Dynamic Systems and Applications 23 (2014) 155-162

A NON-LOCAL PROBLEM FOR A DIFFERENTIAL EQUATION OF FRACTIONAL ORDER

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. We discuss the existence and uniqueness of a solution to the non-local problem for a fractional differential equation

$$D_{0^+}^{\alpha} u(t) = f(t, u(t)), \quad \text{a. e. in} \quad (0,1),$$

$$I_{0^+}^{1-\alpha} u(0) = \beta I_{c^+}^{1-\alpha} u(\xi),$$

using the contraction principle and a continuation method.

AMS (MOS) Subject Classification. 34B10, 34B15.

1. INTRODUCTION

Fractional differential equations and initial and boundary value problems have been studied actively for the past two decades [6, 7]. This paper is a study of Riemann-Liouville integral equation associated with a non-local problem of fractional order differential equation admitting singular solutions. The problems involving the Riemann-Liouville derivative have been considered in [1, 2, 3] among many other references. Recently there have been several works extending [10, 11], where singular solutions of fractional order problems were obtained. In particular, such extensions were obtained in [5]. Bai considered an impulsive fractional problem at resonance, where the solutions $u \in X$ and

$$X = \left\{ u : t^{2-\alpha} u, D_{0^+}^{1-\alpha} u \in PC[0,1] \right\}.$$

In this note we consider a type of a non-local problem of fractional order $0 < \alpha < 1$, which involves unequal terminals:

$$D_{0^+}^{\alpha} u(t) = f(t, u(t)), \quad \text{a. e. in} \quad (0,1), \tag{1.1}$$

under the non-local condition

$$I_{0^+}^{1-\alpha} u(0) = \beta I_{c^+}^{1-\alpha} u(\xi), \tag{1.2}$$

Received June 2013

where $0 < c < \xi < 1$. We study the non-resonant case

$$\Gamma(\alpha) \neq \beta I_{c^+}^{1-\alpha}(t^{\alpha-1})(\xi).$$

To the best of our knowledge the solvability of this type of fractional order problem has not been previously studied.

2. PRELIMINARIES

Within this paper we will use the Reimann-Liouville formulation of the fractional order integral and derivative defined, respectively, as

$$I_{a^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} u(x) dx, \tag{2.1}$$

$$D_{a^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t u(x)(t-x)^{-\alpha} dx, \tag{2.2}$$

where $n = [\alpha] + 1$. The following results show the relationship between (2.1) and (2.2) and can be found in [1].

Theorem 2.1. *If $\alpha > 0$, then*

1. $D_{0^+}^{\alpha} I_{0^+}^{\alpha} u(t) = u(t)$ *for all* $u \in L^1(0,1)$.
2. *For* u, $D_{0^+}^{\alpha} u \in L^1(0,1)$, $I_{0^+}^{n-\alpha} u \in AC^{n-1}[0,1]$, *where* $n = [\alpha] + 1$, *we have*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{t^{\alpha-k-1}}{\Gamma(\alpha-k)} \frac{d^{n-k-1}}{dt^{n-k-1}} I_{0^+}^{n-\alpha} u(0).$$

If $0 < \alpha < 1$, then

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} I_{0^+}^{1-\alpha} u(0).$$

Theorem 2.2. *Let $\alpha + \beta \geq 1$. If $u \in L^1(0,1)$, then $I_{a^+}^{\alpha} I_{a^+}^{\beta} u = I_{a^+}^{\alpha+\beta} u$.*

Suppose that the function $f : (0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions with respect to $L^p(0,1] : f(\cdot, x)$ is Lebesgue measurable in $[0,1]$ for all $x \in \mathbb{R}$, $f(t,\cdot)$ is continuous on $\mathbb{R}$ for almost all $t \in (0,1]$, and a boundedness condition holds. That is, for each $r > 0$, there exists a real-valued function $\mu_r \in L^p(0,1]$ such that $|f(t,x)| \leq \mu_r(t)$, for almost all $t \in (0,1]$ and all $|x| \leq r$.

3. THE EXISTENCE OF A UNIQUE SOLUTION

We assume that f is Carathéodory with respect to $L^p(0,1]$, where $p > \frac{1}{\alpha}$ and $q = \frac{p}{p-1}$. We work in the Banach space $X = \{u \in C(0,1] : \lim_{t\to 0^+} t^{1-\alpha} u \text{ exists}\}$ with the norm $\|u\| = \sup_{t\in(0,1]} |t^{1-\alpha} u|$. By a solution of the non-local problem (1.1), (1.2) we understand a function u satisfying (1.1), (1.2) and such that $D_{0^+}^{\alpha} u \in L^p(0,1]$. The first result relates the non-local problem (1.1), (1.2) to an integral equation in X.

Lemma 3.1. *A function $u \in X$ is a solution of the non-local problem* (1.1), (1.2) *if and only if $u \in X$ satisfies the fixed point problem*

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x, u(x))\, dx$$

$$+ \frac{\beta t^{\alpha-1}}{(\Gamma(\alpha) - \beta\gamma)\Gamma(1-\alpha)} \int_c^\xi (\xi - x)^{-\alpha} \int_0^x f(y, u(y))(x-y)^{\alpha-1}\, dy\, dx, \tag{3.1}$$

where $\gamma = I_{c^+}^{1-\alpha}(s^{\alpha-1})(\xi)$.

Proof. Let u be a solution to the non-local problem (1.1), (1.2). Then $D_{0^+}^\alpha u$ is integrable in $(0,1]$ and, as a result $I_{0^+}^{1-\alpha} u \in AC(0,1]$. Thus, by Theorem 2.1,

$$I_{0^+}^\alpha f(t, u(t)) = I_{0^+}^\alpha D_{0^+}^\alpha u(t) = u(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} I_{0^+}^{1-\alpha} u(0).$$

In order to use the non-local condition (1.2), we apply the fractional integral of order $1-\alpha$ with terminal at c. Thus,

$$I_{c^+}^{1-\alpha} u(t) = I_{c^+}^{1-\alpha} I_{0^+}^\alpha f(t, u(t)) + \frac{I_{0^+}^{1-\alpha} u(0)}{\Gamma(\alpha)} I_{c^+}^{1-\alpha}(s^{\alpha-1})(t).$$

We evaluate both sides at ξ and apply (1.2) to obtain

$$\frac{1}{\beta} I_{0^+}^{1-\alpha} u(0) = I_{c^+}^{1-\alpha} u(\xi) = I_{c^+}^{1-\alpha} I_{0^+}^\alpha f(\cdot, u(\cdot))(\xi) + \frac{I_{0^+}^{1-\alpha} u(0)}{\Gamma(\alpha)} \gamma.$$

Hence

$$I_{0^+}^{1-\alpha} u(0) = \frac{\beta\Gamma(\alpha)}{\Gamma(\alpha) - \beta\gamma} I_{c^+}^{1-\alpha} I_{0^+}^\alpha f(\cdot, u(\cdot))(\xi).$$

Thus u satisfies the integral equation (3.1).

Conversely, if u satisfies the integral equation (3.1), we have

$$u(t) = I_{0^+}^\alpha F(t) + K t^{\alpha-1},$$

where K and $F(t) = f(t, u(t))$ are introduced for convenience. Clearly, the second term is in X. Let $t_1, t_2 \in (0,1]$ with $t_1 < t_2$. Then, since $0 < q(\alpha-1)+1 < 1$,

$$\begin{aligned}
|I_{0^+}^\alpha F(t_2) - I_{0^+}^\alpha F(t_1)| &= \left| \int_0^{t_2} (t_2 - x)^{\alpha-1} F(x) dx - \int_0^{t_1} (t_1 - x)^{\alpha-1} F(x) dx \right| \\
&= \left| \int_{t_1}^{t_2} (t_2 - x)^{\alpha-1} F(x) dx + \int_0^{t_1} ((t_2 - x)^{\alpha-1} - (t_1 - x)^{\alpha-1}) F(x) dx \right| \\
&\le \int_{t_1}^{t_2} (t_2 - x)^{\alpha-1} |F(x)|\, dx + \int_0^{t_1} ((t_1 - x)^{\alpha-1} - (t_2 - x)^{\alpha-1}) |F(x)|\, dx \\
&= C_1 (t_2 - t_1)^{\alpha - 1 + \frac{1}{q}} \|F\|_p + \left[\int_0^{t_1} ((t_1 - x)^{\alpha-1} - (t_2 - x)^{\alpha-1})^q dx \right]^{\frac{1}{q}} \|F\|_p \\
&= C_1 (t_2 - t_1)^{\alpha - 1 + \frac{1}{q}} \|F\|_p + C_1 \left(t_1^{q(\alpha-1)+1} - t_2^{q(\alpha-1)+1} + (t_2 - t_1)^{q(\alpha-1)+1} \right)^{\frac{1}{q}} \|F\|_p,
\end{aligned}$$

where $C_1 > 0$ is a generic constant that depends on α and p.

Thus $u \in C(0,1]$ and, moreover,

$$\lim_{t\to 0^+} |t^{1-\alpha} I_{0^+}^{\alpha} F(t)| \le \lim_{t\to 0^+} C_1 t^{\frac{1}{q}} \|F\|_p = 0.$$

That is,

$$\lim_{t\to 0^+} t^{1-\alpha} u(t) = K$$

and $u \in X$. By Theorem 2.2, $I_{0^+}^{1-\alpha} u = I_{0^+}^{1} F + \Gamma(\alpha) K \in AC(0,1]$. We apply $D_{0^+}^{\alpha}$ to both sides of (3.1) and see from Theorem 2.1 that (1.1) is satisfied. It is easily verified that (1.2) also holds. □

We define the mapping $T : X \to X$ as

$$Tu(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x, u(x))\, dx + \frac{\beta t^{\alpha-1}}{(\Gamma(\alpha) - \beta\gamma)\Gamma(1-\alpha)} \int_c^{\xi} (\xi - x)^{-\alpha} \int_0^x f(y, u(y)) (x-y)^{\alpha-1}\, dy\, dx.$$

From Lemma 3.1 we have $Tu \in C(0,1]$ and $\lim_{t\to 0^+} t^{1-\alpha} Tu(t)$ exists. Thus T is a self-map. We can now state and prove a uniqueness result. First we recall the incomplete beta-function

$$B(z; a, b) = \int_0^z t^{a-1} (1-t)^{b-1}\, dt,$$

which can be found in [12].

Theorem 3.2. *The non-local problem* (1.1), (1.2) *has a unique solution provided*

$$|f(t,u) - f(t,v)| \le |u - v|, \quad u, v \in \mathbb{R}, \quad a.\ e.\ in\ (0,1],$$

where

$$q = L\left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)} + \frac{\beta \Gamma^2(\alpha)}{(\Gamma(\alpha) - \beta\gamma)\,\Gamma(1-\alpha)\Gamma(2\alpha)} \xi^{\alpha} B\left(1 - \frac{c}{\xi}; 1-\alpha, 2\alpha\right)\right) < 1.$$

Proof. Let $u, v \in X$ and introduce for convenience

$$C_2 = \frac{\beta}{(\Gamma(\alpha) - \beta\gamma)\,\Gamma(1-\alpha)}.$$

Then

$$\begin{aligned}
&\|Tu - Tv\| \\
&\le \sup_{t\in(0,1]} t^{1-\alpha} \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} (f(x,u(x)) - f(x,v(x))) dx \right| \\
&\quad + C_2 \left| \int_c^{\xi} (\xi - x)^{-\alpha} \int_0^x (f(y,u(y)) - f(y,v(y)))(x-y)^{\alpha-1} dy\, dx \right| \\
&= \sup_{t\in(0,1]} t^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} |f(x,u(x)) - f(x,v(x))| x^{1-\alpha} x^{\alpha-1} dx
\end{aligned}$$

$$
\begin{aligned}
&+ C_2 \int_c^\xi (\xi - x)^{-\alpha} \int_0^x (x-y)^{\alpha-1} |f(y,u(y)) - f(y,v(y))| y^{1-\alpha} y^{\alpha-1} dy\, dx \\
&\le \sup_{t\in(0,1]} t^{1-\alpha} \frac{L}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} |u(x)-v(x)| x^{\alpha-1} x^{1-\alpha} dx \\
&\quad + LC_2 \int_c^\xi (\xi-x)^{-\alpha} \int_0^x (x-y)^{\alpha-1} |u(y)-v(y)| y^{\alpha-1} y^{1-\alpha} dy\, dx \\
&\le L \left(\sup_{t\in(0,1]} t^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} x^{\alpha-1} dx \right. \\
&\quad \left. + C_2 \int_c^\xi (\xi-x)^{-\alpha} \int_0^x (x-y)^{\alpha-1} y^{\alpha-1} dy\, dx \right) \|u-v\| \\
&= L \left(\sup_{t\in(0,1]} \frac{t^\alpha \Gamma(\alpha)}{\Gamma(2\alpha)} + C_2 \int_c^\xi (\xi-x)^{-\alpha} \frac{\Gamma(\alpha)^2 x^{2\alpha-1}}{\Gamma(2\alpha)} dx \right) \|u-v\| \\
&= L \left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)} + \frac{\Gamma^2(\alpha) C_2}{\Gamma(2\alpha)} \xi^\alpha \int_{\frac{c}{\xi}}^1 (1-s)^{-\alpha} s^{2\alpha-1} ds \right) \|u-v\| \\
&= L \left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)} + \frac{\Gamma^2(\alpha) C_2}{\Gamma(2\alpha)} \xi^\alpha \int_0^{1-\frac{c}{\xi}} s^{-\alpha} (1-s)^{2\alpha-1} ds \right) \|u-v\| \\
&= L \left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)} + \frac{\Gamma^2(\alpha) C_2}{\Gamma(2\alpha)} \xi^\alpha B\left(1 - \frac{c}{\xi}; 1-\alpha, 2\alpha\right) \right) \|u-v\| \\
&= q\|u-v\|,
\end{aligned}
$$

where $q < 1$.

Thus T is a contractive mapping and, by the Banach fixed point theorem, T has a unique fixed point, which is a solution of (1.1), (1.2). □

4. AN EXISTENCE CRITERION

An existence result can be obtained by the Leray-Schauder continuation principle (see, e.g., [13]):

Theorem 4.1. *Let X be a Banach space and $T : X \to X$ be a compact map. Suppose that there exists an $R > 0$ such that if $u = \lambda T u$ for $\lambda \in (0,1)$, then $\|u\| \le R$. Then T has a fixed point.*

Again, we assume that f is Carathéodory with respect to $L^p(0,1]$, where $p\alpha > 1$ and $q = \frac{p}{p-1}$. The Banach space X is the same as before. By a solution of the non-local problem (1.1), (1.2) we understand a function

$$
u \in \operatorname{dom} L = \{u : I_{0^+}^{1-\alpha} u \in AC(0,1],\ D_{0^+}^\alpha u \in L^p(0,1],\ \text{and (1.2) holds}\},
$$

which satisfies (1.1), and $L = D_{0^+}^\alpha$.

Lemma 4.2. *Let $g \in L_p[0,1]$, where $p > \frac{1}{\alpha}$ and $q = \frac{p}{p-1}$. Then the solution of the differential equation $Lu = g$ subject to the boundary conditions (1.2) satisfies*

$$\|u\| \leq A\|g\|_p, \tag{4.1}$$

where

$$A = ((\alpha-1)q+1)^{-1/q}\left(\frac{1}{\Gamma(\alpha)} + \frac{\beta}{(\Gamma(\alpha)-\beta\gamma)\,\Gamma(1-\alpha)}\xi^{1/q}B\left(1-\frac{c}{\xi};1-\alpha,\alpha+\frac{1}{q}\right)\right). \tag{4.2}$$

Proof. As in Lemma 3.1,

$$\begin{aligned}
|t^{1-\alpha}u(t)| &\leq \frac{1}{\Gamma(\alpha)}t^{1-\alpha}\int_0^t (t-x)^{\alpha-1}|g(x)|\,dx \\
&\quad + C_2 t^{1-\alpha}\int_c^\xi (\xi-x)^{-\alpha}\int_0^x (x-y)^{\alpha-1}|g(y)|\,dy\,dx \\
&\leq \frac{1}{\Gamma(\alpha)}\frac{t^{1-\alpha}}{((\alpha-1)q+1)^{1/q}}t^{\alpha-1+\frac{1}{q}}\|g\|_p \\
&\quad + C_2\frac{t^{1-\alpha}}{((\alpha-1)q+1)^{1/q}}\int_c^\xi (\xi-x)^{-\alpha}x^{\alpha-1+\frac{1}{q}}\,dx\|g\|_p \\
&\leq ((\alpha-1)q+1)^{-1/q}\left(\frac{1}{\Gamma(\alpha)} + C_2\int_c^\xi (\xi-x)^{-\alpha}x^{\alpha-1+\frac{1}{q}}\,dx\right)\|g\|_p \\
&= ((\alpha-1)q+1)^{-1/q}\left(\frac{1}{\Gamma(\alpha)} + C_2\xi^{1/q}\int_{\frac{c}{\xi}}^1 (1-s)^{-\alpha}s^{\alpha-1+\frac{1}{q}}\,ds\right)\|g\|_p \\
&= ((\alpha-1)q+1)^{-1/q}\left(\frac{1}{\Gamma(\alpha)} + C_2\xi^{1/q}\int_0^{1-\frac{c}{\xi}} s^{-\alpha}(1-s)^{\alpha-1+\frac{1}{q}}\,ds\right)\|g\|_p \\
&= ((\alpha-1)q+1)^{-1/q}\left(\frac{1}{\Gamma(\alpha)} + C_2\xi^{1/q}B\left(1-\frac{c}{\xi};1-\alpha,\alpha+\frac{1}{q}\right)\right)\|g\|_p.
\end{aligned}$$

The assertion follows. □

Theorem 4.3. *Assume that f is Carathéodory with respect to $L^p(0,1]$, where $p\alpha > 1$ and $q = \frac{p}{p-1}$ and satisfies*

$$|f(t,u)| \leq \beta(t)|u| + \gamma(t), \tag{4.3}$$

where $A\|t^{\alpha-2}\beta\|_p < 1$ and A is given by (4.2).

Then the boundary value problem (1.1), (1.2) has at least one solution.

Proof. We consider, for $\lambda \in (0,1)$,

$$D_{0+}^{\alpha}u(t) = \lambda f(t,u(t)), \quad \text{a. e. in} \quad (0,1), \tag{4.4}$$

subject to the boundary conditions (1.2). It readily follows from the results in Section 3 that the function $u \in X$ is a solution of the boundary value problem (4.4), (1.2) if $u \in X$ is a solution of $u = \lambda Tu$.

Using (4.3) and (4.1), we obtain

$$\begin{aligned}\|u\| &= \lambda A\|f(\cdot,u)\|_p \leq A\|\beta u\|_p + A\|\gamma\|_p \\ &\leq A\|t^{\alpha-1}\beta t^{1-\alpha}u\|_p + A\|\gamma\|_p \leq A\|t^{\alpha-1}\beta\|_p\|u\| + A\|\gamma\|_p.\end{aligned}$$

Hence,

$$\|u\| \leq \frac{A\|\gamma\|_p}{1 - A\|t^{\alpha-1}\beta\|_p},$$

that is, the solution set of $u = \lambda Tu$, $\lambda \in (0,1)$, is *a priori* bounded in X by a constant independent of λ. The mapping T is compact. Since the *a priori* boundedness condition of Theorem 4.1 is verified, the assertion follows. □

REFERENCES

[1] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives*. Gordon and Breach Science Publishers, Yverdon, Switzerland 1993.

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North Holland Mathematics Studies, 204, Elsevier.

[3] I. Podlubny, *Fractional Differential Equations, Mathematics in Sciences and Applications*, Academic Press, New York, 1999.

[4] A. B. Malinowska, D. F. Torres, Generalized natural boundary conditions for fractional variational problems in terms of the Caputo derivative, *Comput. Math. Appl.*, **59** (2010), 3110–3116.

[5] C. Bai, Solvability of multi-point boundary value problem of nonlinear impulsive fractional differential equation at resonance, *Electron. J. of Qual. Theory Differential Equ.*, **89** (2011), 1–19.

[6] L. M. B. C. Campos, On the solution of some simple fractional differential equations. *Internat. J. Math. Math. Sci.*, **13** (1990), 481–496.

[7] Y. F. Luchko, H. M. Srivastava, The exact solution of certain differential equations of fractional order by using operational calculus, *Comp. Math. Appl.*, **29** (1995), 73–85.

[8] S. Sun, Y. Zhao, Z. Han, and M. Xu, Uniqueness of positive solutions for boundary value problems of singular fractional differential equations, *Inverse Probl. Sci. Eng.*, **20** (2012), 299–309.

[9] Y. Liu, W. Zhang, Z., and X. Liu, A sufficient condition for existence of positive solution for nonlinear fractional differential equation with the Riemann-Liouville derivative, *Appl. Math. Lett.* **25** (2011), 1986–1992.

[10] N. Kosmatov, A singular boundary value problem for nonlinear differential equations of fractional order, *J. Appl. Math. Comput.*, **29** (2009), 125–135.

[11] N. Kosmatov, A boundary value problem of fractional order at resonance, *Electron. J. Differential Equations* **2010**, No. 135, 1–10.

[12] J. Dutka, The incomplete Beta function - historical profile, *Arch. Hist. Exact Sci.*, **24** (1981), 11–29.

[13] E. Zeidler, *Nonlinear Functional Analysis and Applications, I: Fixed Point Theorems*, Springer-Verlag, New York, 1986.