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BOUNDARY DATA SMOOTHNESS FOR SOLUTIONS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. Under certain conditions, solutions of the boundary value problem, $y'' = f(x, y, y')$, $a < x < b$, $y(x_1) = y_1, \int_{x_1}^{x_2} y(x)dx = y_2$, $a < x_1 < x_2 < b$, are differentiated with respect to the boundary conditions.

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1. INTRODUCTION

In this paper, we will be concerned with differentiating solutions of boundary value problems with respect to boundary data for the second order ordinary differential equation,

$$y'' = f(x, y, y'), \quad a < x < b, \tag{1.1}$$

satisfying

$$y(x_1) = y_1, \ \int_{x_1}^{x_2} y(x)dx = y_2, \tag{1.2}$$

where $a < x_1 < x_2 < b$, and $y_1, y_2 \in \mathbb{R}$, and where we assume:

(i) $f(x, u_1, u_2) : (a, b) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous,
(ii) $\frac{\partial f}{\partial u_i}(x, u_1, u_2) : (a, b) \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, $i = 1, 2$, and
(iii) Solutions of initial value problems for (1.1) extend to (a, b).

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We remark that condition (iii) is not necessary for the spirit of this work's results, however, by assuming (iii), we avoid continually making statements in terms of solutions' maximal intervals of existence.

Under uniqueness assumptions on solutions of (1.1), (1.2), we will establish analogues of a result that Hartman [11] attributes to Peano concerning differentiation of solutions of (1.1) with respect to initial conditions. For our differentiation with respect to boundary conditions results, given a solution $y(x)$ of (1.1), we will give much attention to the *variational equation for* (1.1) *along* $y(x)$, which is defined by

$$z'' = \frac{\partial f}{\partial u_1}(x, y(x),\ y'(x))z + \frac{\partial f}{\partial u_2}(x, y(x), y'(x))z'. \tag{1.3}$$

Interest in nonlocal boundary value problems for ordinary differential equations involving integral boundary conditions has been ongoing for several years. To see only few of these papers, we refer the reader to the papers [1, 2, 3, 19, 21, 22].

Likewise, many papers have been devoted to smoothness of solutions of boundary value problems in regard to smoothness of the differential equation's nonlinearity, as well as the smoothness of the boundary conditions. For a view of how this work has evolved, involving not only boundary value problems for ordinary differential equations, but also discrete versions, we suggest the manifold results in the classical papers [4]–[9], [12], [13], [15], [20] and [23]–[28], as well as the more current papers [10], [14] and [16]-[18].

The theorem for which we seek an analogue and attributed to Peano by Hartman can be stated in the context of (1.1) as follows:

Theorem 1.1 ([Peano])**.** *Assume that with respect of* (1.1), *conditions (i)–(iii) are satisfied. Let* $x_0 \in (a, b)$ *and* $y(x) \equiv y(x, x_0, c_1, c_2)$ *denote the solution of* (1.1) *satisfying the initial conditions* $y(x_0) = c_1$, $y'(x_0) = c_2$. *Then,*

(a) $\frac{\partial y}{\partial c_1}$ *and* $\frac{\partial y}{\partial c_2}$ *exist on* (a, b), *and* $\alpha_i \equiv \frac{\partial y}{\partial c_i}$, $i = 1, 2$, *are solutions of the variational equation* (1.3) *along* $y(x)$ *satisfying the respective initial conditions,*

$$\alpha_1(x_0) = 1, \quad \alpha_1'(x_0) = 0,$$
$$\alpha_2(x_0) = 0, \quad \alpha_2'(x_0) = 1.$$

(b) $\frac{\partial y}{\partial x_0}$ *exists on* (a, b), *and* $\beta \equiv \frac{\partial y}{\partial x_0}$ *is the solution of the variational equation* (1.3) *along* $y(x)$ *satisfying the initial conditions,*

$$\beta(x_0) = -y'(x_0),$$
$$\beta'(x_0) = -y''(x_0).$$

(c) $\frac{\partial y}{\partial x_0}(x) = -y'(x_0)\frac{\partial y}{\partial c_1}(x) - y''(x_0)\frac{\partial y}{\partial c_2}(x)$.

In addition, our analogue of Theorem 1.1 depends on uniqueness of solutions of (1.1), (1.2), a condition we list as an assumption:

(iv) Given $a < x_1 < x_2 < b$, if $y(x_1) = z(x_1)$ and $\int_{x_1}^{x_2} y(x)dx = \int_{x_1}^{x_2} z(x)dx$, where $y(x)$ and $z(x)$ are solutions of (1.1), then $y(x) \equiv z(x)$.

We will also make extensive use of a similar uniqueness condition on (1.3) along solutions $y(x)$ of (1.1).

(v) Given $a < x_1 < x_2 < b$ and a solution $y(x)$ of (1.1), if $u(x_1) = 0$ and $\int_{x_1}^{x_2} u(x)dx = 0$, where $u(x)$ is a solution of (1.3) along $y(x)$, then $u(x) \equiv 0$.

2. AN ANALOGUE OF PEANO'S THEOREM FOR (1.1), (1.2)

In this section, we derive our analogue of Theorem 1.1 for boundary value problem (1.1), (1.2). For such a differentiation result, we need continuous dependence of solutions on boundary conditions. Proof of continuous dependence usually makes application of the Brouwer theorem on invariance of domain. The spirit of such arguments can be found in [14] or [17]; we state the continuity result here, but we omit the details.

Theorem 2.1. *Assume (i)–(iv) are satisfied with respect to* (1.1). *Let* $u(x)$ *be a solution of* (1.1) *on* (a, b), *and let* $a < c < x_1 < x_2 < d < b$ *be given. Then, there exists a* $\delta > 0$ *such that, for* $|x_i - t_i| < \delta$, $i = 1, 2$, $|u(x_1) - y_1| < \delta$, *and* $|\int_{x_1}^{x_2} u(x)dx - y_2| < \delta$, *there exists a unique solution* $u_\delta(x)$ *of* (1.1) *such that* $u_\delta(t_1) = y_1$, $\int_{t_1}^{t_2} u_\delta(x)dx = y_2$, *and* $\{u_\delta^{(j)}(x)\}$ *converges uniformly to* $u^{(j)}(x)$, *as* $\delta \to 0$, *on* $[c, d]$, *for* $j = 0, 1$.

We now present the result of this paper.

Theorem 2.2. *Assume conditions (i)–(v) are satisfied. Let* $u(x)$ *be a solution of* (1.1) *on* (a, b). *Let* $a < x_1 < x_2 < b$ *be given, so that* $u(x) = u(x, x_1, x_2, u_1, u_2)$, *where* $u(x_1) = u_1$ *and* $\int_{x_1}^{x_2} u(x)dx = u_2$. *Then,*

(a) $\frac{\partial u}{\partial u_1}$ *and* $\frac{\partial u}{\partial u_2}$ *exist on* (a, b), *and* $r_i := \frac{\partial u}{\partial u_i}$, $i = 1, 2$, *are solutions of* (1.3) *along* $u(x)$ *and satisfy the respective boundary conditions,*

$$r_1(x_1) = 1, \quad \int_{x_1}^{x_2} r_1(x)dx = 0,$$
$$r_2(x_1) = 0, \quad \int_{x_1}^{x_2} r_2(x)dx = 1.$$

(b) $\frac{\partial u}{\partial x_1}$ *and* $\frac{\partial u}{\partial x_2}$ *exist on* (a, b), *and* $z_i := \frac{\partial u}{\partial x_i}$, $i = 1, 2$, *are solutions of* (1.3) *along* $u(x)$ *and satisfy the respective boundary conditions,*

$$z_1(x_1) = -u'(x_1), \quad \int_{x_1}^{x_2} z_1(x)dx = u(x_1),$$
$$z_2(x_1) = 0, \quad \int_{x_1}^{x_2} z_2(x)dx = -u(x_2).$$

(c) *The partial derivatives satisfy,*

$$\frac{\partial u}{\partial x_1}(x) = -u'(x_1)\frac{\partial u}{\partial u_1}(x) + u(x_1)\frac{\partial u}{\partial u_2}(x),$$
$$\frac{\partial u}{\partial x_2}(x) = -u(x_2)\frac{\partial u}{\partial u_2}(x).$$

Proof. For part (a) we first give the argument for $\frac{\partial u}{\partial u_1}$. Let $\delta > 0$ be as in Theorem 2.1. Let $0 < |h| < \delta$ be given and define

$$r_{1h}(x) = \frac{1}{h}[u(x, x_1, x_2, u_1 + h, u_2) - u(x, x_1, x_2, u_1, u_2)].$$

Note that $u(x_1, x_1, x_2, u_1 + h, u_2) = u_1 + h$, and $u(x_1, x_1, x_2, u_1, u_2) = u_1$, so that for every $h \neq 0$,

$$r_{1h}(x_1) = \frac{1}{h}[u_1 + h - u_1] = 1,$$

and

$$\int_{x_1}^{x_2} r_{1h}(x)dx = \frac{1}{h}\int_{x_1}^{x_2} [u(x, x_1, x_2, u_1 + h, u_2) - u(x, x_1, x_2, u_1, u_2)]dx$$
$$= \frac{1}{h}[u_2 - u_2] = 0.$$

Let

$$\beta_2 = u'(x_1, x_1, x_2, u_1, u_2),$$

and

$$\epsilon_2 = \epsilon_2(h) = u'(x_1, x_1, x_2, u_1 + h, u_2) - \beta_2.$$

By Theorem 2.1, $\epsilon_2 = \epsilon_2(h) \to 0$, as $h \to 0$. Using the notation of Theorem 1.1 for solutions of initial value problems for (1.1) and viewing the solutions u as solutions of initial value problems, we have

$$r_{1h}(x) = \frac{1}{h}[y(x, x_1, u_1 + h,\ \beta_2 + \epsilon_2) - y(x, x_1, u_1, \beta_2)].$$

Then, by utilizing a telescoping sum, we have

$$r_{1h}(x) = \frac{1}{h}\Big[\{y(x, x_1, u_1 + h,\ \beta_2 + \epsilon_2) - y(x, x_1, u_1, \beta_2 + \epsilon_2)\}$$
$$+ \{y(x, x_1, u_1, \beta_2 + \epsilon_2) - y(x, x_1, u_1, \beta_2)\}\Big].$$

By Theorem 1.1 and the Mean Value Theorem, we obtain

$$r_{1h}(x) = \frac{1}{h}\alpha_1(x, y(x, x_1, u_1 + \bar{h},\ \beta_2 + \epsilon_2))(u_1 + h - u_1)$$
$$+ \frac{1}{h}\alpha_2(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2))(\beta_2 + \epsilon_2 - \beta_2),$$

where $\alpha_i(x, y(\cdot))$, $i = 1, 2$, is the solution of the variational equation (1.3) along $y(\cdot)$ and satisfies in each case,

$$\alpha_1(x_1) = 1, \qquad \alpha_1'(x_1) = 0,$$
$$\alpha_2(x_1) = 0, \qquad \alpha_2'(x_1) = 1.$$

Furthermore, $u_1 + \bar{h}$ is between u_1 and $u_1 + h$, and $\beta_2 + \bar{\epsilon}_2$ is between β_2 and $\beta_2 + \epsilon_2$. Now simplifying,

$$\begin{aligned} r_{1h}(x) &= \alpha_1(x, y(x, x_1, u_1 + \bar{h},\ \beta_2 + \epsilon_2)) \\ &\quad + \frac{\epsilon_2}{h}\alpha_2(x, y(x, x_1, u_1,\ \beta_2 + \bar{\epsilon}_2)). \end{aligned}$$

Thus, to show $\lim_{h\to 0} r_{1h}(x)$ exists, it suffices to show $\lim_{h\to 0}\frac{\epsilon_2}{h}$ exists.

Now $\alpha_2(x, y(\cdot))$ is a nontrivial solution of (1.3) along $y(\cdot)$, and $\alpha_2(x_1, y(\cdot)) = 0$. So, by assumption (v),

$$\int_{x_1}^{x_2} \alpha_2(x, y(\cdot))dx \neq 0.$$

However, we observed that $\int_{x_1}^{x_2} r_{1h}(x)dx = 0$, from which we obtain

$$\frac{\epsilon_2}{h} = \frac{-\int_{x_1}^{x_2} \alpha_1(x, y(x, x_1, u_1 + \bar{h},\ \beta_2 + \epsilon_2))dx}{\int_{x_1}^{x_2} \alpha_2(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2))dx}.$$

As a consequence of continuous dependence, we can let $h \to 0$, so that

$$\begin{aligned} \lim_{h\to 0}\frac{\epsilon_2}{h} &= \frac{-\int_{x_1}^{x_2} \alpha_1(x, y(x, x_1, u_1, \beta_2))dx}{\int_{x_1}^{x_2} \alpha_2(x, y(x, x_1, u_1, \beta_2))dx} \\ &= \frac{-\int_{x_1}^{x_2} \alpha_1(x, u(\cdot))dx}{\int_{x_1}^{x_2} \alpha_2(x, u(\cdot))dx} \\ &:= D. \end{aligned}$$

Let $r_1(x) = \lim_{h\to 0} r_{1h}(x)$, and note by construction of $r_{1h}(x)$,

$$r_1(x) = \frac{\partial u}{\partial u_1}(x, x_1, x_2, u_1, u_2).$$

Furthermore,

$$\begin{aligned} r_1(x) &= \lim_{h\to 0} r_{1h}(x) \\ &= \alpha_1(x, y(x, x_1, u_1, \beta_2)) + D\alpha_2(x, y(x, x_1, u_1, \beta_2)) \\ &= \alpha_1(x, u(x, x_2, x_2, u_1, u_2)) + D\alpha_2(x, u(x, x_1, x_2, u_1, u_2)), \end{aligned}$$

which is a solution of the variational equation (1.3) along $u(x)$. In addition because of the boundary conditions satisfied by $r_{1h}(x)$, we also have,

$$r_1(x_1) = 1 \text{ and } \int_{x_1}^{x_2} r_1(x)dx = 0.$$

This completes the argument for $\frac{\partial u}{\partial u_1}$.

While there are similarities with the previous arguments, there are significant enough differences for us to include the details concerning the characterization of $\frac{\partial u}{\partial u_2}$. Again, let $\delta > 0$ be as in Theorem 2.1. Let $0 < |h| < \delta$ be given and define

$$r_{2h}(x) = \frac{1}{h}[u(x, x_1, x_2, u_1, u_2 + h) - u(x, x_1, x_2, u_1, u_2)].$$

This time, for $h \neq 0$,

$$r_{2h}(x_1) = \frac{1}{h}[u_1 - u_1] = 0,$$

and

$$\begin{aligned}\int_{x_1}^{x_2} r_{2h}(x)dx &= \frac{1}{h}\int_{x_1}^{x_2} [u(x, x_1, x_2, u_1, u_2 + h) - u(x, x_1, x_2, u_1, u_2)]dx \\ &= \frac{1}{h}[u_2 + h - u_2] = 1.\end{aligned}$$

Let

$$\beta_2 = u'(x_1, x_1, x_2, u_1, u_2),$$

and

$$\epsilon_2 = \epsilon_2(h) = u'(x_1, x_1, x_2, u_1, u_2 + h) - \beta_2.$$

As before, $\epsilon_2 = \epsilon_2(h) \to 0$, as $h \to 0$. Employing the notation of Theorem 1.1 for solutions of initial value problems for (1.1) and viewing the solutions u as solutions of initial value problems, we have

$$r_{2h}(x) = \frac{1}{h}[y(x, x_1, u_1,\ \beta_2 + \epsilon_2) - y(x, x_1, u_1, \beta_2)].$$

By Theorem 1.1 and the Mean Value Theorem, we obtain

$$\begin{aligned}r_{2h}(x) &= \frac{1}{h}\alpha_2(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2))(\beta_2 + \epsilon_2 - \beta_2) \\ &= \frac{\epsilon_2}{h}\alpha_2(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2)),\end{aligned}$$

where $\beta_2 + \bar{\epsilon}_2$ is between β_2 and $\beta_2 + \epsilon_2$, and $\alpha_2(x, y(\cdot))$ is the solution of the variational equation (1.3) along $y(\cdot)$ and satisfies,

$$\alpha_2(x_1) = 0, \qquad \alpha_2'(x_1) = 1.$$

Thus, to show $\lim_{h\to 0} r_{2h}(x)$ exists, it suffices to show $\lim_{h\to 0} \frac{\epsilon_2}{h}$ exists.

Now by assumption (v),

$$\int_{x_1}^{x_2} \alpha_2(x, y(\cdot))dx \neq 0,$$

and we have above that $\int_{x_1}^{x_2} r_{2h}(x)dx = 1$, from which we obtain

$$\frac{\epsilon_2}{h} = \frac{1}{\int_{x_1}^{x_2} \alpha_2(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2))dx}.$$

By continuous dependence, we can let $h \to 0$, so that

$$\begin{aligned}\lim_{h\to 0}\frac{\epsilon_2}{h} &= \frac{1}{\int_{x_1}^{x_2}\alpha_2(x,y(x,x_1,u_1,\beta_2))dx}\\ &= \frac{1}{\int_{x_1}^{x_2}\alpha_2(x,u(\cdot))dx}\\ &:= E.\end{aligned}$$

Let $r_2(x) = \lim_{h\to 0} r_{2h}(x)$, and then by construction of $r_{2h}(x)$,

$$r_2(x) = \frac{\partial u}{\partial u_2}(x,x_1,x_2,u_1,u_2).$$

Moreover,

$$\begin{aligned}r_2(x) &= \lim_{h\to 0} r_{2h}(x)\\ &= E\alpha_2(x,y(x,x_1,u_1,\beta_2))\\ &= E\alpha_2(x,u(x,x_1,x_2,u_1,u_2)),\end{aligned}$$

which is a solution of the variational equation (1.3) along $u(x)$. Because of the boundary conditions satisfied by $r_{2h}(x)$, we also have,

$$r_2(x_1) = 0 \text{ and } \int_{x_1}^{x_2} r_2(x)dx = 1.$$

And this completes the argument for $\frac{\partial u}{\partial u_2}$.

In part (b) of the theorem, we will produce the details for $\frac{\partial u}{\partial x_1}$, with the arguments for $\frac{\partial u}{\partial x_2}$ being somewhat along the same lines. Again, let $\delta > 0$ be as in Theorem 2.1, let $0 < |h| < \delta$ be given, and define

$$z_{1h}(x) = \frac{1}{h}[u(x,x_1+h,x_2,u_1,u_2) - u(x,x_1,x_2,u_1,u_2)].$$

First, we consider boundary conditions. By employing the Mean Value Theorem for integrals, we have, for $h \neq 0$,

$$\begin{aligned}
\int_{x_1}^{x_2} z_{1h}(x)dx &= \frac{1}{h}\int_{x_1}^{x_2}[u(x,x_1+h,x_2,u_1,u_2) - u(x,x_1,x_2,u_1,u_2)]dx \\
&= \frac{1}{h}\left\{\int_{x_1}^{x_1+h} u(x,x_1+h,x_2,u_1,u_2)dx\right. \\
&\quad + \int_{x_1+h}^{x_2} u(x,x_1+h,x_2,u_1,u_2)dx \\
&\quad \left. - \int_{x_1}^{x_2} u(x,x_1,x_2,u_1,u_2)dx\right\} \\
&= \frac{1}{h}\left\{\int_{x_1}^{x_1+h} u(x,x_1+h,x_2,u_1,u_2)dx + u_2 - u_2\right\} \\
&= \frac{1}{h}u(c_h,x_1+h,x_2,u_1,u_2)\cdot h \\
&= u(c_h,x_1+h,x_2,u_1,u_2)
\end{aligned}$$

for some c_h inclusively between x_1 and x_1+h. By Theorem 2.1, we can compute the limit,

$$\lim_{h\to 0}\int_{x_1}^{x_2} z_{1h}(x)dx = u(x_1,x_1,x_2,u_1,u_2) = u(x_1).$$

Next, we apply the Mean Value Theorem in looking at

$$\begin{aligned}
z_{1h}(x_1) &= \frac{1}{h}[u(x_1,x_1+h,x_2,u_1,u_2) - u(x_1,x_1,x_2,u_1,u_2)] \\
&= \frac{1}{h}[u(x_1,x_1+h,x_2,u_1,u_2) - u(x_1+h,x_1+h,x_2,u_1,u_2)] \\
&= -\frac{1}{h}u'(\zeta_h,x_1+h,x_2,u_1,u_2)\cdot h \\
&= -u'(\zeta_h,x_1+h,x_2,u_1,u_2),
\end{aligned}$$

where ζ_h is between x_1 and x_1+h. And so, in passing to the limit, we have

$$\lim_{h\to 0} z_{1h}(x_1) = -u'(x_1,x_1,x_2,u_1,u_2) = -u'(x_1).$$

Finally, we deal with the $\lim_{h\to 0} z_{1h}(x)$. This time, let

$$\beta_2 = u'(x_1,x_1,x_2,u_1,u_2),$$

and

$$\epsilon_2 = \epsilon_2(h) = u'(x_1+h,x_1+h,x_2,u_1,u_2) - \beta_2.$$

By Theorem 2.1, $\epsilon_2 = \epsilon_2(h) \to 0$, as $h \to 0$. As in part (a), we use the notation of Theorem 1.1 for solutions of initial value problems for (1.1) and viewing the solutions

u as solutions of initial value problems, we have

$$\begin{aligned}
z_{1h}(x) &= \frac{1}{h}[u(x, x_1 + h, x_2, u_1, u_2) - u(x, x_1, x_2, u_1, u_2)] \\
&= \frac{1}{h}[y(x, x_1 + h, u_1, \beta_2 + \epsilon_2) - y(x, x_1, u_1, \beta_2)] \\
&= \frac{1}{h}[y(x, x_1 + h, u_1, \beta_2 + \epsilon_2) - y(x, x_1, u_1, \beta_2 + \epsilon_2) \\
&\quad + y(x, x_1, u_1, \beta_2 + \epsilon_2) - y(x, x_1, u_1, \beta_2)] \\
&= \frac{1}{h}\{\beta(x, y(x, x_1 + \bar{h}, u_1, \beta_2 + \epsilon_2)) \cdot h \\
&\quad + \alpha_2(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2)) \cdot \epsilon_2\} \\
&= \beta(x, y(x, x_1 + \bar{h}, u_1, \beta_2 + \epsilon_2)) + \frac{\epsilon_2}{h}\alpha_2(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2)),
\end{aligned}$$

where $\beta(x, y(\cdot))$ is the solution of (1.3) along $y(\cdot)$ and satisfies

$$\beta(x_1) = -y'(x_1) = -u'(x_1) \text{ and } \beta'(x_1) = -y''(x_1) = -u''(x_1),$$

and $\alpha_2(x, y(\cdot))$ is the solution of (1.3) along $y(\cdot)$ and satisfies

$$\alpha_2(x_1) = 0, \quad \alpha_2'(x_1) = 1,$$

and moreover, $\beta_2 + \bar{\epsilon}_2$ lies between β_2 and $\beta_2 + \epsilon_2$, and $x_1 + \bar{h}$ lies between x_1 and $x_1 + h$. As before, to show $\lim_{h\to 0} z_{1h}(x)$ exists, it suffices to show $\lim_{h\to 0} \frac{\epsilon_2}{h}$ exists.

Since $\alpha_2(x, y(\cdot))$ is a nontrivial solution of (1.3) along $y(\cdot)$ and since $\alpha_2(x_1, y(\cdot)) = 0$, it follows from assumption (v) that

$$\int_{x_1}^{x_2} \alpha_2(x, y(\cdot))dx \neq 0.$$

Hence,

$$\frac{\epsilon_2}{h} = \frac{\int_{x_1}^{x_2} z_{1h}(x)dx - \int_{x_1}^{x_2} \beta(x, y(x, x_1 + \bar{h}, u_1, \beta_2 + \epsilon_2))dx}{\int_{x_1}^{x_2} \alpha_2(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2))dx}.$$

And so in passing to the limit, we have from above,

$$\lim_{h\to 0} \frac{\epsilon_2}{h} = \frac{u(x_1) - \int_{x_1}^{x_2} \beta(x, u(\cdot))dx}{\int_{x_1}^{x_2} \alpha_2(x, u(\cdot))dx} := H.$$

From above,

$$z_{1h}(x) = \beta(x, y(x, x_1 + \bar{h}, u_1, \beta_2 + \epsilon_2)) + \frac{\epsilon_2}{h}\alpha_2(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2)),$$

and so we can evaluate the limit as $h \to 0$, and if we let $z_1(x) = \lim_{h\to 0} z_{1h}(x)$, we have $z_1(x) = \frac{\partial u}{\partial x_1}$. We obtain

$$z_1(x) = \frac{\partial u}{\partial x_1}(x) = \lim_{h\to 0} z_{1h}(x) = \beta(x, u(x)) + H\alpha_2(x, u(x))$$

which is a solution of (1.3) along $u(x)$. In addition, from above computations, $z_1(x)$ satisfies the boundary conditions,

$$z_1(x_1) = \lim_{h\to 0} z_{1h}(x_1) = -u'(x_1),$$
$$\int_{x_1}^{x_2} z_1(s)dx = \lim_{h\to 0}\int_{x_1}^{x_2} z_{1h}(s)dx = u(x_1).$$

This completes the proof of part (b).

Part (c) of the theorem is immediate by verifying that each side of the respective equations are solutions of (1.3) along $u(x)$ and satisfy the same boundary conditions, and then assumption (v) establishes the equalities. □

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