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SOLVABILITY OF SECOND ORDER NONLINEAR MULTI-POINT BOUNDARY VALUE PROBLEMS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. We are interested in the existence of non-trivial solutions for second order boundary value problem: (E) $y'' + f(t, y) = 0$, $0 < t < 1$, subject to multi-point boundary condition at $t = 1$ and Robin boundary condition at $t = 0$. Our results extend similar results of Sun and Liu [12], Sun [11] Li and Sun [8] and Guezane-Lakoud and Kelaiaia [2] for the three point problem.

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1. **Introduction**

We consider second order nonlinear differential equation

$$y'' + f(t, y) = 0, \quad 0 < t < 1, \tag{1.1}$$

subject to the multi-point boundary condition

$$\cos\theta y(0) = \sin\theta y'(0), \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \tag{1.2}$$

$$y(1) = \sum_{i=1}^{m} \beta_i y(\eta_i) + \sum_{i=1}^{m} \gamma_i y'(\eta_i), \tag{1.3}$$

where $\beta_i, \gamma_i \in \mathbb{R}$ and $0 < \eta_1, < \cdots < \eta_m < 1$. Condition (1.2) is sometimes referred to as the Robin boundary condition at $t = 0$ because the choices of $\theta = 0$ and $\theta = \frac{\pi}{2}$ correspond to Dirichlet and Neumann conditions, i.e., $y(0) = 0$ and $y'(0) = 0$, respectively. Condition (1.3) is often known as the m-point boundary condition, see Kwong and Wong [6] for a discussion concerning Robin boundary condition.

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Many existence results have been proved for the three point problem, $m = 1$ in (1.3) by Gupta [3], Sun and Liu [12], Sun [11], Li and Sun [8] and Kwong and Wong [7].

In [12], Sun and Liu studied the three point problem, equation (1.1) subject to

$$y'(0) = 0, \quad y(1) = \beta y(\eta), \tag{1.4}$$

where $0 < \eta < 1$ and $\beta \neq 1$. It is assumed that $f(t, y) \in C([0, 1], \mathbb{R}, \mathbb{R})$ and satisfies

$$|f(t, y)| \leq k(t)|y| + h(t), \tag{1.5}$$

where $k, h \in L^1(0, 1)$ are non-negative functions. They obtained the following result.

Proposition 1.1. *Suppose that* $f(t, 0) \not\equiv 0$ *in* $[0, 1]$ *and satisfies* (1.5)*. If* $k(t)$ *satisfies*

$$\left(1 + \left|\frac{1}{1-\beta}\right|\right) \int_0^1 (1-s)k(s)ds + \left|\frac{\beta}{1-\beta}\right| \int_0^\eta (\eta - s)k(s)ds < 1, \tag{1.6}$$

then the boundary value problem (1.1)*,* (1.4) *has a non-trivial solution.*

Subsequent to [12], Sun [11] considered equation (1.1) subject to the following three point boundary condition:

$$y'(0) = 0, \quad y(1) = \gamma y'(\eta), \tag{1.7}$$

and proved

Proposition 1.2. *Suppose that* $f(t, y)$ *satisfies the same assumptions as in Proposition* 1.1*. If* $k(t)$ *satisfies*

$$2\int_0^1 (1-s)k(s)ds + \frac{|\gamma|}{|1-\gamma|} \int_0^\eta k(s)ds < 1, \tag{1.8}$$

then the boundary value problem (1.1)*,* (1.7) *has a non-trivial solution.*

Further to [12], Li and Sun [8] studied equation (1.1) subject to the following three point boundary condition

$$ay(0) = by'(0), \quad y(1) = \beta u(\eta). \tag{1.9}$$

and obtained

Proposition 1.3. *Suppose that* $f(t, y)$ *satisfies the same assumptions as in Proposition* 1.1*. If* $k(t)$ *satisfies*

$$\left\{1 + \frac{1}{|\rho|}(|b| + |a|)\right\} \int_0^1 (1-s)k(s)ds + \frac{|\beta|(|b| + |a|)}{|\rho|} \int_0^\eta (\eta - s)k(s)ds < 1, \tag{1.10}$$

where $\rho = a(1 - \beta\eta) + b(1 - \beta) \neq 0$*, then the boundary value problem* (1.1)*,* (1.9) *has a non-trivial solution.*

Let $a = 0$, $b = 1$, $\rho = 1 - \beta$, (1.10) becomes (1.6), so Proposition 1.3 includes Proposition 1.1 as a special case.

Very recently, Guezane-Lakoud and Kelaiaia [2] consider the following boundary condition for the three point problem

$$y(0) = \alpha y'(0), \quad y(1) = \gamma y'(\eta), \tag{1.11}$$

and proved the following result.

Proposition 1.4. *Assume that $f(t, y)$ satisfies condition* (1.5) *as in Proposition* 1.1 *and that $k(t)$ satisfies*

$$\left(1 + \frac{|1+\alpha|}{|1+\alpha-\gamma|}\right) \int_0^1 (1-s)k(s)ds + \frac{|\gamma(1+\alpha)|}{|1+\alpha-\gamma|} \int_0^\eta k(s)ds < 1, \tag{1.12}$$

Then the boundary value problem (1.1), (1.11) *has a non-trivial solution.*

Let $\alpha \to \infty$, then condition (1.11) reduces to (1.7) and (1.12) implies (1.8), so Proposition 1.4 includes Proposition 1.2 as a special case.

We shall show how the m-point boundary value problem (1.1), (1.2), (1.3) can be treated like the three point problems (1.1), (1.4); (1.1), (1.7); (1.1), (1.9) and prove extensions of Propositions 1.3 and 1.4. We first introduce a convenient notation for the multi-point boundary condition at $t = 1$ in (1.3). Consider the set of m interior points η_i, $i = 1, 2, \ldots, m$ as a m-vector and likewise for β_i, γ_i, and $y(\eta_i)$, $i = 1, 2, \ldots, m$. Now denote the scalar product of two vectors $\beta = (\beta_1, \ldots, \beta_m)$ and $y(\eta) = (y(\eta_1), \ldots, y(\eta_m))$ by

$$\langle \beta, y(\eta) \rangle = \sum_{i=1}^{m} \beta_i y(\eta_i). \tag{1.13}$$

Using the notation (1.13), we can rewrite the boundary conditions (1.2), (1.3) with $\alpha = \tan\theta$ as

$$y(0) = \alpha y'(0), \quad y(1) = \langle \beta, y(\eta) \rangle + \langle \gamma, y'(\eta) \rangle. \tag{1.14}$$

Here we adopt the simpler Robin condition $y(0) = \alpha y'(0)$ in (1.14) instead of (1.2) for ease of comparison with Propositions 1.3 and 1.4.

2. Main Results

In this section, we prove two theorems both generalize Propositions 1.3 and 1.4 for the three point problem. We first define for $q(t) \in L^1(0, 1)$, the integral operator $I[q](t)$ by

$$I[q](t) = \int_0^t (t-s)q(s)ds. \tag{2.1}$$

Lemma 2.1. *Suppose* $\triangle = 1 + \alpha(1 - \overline{\beta}) - \overline{\gamma} - \langle \beta, \eta \rangle \neq 0$ *where* $\overline{\beta} = \sum_{i=1}^{m} \beta_i$, $\overline{\gamma} = \sum_{i=1}^{m} \gamma_i$, $\langle \gamma, \eta \rangle = \sum_{i=1}^{m} \gamma_i \eta_i$. *Then the solution to the multi-point boundary value problem* $y'' + q(t) = 0$ *satisfying boundary condition* (1.14) *has a unique solution represented by the fixed point of the operator* A *given by*

$$Ay(t) = -I[q](t) + \frac{t+\alpha}{\triangle} \left\{ I[q](1) - \langle \beta, I[q](\eta) \rangle - \langle \gamma, I'[q](\eta) \rangle \right\}. \tag{2.2}$$

Proof. A solution $y(t)$ of $y'' + q(t) = 0$ can be written as

$$y(t) = -\int_0^t (t-s)q(s)ds + C_1 t + C_2.$$

Now, $y(0) = \alpha y'(0)$ implies $C_2 = \alpha C_1$. The multi-point boundary condition (1.14) implies that $y(t)$ satisfies

$$y(t) = -I[q](t) + \frac{t+\alpha}{\triangle} \left\{ I[q](1) - \langle \beta, I[q](\eta) \rangle - \langle \gamma, I'[q](\eta) \rangle \right\}, \tag{2.3}$$

which is the fixed point of the operator A defined by (2.2). □

Since $q(t) \in L^1(0,1)$, the operator A defined by (2.2) is completely continuous. We shall use the following Schauder Fixed Point Theorem, known as the Leray Schauder Nonlinear Alternative:

Lemma 2.2 (Deimling [1]). *Let* X *be a real Banach space and* Ω *a bounded open subset of* X *with* $O \in \Omega$. *Suppose that* $A : \overline{\Omega} \to X$ *is a completely continuous operator. Then either there exists* $x \in \partial\Omega$ *such that* $Ax = \lambda x$, *for some* $\lambda > 1$, *or there exists a fixed point* $\widehat{x}$ *of* A *in* $\overline{\Omega}$, *i.e.* $A\widehat{x} = \widehat{x}$.

For m-vectors $\beta = (\beta_1, \ldots, \beta_m)$, $|\beta| = (|\beta_1|, \ldots, |\beta_m|)$, denote $\overline{\beta} = \sum_{i=1}^{m} \beta_i$, $\overline{\gamma} = \sum_{i=1}^{m} \gamma_i$, $|\widehat{\beta}| = \sum_{i=1}^{m} |\beta_i|$ and $|\widehat{\gamma}| = \sum_{i=1}^{m} \gamma_i$ for short. Note that $|\overline{\beta}| \leq |\widehat{\beta}|$, $|\overline{\gamma}| \leq |\widehat{\gamma}|$. Let $\triangle = 1 + \alpha(1 - \overline{\beta}) - \overline{\gamma} - \langle \beta, \eta \rangle$.

Theorem 2.1. *Suppose that* $\triangle \neq 0$ *and* $f(t,y) \in C((0,1) \times \mathbb{R}, \mathbb{R})$ *satisfies* (1.5) *almost everywhere in* t *with* $k(t), h(t) \in L^1(0,1)$. *If* $k(t)$ *satisfies*

$$\Gamma(k) = \left(1 + \left|\frac{1+\alpha}{\triangle}\right|\right) I[k](1) + \left|\frac{1+\alpha}{\triangle}\right| \left\{ \langle |\beta|, I[k](\eta) \rangle + \langle |\gamma|, I'(k)(\eta) \rangle \right\} < 1, \tag{2.4}$$

then the boundary value problem (1.1), (1.14) *has a non-trivial solution.*

Note that by relaxing the requirement $f(t,y) \in C([0,1] \times \mathbb{R}, \mathbb{R})$ we allow $f(t,y) = q(t)g(y)$ with $q(t) \in L_1(0,1)$ which can be singular at $t = 0$ or 1 or both.

Proof. Since $f(t,0) \not\equiv 0$ for $t \in [0,1]$, there exists an interval $[\sigma, \tau] \subseteq [0,1]$ such that by (1.5) we have $h(t) \geq |f(t,0)| \geq \min_{\sigma \leq t \leq \tau} |f(t,0)| > 0$, so we can define the positive constant $r = \Gamma(h)(1 - \Gamma(k))^{-1} > 0$, where $\Gamma(h)$ is defined similarly as in (2.4) for $k(t)$.

Denote $B_r = \{y \in C[0,1] : \|y\| < r\}$ and its boundary $\partial B_r = \{y \in \overline{B_r} : \|y\| = r\}$. Suppose that there exists $\overline{y} \in \partial B_r$ such that $A\overline{y} = \lambda\overline{y}$ for some $\lambda > 1$. Applying (1.5) and (2.4) to the operator A given by (2.2), we have

$$\lambda r = \|A\overline{y}\| \leq \Gamma(k)\|\overline{y}\| + \Gamma(h) \leq \Gamma(k)r + \Gamma(h) = r,$$

which contradicts the assumption that $\lambda > 1$. By Lemma 2.2, we conclude that A has a fixed point $\widehat{y} \in C[0,1]$, which is a solution to the boundary value problem (1.1), (1.14) by Lemma 2.1. Since $f(t,0) \not\equiv 0$, $\widehat{y}$ cannot be the identically zero solution, which incidentically also satisfies (1.1) and (1.14). This completes the proof of the theorem. □

In case of three point BVP (1.1), (1.9), if $a = 0$, $b = 1$, then we have Neumann boundary condition at $t = 0$, i.e. it reduces to (1.1), (1.4), so Proposition 1.3 reduces to Proposition 1.1 as mentioned before.

However, when $a \neq 0$, and $\beta_i = 0$, $\gamma_i = \gamma$ for all $i - 1, 2, \ldots, m$, then $\rho = a\triangle$, with $\alpha = b/a$. We can divide (1.10) by $|a|$ and obtain

$$\left(1 + \frac{1}{|\triangle|}\big|1 + |\alpha|\big|\right) I[k](1) + \frac{1}{|\triangle|}\,|\beta|(1 + |\alpha|)|I[k](\eta)| < 1.$$

Also, $\triangle = 1 + \alpha - \gamma$ and $\beta = 0$, so (2.4) becomes (1.10).

This shows that Theorem 2.1 also generalizes Proposition 1.4 from three-point BVP's to m-point BVP's.

Next we modify the proof of Theorem 2.1 and prove a result which improves upon Proposition 1.1 for the three point boundary value problem (1.1), (1.7). Recall that the Green's function for the two-point boundary value problem:

$$y'' + q(t) = 0, \quad y'(0) = 0, \quad y(1) = 0 \tag{2.5}$$

is given by

$$G(t,s) = \begin{cases} 1 - s, & 0 \leq t \leq s \leq 1, \\ 1 - t, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.6}$$

In other words, the unique solution to (2.5) can also be represented by

$$y(t) = \int_0^1 G(t,s)q(s)ds = G[q](t). \tag{2.7}$$

Note that the operator $G[q]$ is defined for any $q \in L^1(0,1)$ so $G[k]$, $G[h]$ are likewise defined for k, $h \in L^1(0,1)$. In fact, we have for $t \in [0,1]$ the following identify

$$-G[q](t) = I[q](t) - I[q](1) \tag{2.8}$$

for any $q(t) \in C[0,1]$. Using this in (2.3), we observe that the operator A defined by (2.2) can also be represented by

$$
\begin{aligned}
Ay(t) = G[q](t) + \frac{(t+\alpha)}{\triangle}\{\langle \beta, G[q](\eta)\rangle - \langle \gamma, G'[q](\eta)\rangle\} \\
+ I[q](1)\left\{\frac{(1-\overline{\beta})}{\triangle}(t+\alpha) - 1\right\}.
\end{aligned}
\tag{2.9}
$$

We can now state and prove another result for the boundary value problem (1.1), (1.14).

Theorem 2.2. *Suppose that* $\triangle \neq 0$ *and* $f(t,y)$ *satisfies* (1.5) *as in Proposition* 1.1. *If* $k(t)$ *satisfies*

$$
\begin{aligned}
\Lambda[k] = \left(1 + \frac{1}{|\triangle|}\left|\overline{\gamma} - \overline{\beta} + \langle \beta, \eta\rangle\right|\right) I[k](1) \\
+ \frac{|1+\alpha|}{|\triangle|}\{\langle |\beta|, G[k](\eta)\rangle + |\langle |\gamma|, G'[k](\eta)\rangle|\} < 1,
\end{aligned}
\tag{2.10}
$$

then the boundary value problem (1.1), (1.14) *has a non-trivial solution.*

Proof. We use the integral representation (2.9) for the operator A instead of (2.2) as in Theorem 2.1. Since $f(t,0) \not\equiv 0$ in $[0,1]$, we can define $R = \Lambda(h)\big(1-\Lambda(k)\big)^{-1} > 0$. Now we apply the Leray-Schauder nonlinear alternative in the same manner as in Theorem 2.1 to $B_R = \{y \in C[0,1] : \|y\| < R\}$ and obtain a fixed point $\widehat{y} \in \overline{B}_R$ which is, by Lemma 2.1, a solution to the boundary value problem (1.1), (1.14). This solution is non-trivial because $f(t,0) \not\equiv 0$. This proves Theorem 2.2. □

If, in addition, $\beta_i = 0$, $i = 1, 2, \ldots, m$ in (1.14), with $\triangle = 1 + \alpha - \overline{\gamma}$, then condition (2.10) reduces to

$$
\left(1 + \frac{|\overline{\gamma}|}{|1+\alpha-\overline{\gamma}|}\right) I[k](1) + \frac{|(1+\alpha)\overline{\gamma}|}{|1+\alpha-\overline{\gamma}|}\int_0^\eta k(s)ds < 1. \tag{2.11}
$$

Let $\alpha \to \infty$ in (2.11), we have

$$
I[k](1) + |\overline{\gamma}| \int_0^\eta k(s)ds < 1, \tag{2.12}
$$

showing that the BVP (1.1), (1.7) has a non-trivial solution. Clearly (2.12) is a substantive improvement upon (1.8) in Proposition 1.2.

3. Discussion

We give two examples involving $k(t) \in L^1(0,1)$ where $k(t)$ can be singular at $t = 0$ or 1 or both where Propositions 1.1–1.4 stated for $f(t,y) \in C([0,1] \times \mathbb{R}, \mathbb{R})$ are inapplicable. Examples 3.3 and 3.4 in [8] used $f(t,y)$ which is singular at $t = 0$, so in fact Proposition 1.3 is inapplicable.

Example 3.1. Consider second order differential equation

$$y'' + \frac{1}{5}\frac{y\sin y}{\sqrt{t(1-t)}} + h(t) = 0, \quad 0 < t < 1, \tag{3.1}$$

where $h(t) \in L^1(0,1)$, subject to the boundary condition:

$$y(0) = \frac{1}{2}y'(0), \quad y(1) = \frac{1}{2}y'\left(\frac{1}{4}\right). \tag{3.2}$$

Here, $\alpha = \gamma = \frac{1}{2}, \beta = 0$ and $\eta = \frac{1}{4}$, so $\triangle = 1 + \alpha - \gamma = 1$. To apply Theorem 2.1 for this three point problem, we need to verify condition (1.12) in Proposition 1.4. In this case $k(t) = \frac{1}{5}[t(1-t)]^{-1/2}$ so Proposition 1.4 does not apply. We compute $I[k](1)$ and $I'[k]\left(\frac{1}{2}\right)$ and find

$$I[k](1) = \frac{1}{5}\int_0^1 (1-s)k(s)ds = \frac{1}{5}\int_0^1 \sqrt{1-s}\frac{ds}{s} = \frac{\pi}{10},$$

and

$$I'[k]\left(\frac{1}{4}\right) = \frac{1}{5}\int_0^{1/3} s^{-1/2}(1-s)^{-1/2}ds = \frac{1}{5}B\left(\frac{1}{2},\frac{1}{2}\right)\left(\frac{1}{4}\right),$$

where $B(p,q)(t) = \int_0^t s^{p-1}(1-s)^{q-1}ds$ is the Beta Function. We know that $B\left(\frac{1}{2},\frac{1}{2}\right)(t) = \sin^{-1}\sqrt{t}$, so $B\left(\frac{1}{2},\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{\pi}{6}$. Using this in (2.4) we find

$$\left(1 + \frac{3}{2}\right)I[k](1) + \frac{3}{4}I'[k]\left(\frac{1}{4}\right) = \frac{\pi}{4} + \frac{\pi}{40} < 1.$$

Thus, we can conclude from Theorem 2.1 that the three problem (3.1), (3.2) has a non-trivial solution.

The next example is a four-point boundary value problem again with coefficient function singular at $t = 1$ but belong to $L^1(0,1)$.

Example 3.2. Consider the second order nonlinear differential equation

$$y'' + \frac{1}{2}(1-t)^{-1/2}y\sin y + e^t\cos t/\sqrt{t} = 0, \tag{3.3}$$

where $h(t) = e^t\cos t/\sqrt{t}$ and $k(t) = [2(1-t)]^{-1/2} \in L^1(0,1)$, subject to the four point boundary condition

$$y(0) = \frac{1}{3}y'(0), \quad y(1) = \frac{1}{2}y\left(\frac{1}{3}\right) - 3y'\left(\frac{2}{3}\right). \tag{3.4}$$

To apply Theorem 2.1, we wish to verify (2.4). Here $\eta_1 = \frac{1}{3}$, $\eta_2 = \frac{2}{3}$, $\alpha = \frac{1}{3}$, $\beta_1 = \frac{1}{2}$, $\beta_2 = 0$, $\gamma_1 = 0$, $\gamma_2 = -3$, so $\triangle = 1 + \alpha(1-\beta_1) - \gamma_2 - \beta_1\eta_1 = 4$ and

$$\Gamma(k) = \frac{4}{3}I[k](1) + \frac{1}{6}I[k]\left(\frac{1}{3}\right) + I'[k]\left(\frac{2}{3}\right). \tag{3.5}$$

Note that $I[k](1) = \frac{1}{3}$, $I[k]\left(\frac{1}{3}\right) \le \int_0^{1/3}(1-s)k(s)ds = \frac{1}{3}\left(1 - \frac{2}{3}\sqrt{\frac{2}{3}}\right)$, and $I'[k]\left(\frac{2}{3}\right) = 1 - 1/\sqrt{3}$. Computing $\Gamma(k)$ in (3.5), we find

$$\begin{aligned}\Gamma(k) &= \frac{4}{9} + \frac{1}{18}\left[1 - \left(\frac{2}{3}\right)^{3/2}\right] + \left(1 - \frac{1}{\sqrt{3}}\right) \\ &= 0.44444 + 0.02532 + 0.42266 = 0.89242 < 1,\end{aligned}$$

so (2.4) is satisfied. Hence the 4-pt boundary value problem (3.3), (3.4) has a non-trivial solution.

We now close our discussion with a few remarks.

Remark 3.1. The requirement that $\triangle \neq 0$ in both Theorems 2.1 and 2.2 is known as the non-resonance condition. This is equivalent to the statement that $y''(t) = 0$, $0 \le t \le 1$, has no non-trivial solutions satisfying multi-point boundary condition (1.14). We refer the reader to papers by. Infante and Webb [5] and Ma [10] which discussed 3-point boundary value problems in non-resonant cases.

Remark 3.2. Condition (1.12) in Proposition 1.4 is not optimal in the sense that for the boundary condition (1.9), i.e. $y(0) = \alpha y'(0)$, $y(1) = \beta y'(\frac{1}{2})$, it is easy to construct examples so that (1.12)is not satisfied. Consider the simple equation $y'' + \pi^2 y = \pi^2$ which has

$$y(t) = \frac{1}{\alpha\pi}\left(\frac{1}{1-\beta\pi} + 1\right)\sin\pi t + \left(\frac{1}{1-\beta\pi}\right)\cos\pi t + 1$$

as a solution satisfying the three point boundary condition (1.9) for any $\alpha \neq 0$, $\beta\pi \neq 1$.

Remark 3.3. We refer readers to recent papers by Han and Wu [4], Liu, Liu and Wu [9] which dealt with sign-changing nonlinearities using the Krasnosel'skii Fixed Point Theorem on cones and prove existence of positive solutions for multi-point boundary value problems.

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