n):=\frac{1}{f(n)}\sum_{s=a}^{n-1}\frac{f(s)g(s)}{s}. \tag{2.46}$$

If

$$1+\frac{2^{\lambda-2}\lambda K^{\gamma}}{(\gamma-1)}\frac{n\Delta f(n)}{f(n+1)}\geq\frac{1}{m}>0$$

for some constant $m>0$, then

$$\sum_{n=a}^{\infty}\frac{\Lambda^{\lambda}(n)}{n^{\gamma}}-\frac{2^{\lambda-2}mK^{\gamma}}{\gamma-1}\sum_{n=a}^{\infty}n\left(\Delta\Lambda(n)\right)^{\lambda}$$
$$\leq\quad\frac{\lambda 2^{\lambda-2}mK^{\gamma}}{\gamma-1}\left[\sum_{n=a}^{\infty}n^{(\lambda-\gamma)}\left(\frac{f(n)g(n)}{nf(n+1)}\right)^{\lambda}\right]^{\frac{1}{\lambda}}\left[\sum_{n=a}^{\infty}\frac{\Lambda^{\lambda}(n)}{n^{\gamma}}\right]^{1-\frac{1}{\lambda}}.$$

If $\mathbb{T}=\mathbb{N}$, we have the following result as a special case of Theorem 2.2.

Corollary 2.11. *Let $a\in\mathbb{N}$, $\lambda\geq 2$, and $\gamma>1$. Let $f(n)$ and $g(n)$ be nonnegative sequences such that $\Delta f(n)\leq 0$. If*

$$\left[1+\frac{\lambda K^{\gamma}}{(\gamma-1)}\left(\frac{\Lambda(n)}{\Lambda(n+1)}\right)^{\lambda}\frac{n\Delta f(n)}{f(n+1)}\right]\geq\frac{1}{m}>0,$$

for some constant $m>0$, then

$$\sum_{n=a}^{\infty}\frac{1}{n^{\gamma}}\left(\frac{\sum_{s=a}^{n}\frac{f(s)g(s)}{s}}{f(n+1)}\right)^{\lambda}\Delta t\leq\left(\frac{\lambda mK^{\gamma}}{\gamma-1}\right)^{\lambda}\sum_{n=a}^{\infty}\frac{1}{n^{\gamma}}\left(\frac{f(n)g(n)}{f(n+1)}\right)^{\lambda}.$$

Finally, the following result is a special case of Theorem 2.4 in case $\mathbb{T}=\mathbb{N}$.

Corollary 2.12. *Let $a\in\mathbb{N}$, $\lambda\geq 2$, and $\gamma>1$, and let $f(n)$ and $g(n)$ be nonnegative sequences such that $\Delta f(n)\leq 0$. If*

$$1+\frac{2^{\lambda-1}K^{\gamma}}{\gamma-1}\frac{n\Delta f(n)\Lambda(n)}{f(n+1)\Lambda(n+1)}\geq\frac{1}{m}>0$$

for some constant $m>0$, then

$$\sum_{n=a}^{\infty}\frac{1}{n^{\gamma}}\left(\frac{1}{f(n+1)}\sum_{s=a}^{n}\frac{f(s)g(s)}{s}\right)^{\lambda}\leq\left(\frac{2^{\lambda-1}mK^{\gamma}}{\gamma-1}\right)^{\lambda}\sum_{n=a}^{\infty}\frac{1}{n^{\gamma}}\left(\frac{f(n)g(n)}{f(n+1)}\right)^{\lambda}.$$

Next, we consider the case where $p/q \leq 2$. To prove these results, we need the inequality

$$2^{r-1}\left(a^r+b^r\right) \leq (a+b)^r \leq a^r+b^r, \text{ where } a, b \geq 0 \text{ and } 0 \leq r \leq 1. \tag{2.47}$$

Applying this inequality with $r = p/q - 1 < 1$ instead of (2.17), we see that

$$(p/q)\int_0^1 \left[\Lambda + h\mu\Lambda^{\Delta}\right]^{(p/q)-1} dh \leq (p/q)\Lambda^{p/q-1} + (\mu\Lambda^{\Delta})^{p/q-1}, \ \ p/q \leq 2.$$

Proceeding as in the proof of Theorem 2.1, we can prove the following result.

Theorem 2.13. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$, $\gamma > 1$, and p and q be positive constants with $p/q \leq 2$, and f be nonincreasing. If*

$$1 + \frac{pK^{\gamma}}{q\left(\gamma-1\right)}\frac{tf^{\Delta}(t)}{f^{\sigma}(t)} \geq \frac{1}{m} > 0 \tag{2.48}$$

for some constant $m > 0$, then

$$\int_a^{\infty} \frac{\Lambda^{p/q}(t)}{t^{\gamma}}\Delta t - \int_a^{\infty} \frac{mK^{\gamma}\mu^{p/q-1}(t)}{t^{\gamma-1}}(\Lambda^{\Delta}(t))^{p/q}\Delta t$$
$$\leq \frac{pmK^{\gamma}}{q(\gamma-1)}\left[\int_a^{\infty} \frac{1}{t^{\gamma}}\left(\frac{f(t)}{f^{\sigma}(t)}\right)^{p/q} g^{p/q}(t)\Delta t\right]^{\frac{q}{p}}\left[\int_a^{\infty} \frac{\Lambda^{p/q}(t)}{t^{\gamma}}\Delta t\right]^{\frac{p-q}{p}}.$$

Similar to the proof of Theorem 2.2 we can show that the following result holds.

Theorem 2.14. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$, $\gamma > 1$, and p and q be positive constants with $p/q \leq 2$, and f be nonincreasing. If*

$$1 + \frac{2K^{\gamma}}{(\gamma-1)}\frac{tf^{\Delta}(t)\Lambda(t)}{f^{\sigma}(t)\Lambda^{\sigma}(t)} \geq \frac{1}{m} > 0$$

for some constant $m > 0$, then

$$\int_a^{\infty} \frac{1}{t^{\gamma}}(\Lambda^{\sigma}(t))^{p/q}\Delta t \leq \left(\frac{2mK^{\gamma}}{\gamma-1}\right)^{\frac{p}{q}} \int_a^{\infty} \left(\frac{f(t)}{f^{\sigma}(t)}\right)^{p/q} \frac{1}{t^{\gamma}} g^{p/q}(t)\Delta t.$$

Next, we prove a new class of inequalities on time scales for the case $\gamma < 1$ by using the new function

$$\Omega(t) := \frac{1}{f(t)}\int_t^{\infty} \frac{f(s)g(s)}{s}\Delta s \quad \text{for any} \quad t \in [a,\infty)_{\mathbb{T}} \tag{2.49}$$

instead of $\Lambda(t)$.

Theorem 2.15. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$, $\gamma < 1$, and p and q be positive constants with $p/q \geq 2$, and f be nondecreasing. Then*

$$\int_a^\infty \frac{(\Omega^\sigma(t))^{p/q}}{t^\gamma}\left[1 - \frac{(p/q)K}{(1-\gamma)}\left(\frac{\Omega(t)}{(\Omega^\sigma(t))}\right)^{p/q}\frac{tf^\Delta(t)}{f^\sigma(t)}\right]\Delta t$$
$$\leq \frac{(p/q)K}{1-\gamma}\left[\int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)g(t)}{f^\sigma(t)}\right)^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty \frac{\Omega^{p/q}(t)}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}}. \tag{2.50}$$

Proof. First note that applying the chain rule (2.1) and using the fact that $\sigma(s) \geq s$ and property (2.5), we have

$$\begin{aligned}
\left(s^{1-\gamma}\right)^\Delta &= (1-\gamma)\int_0^1 \frac{1}{[h\sigma(s) + (1-h)s]^\gamma}dh \\
&\geq (1-\gamma)\int_0^1 \frac{1}{[h\sigma(s) + (1-h)\sigma(s)]^\gamma}dh = \frac{(1-\gamma)}{\sigma^\gamma(s)} \\
&= \frac{(1-\gamma)s^\gamma}{\sigma^\gamma(s)s^\gamma} \geq \frac{(1-\gamma)}{s^\gamma}\frac{1}{K^\gamma}.
\end{aligned}$$

This implies

$$\begin{aligned}
v^\sigma(t) &= \int_a^{\sigma(t)} \frac{1}{s^\gamma}\Delta s \leq \frac{K^\gamma}{1-\gamma}\int_a^{\sigma(t)}\left(\frac{1}{s^{\gamma-1}}\right)^\Delta \Delta t \\
&= \frac{K^\gamma}{1-\gamma}\frac{1}{(\sigma(t))^{\gamma-1}} - \frac{K^\gamma}{1-\gamma}\frac{1}{a^{\gamma-1}} \leq \frac{K^\gamma}{1-\gamma}(\sigma(t))^{1-\gamma}, \qquad (2.51)
\end{aligned}$$

and in view of (2.5), we have

$$v^\sigma(t) \leq \frac{K^\gamma}{1-\gamma}(Kt)^{1-\gamma} = \frac{K}{(1-\gamma)}\frac{1}{t^{\gamma-1}}. \tag{2.52}$$

Integrating by parts and using the facts that $\Omega(\infty) = 0$ and $v(a) = 0$, we obtain

$$\int_a^\infty \frac{1}{t^\gamma}(\Omega^\sigma)^{p/q}(t)\Delta t = \int_a^\infty v^\sigma(t)\left(\Omega^{p/q}(t)\right)^\Delta \Delta t, \tag{2.53}$$

and applying the chain rule (2.2) and inequality (2.52), we have

$$\int_a^\infty \frac{1}{t^\gamma}(\Omega^\sigma)^{p/q}(t)\Delta t \leq \frac{(p/q)K}{1-\gamma}\int_a^\infty \frac{(-\Omega^\Delta(t))}{t^{\gamma-1}}\int_0^1 \left[\Omega + \mu(t)h\Omega^\Delta\right]^{\frac{p}{q}-1} dh\Delta t. \tag{2.54}$$

Now

$$-\Omega^\Delta(t) = -\left[\frac{1}{f(t)}\int_t^b \frac{f(s)g(s)}{s}\Delta s\right]^\Delta = \frac{f(t)g(t)}{tf^\sigma(t)} + \frac{f^\Delta(t)\int_t^b \frac{f(s)g(s)}{s}\Delta s}{f(t)f^\sigma(t)} > 0 \tag{2.55}$$

since $f^\Delta(t) \geq 0$. Also, since $\Omega^\Delta(t) < 0$,

$$\int_0^1 \left[\Omega + \mu(t)h\Omega^\Delta\right]^{\frac{p}{q}-1} dh \leq \Omega^{\frac{p}{q}-1}. \tag{2.56}$$

Combining (2.54), (2.55), and (2.56), we have

$$\begin{aligned}\int_a^\infty \frac{1}{t^\gamma}\left(\Omega^\sigma(t)\right)^{p/q}\Delta t &\le \frac{(p/q)K}{1-\gamma}\int_a^\infty \frac{1}{t^{\gamma-1}}\left(\Omega(t)\right)^{\frac{p}{q}-1}\left[\frac{f(t)g(t)}{tf^\sigma(t)}+\frac{f^\Delta(t)}{f^\sigma(t)}\Omega(t)\right]\Delta t\\ &= \frac{(p/q)K}{1-\gamma}\int_a^\infty \frac{1}{t^\gamma}\left(\Omega(t)\right)^{\frac{p}{q}-1}\frac{f(t)g(t)}{f^\sigma(t)}\Delta t\\ &\quad+\frac{(p/q)K}{1-\gamma}\int_a^\infty \frac{1}{t^{\gamma-1}}\left(\Omega(t)\right)^{\frac{p}{q}}\frac{f^\Delta(t)}{f^\sigma(t)}\Delta t,\end{aligned}$$

so

$$\begin{aligned}\int_a^\infty \frac{1}{t^\gamma}\left(\Omega^\sigma(t)\right)^{p/q}\Delta t &- \frac{(p/q)K}{1-\gamma}\int_a^\infty \frac{1}{t^{\gamma-1}}\left(\Omega(t)\right)^{\frac{p}{q}}\frac{f^\Delta(t)}{f^\sigma(t)}\Delta t\\ &\le \frac{(p/q)K}{1-\gamma}\int_a^\infty \frac{1}{t^\gamma}\left(\Omega(t)\right)^{\frac{p}{q}-1}\frac{f(t)g(t)}{f^\sigma(t)}\Delta t\\ &= \frac{(p/q)K}{1-\gamma}\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{f^\sigma(t)}\right]\left[\frac{(\Omega(t))^{(p-q)/q}}{(t^\gamma)^{(p-q)/p}}\right]\Delta t.\end{aligned}\tag{2.57}$$

An application of Hölder's inequality with indices p/q and $p/(p-q)$ gives

$$\begin{aligned}\int_a^\infty &\left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{f^\sigma(t)}\right]\left[\frac{(\Omega(t))^{(p-q)/q}}{(t^\gamma)^{(p-q)/p}}\right]\Delta t\\ &\le \left[\int_a^\infty \left[\frac{(t^\gamma)^{(p-q)/p}}{t^\gamma}\frac{f(t)g(t)}{f^\sigma(t)}\right]^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty \frac{\Omega^{p/q}(t)}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}}.\end{aligned}\tag{2.58}$$

A substitution of (2.58) into (2.57) yields

$$\begin{aligned}\int_a^\infty \frac{1}{t^\gamma}&\left[\left(\Omega^\sigma(t)\right)^{p/q}-\frac{(p/q)K}{1-\gamma}\Omega^{p/q}(t)\frac{tf^\Delta(t)}{f^\sigma(t)}\right]\Delta t\\ &\le \frac{(p/q)K}{1-\gamma}\left[\int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)g(t)}{f^\sigma(t)}\right)^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty \frac{\Omega^{p/q}(t)}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}}\end{aligned}$$

from which the desired inequality follows. □

If we apply inequality (2.47) to the term $[h\Omega^\sigma+(1-h)\Omega]^{\frac{p}{q}-1}$ with $p/q\le 2$, we obtain

$$\begin{aligned}\frac{p}{q}\int_0^1 \left[h\Omega^\sigma+(1-h)\Omega\right]^{\frac{p}{q}-1}dh &\le \frac{p}{q}\int_0^1 \left[h^{\frac{p}{q}-1}\left(\Omega^\sigma\right)^{\frac{p}{q}-1}+(1-h)^{\frac{p}{q}-1}\Omega^{\frac{p}{q}-1}\right]dh\\ &= \left[\left(\Omega^\sigma\right)^{\frac{p}{q}-1}+\Omega^{\frac{p}{q}-1}\right]\le 2\Omega^{\frac{p}{q}-1}.\end{aligned}\tag{2.59}$$

Then proceeding as in the proof of Theorem 2.15, we can prove the following theorem.

Theorem 2.16. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$, $\gamma < 1$, and p and q be positive constants with $p/q \geq 2$, and f be nondecreasing. Then*

$$\int_a^\infty \frac{(\Omega^\sigma(t))^{p/q}}{t^\gamma}\left[1 - \frac{2K}{1-\gamma}\left(\frac{\Omega(t)}{(\Omega^\sigma(t))}\right)^{p/q}\frac{tf^\Delta(t)}{f^\sigma(t)}\right]\Delta t$$

$$\leq \frac{2K}{1-\gamma}\left[\int_a^\infty \frac{1}{t^\gamma}\left(\frac{f(t)g(t)}{f^\sigma(t)}\right)^{p/q}\Delta t\right]^{\frac{q}{p}}\left[\int_a^\infty \frac{\Omega^{p/q}(t)}{t^\gamma}\Delta t\right]^{\frac{p-q}{p}}.$$

As special cases, Theorems 2.15 and 2.16 can be used to derive some differential and discrete inequalities, i.e., for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$. The details are left to the interested reader.

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