Dynamic Systems and Applications 23 (2014) 179-188

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SECOND ORDER NONLINEAR DYNAMIC EQUATIONS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. This paper is concerned with the asymptotic behavior of solutions of second order nonlinear dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + p\,(t)\,\phi_{\gamma}(x(t)) = 0,$$

on an above-unbounded time scale $\mathbb{T}$ where γ is a positive constant and where, in addition, r and p are real-valued, rd-continuous functions on $\mathbb{T}$ with no explicit sign assumptions on p. Our results are established for a time scale $\mathbb{T}$ without assuming certain restrictive conditions on $\mathbb{T}$. Several examples illustrating our results will be given.

AMS (MOS) Subject Classification. 34K11, 39A10, 39A99.

1. PRELIMINARIES

Following Hilger's landmark paper [18], there have been plenty of references focused on the theory of time scales in order to unify continuous and discrete analysis, where a time scale is an arbitrary nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, e.g., $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ for $q > 1$ (which has important applications in quantum theory), $\mathbb{T} = h\mathbb{N}$ with $h > 0$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{H}_n$ the space of the harmonic numbers. For the notions used below we refer to [6, 7] that provides some basic facts on time scale.

We are concerned with the asymptotic behavior of solutions of second order nonlinear dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)\phi_{\gamma}(x(t)) = 0 \tag{1.1}$$

on an above-unbounded time scale $\mathbb{T}$, where $\varphi_{\gamma}(u) := |u|^{\gamma-1}u$ with $\gamma > 0$; r and p are real-valued, rd-continuous functions on $\mathbb{T}$ with $r > 0$. By a solution of (1.1) we mean a nontrivial real–valued function $x \in C^1_{rd}[T_x, \infty)$, $T_x \geq t_0$ which has the

Received August 2013

property that $rx^{\Delta} \in C_r^1[T_x, \infty)$ and satisfies equation (1.1) on $[T_x, \infty)$, where C_{rd} is the space of $rd-$continuous functions. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. We refer the reader to the papers [2, 3, 11, 12, 13, 14, 15, 16, 17], and the references cited therein.

Note that, in the special case when $\mathbb{T} = \mathbb{R}$, $r = 1$ and γ is quotient of odd positive integrers, (1.1) becomes the second order Emden-Fowler differential equation

$$x''(t) + p(t)x^{\gamma}(t) = 0. \tag{1.2}$$

The Emden-Fowler equation has several physical applications in astrophysics, see Bellman [4] and Fowler [9]. Moore and Nehari [10] have proved the follwoing: If p is a positive and continuous function and $\gamma \geq 1$, then (1.2) has a solution for which

$$\lim_{t\to\infty} \frac{x(t)}{t} = A > 0,$$

if and only if

$$\int^{\infty} t^{\gamma} p(t) dt < \infty.$$

This is related to results of Atkinson [1] who showed that if $\gamma > 1$, $p(t) \geq 0$ and is nonincreasing, then every solution of (1.2) is nonoscillatory. Wong [21] has established the sufficiently part of [10] with no explicit sign on $p(t)$.

Recently, Erbe, Baoguo and Peterson [8] have proved the following: If $[t_0, \infty)_{\mathbb{T}}$ satifies condition (C), $\gamma > 0$ is the quotient of odd positive integers and

$$\int_{t_0}^{\infty} t^{\gamma} |p(t)| \Delta t < \infty,$$

(and if $\gamma = 1$, assume $\lim_{t\to\infty} tp(t)\mu(t) = 0$). Then

$$x^{\Delta\Delta} + p(t)x^{\gamma}(t) = 0,$$

has a solution satisfying $\lim_{t\to\infty} \dfrac{x(t)}{t} = A \neq 0.$

To be precise, we say $\mathbb{T}$ satisfies condition (C), that is there is an $M > 0$ such that $\chi(t) \leq M\mu(t)$, $t \in \mathbb{T}$, where χ is the characteristic function of the set $\hat{\mathbb{T}} = \{t \in \mathbb{T} : \mu(t) > 0\}$. We note that if $\mathbb{T}$ satisfies condition (C), then the subset $\check{\mathbb{T}}$ of $\mathbb{T}$ defined by

$$\check{\mathbb{T}} = \{t \in \mathbb{T} : t > 0 \text{ is right-scattered or left-scattered}\},$$

is necessarily countable and $\hat{\mathbb{T}} \subset \check{\mathbb{T}}$. Then, we can rewrite $\check{\mathbb{T}}$ by

$$\check{\mathbb{T}} = \{t_i \in \mathbb{T} : 0 < t_1 < t_2 < \cdots < t_n < \cdots\},$$

and so

$$\mathbb{T} = \check{\mathbb{T}} \cup [\cup_{n\in A} (t_{n-1}, t_n)],$$

where A is the set of all integers for which the real open interval (t_{n-1}, t_n) is contained in $\mathbb{T}$.

Therefore it will be of great interest to prove the results of [8] for the general nonlinear dynamic equation (1.1). We note that in our results, the equation involves a more γ is a positive constant, the coefficient functions r and p, and, in addition, p may change sign. Our work applies to general time scales without assuming condition (C). Several examples are given to illustrate the main results.

2. MAIN RESULTS

Before stating our main results, we need the following lemmas.

Lemma 2.1 ([8, Lemma 2.2]). *Assume that $P \in \mathcal{R}$, $\lim_{t\to\infty} P(t)\mu(t) = 0$ and $\int_{t_0}^{\infty} |P(t)|\,\Delta t < \infty$. Then $e_P(t,t_0)$ is bounded above on $[t_0,\infty)_{\mathbb{T}}$.*

Next, in particular, Gronwall inequality on time scale which has been proved in [6, Corollary 6.7].

Lemma 2.2. *Assume that y and P are rd-continuous functions on $[t_0,\infty)_{\mathbb{T}}$ with $P \in \mathcal{R}^+$. Then*

$$y\,(t) \le y_0 + \int_{t_0}^{t} P(s)y(s)\Delta s, \quad t \in [t_0,\infty)_{\mathbb{T}},$$

implies

$$y(t) \le y_0 e_P(t,t_0), \quad t \in [t_0,\infty)_{\mathbb{T}}.$$

We prove the next lemma without assuming certain restrictive conditions on $\mathbb{T}$ and for $\gamma \neq 1$.

Lemma 2.3. *Assume that y, P and Q are positive rd-continuous functions on $[t_0,\infty)_{\mathbb{T}}$ and assume $y_0 > 0$ and $\gamma \neq 1$. Then*

$$y\,(t) \le y_0 + \int_{t_0}^{t} P(s)y^{\sigma}(s)\Delta s + \int_{t_0}^{t} Q(s)y^{\gamma}(s)\Delta s, \tag{2.1}$$

implies

$$\frac{y^{1-\gamma}(t)}{1-\gamma} \le [e_P(t,t_0)]^{1-\gamma}\left[\frac{y_0^{1-\gamma}}{1-\gamma} + \int_{t_0}^{t} Q(s)e_{-P}^{1-\gamma}(s,t_0)\Delta s\right],$$

for $t \in [t_0,\infty)_{\mathbb{T}}$.

Proof. Let, for $t \in [t_0,\infty)_{\mathbb{T}}$,

$$z(t) = y_0 + \int_{t_0}^{t} P(s)y^{\sigma}(s)\Delta s + \int_{t_0}^{t} Q(s)y^{\gamma}(s)\Delta s > 0.$$

Then (2.1) can be rewritten as

$$y(t) \le z(t), \qquad \text{for } t \in [t_0,\infty)_{\mathbb{T}},$$

and so

$$z^{\Delta}(t) = P(t)y^{\sigma}(t) + Q(t)y^{\gamma}(t) \le P(t)z^{\sigma}(t) + Q(t)z^{\gamma}(t).$$

Hence

$$z^{\Delta}(t) - P(t)z^{\sigma}(t) \leq Q(t)z^{\gamma}(t), \qquad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.$$

Multilping by the integrating factor $e_{-P}(t, t_0)$, we get

$$e_{-P}(t, t_0)z^{\Delta}(t) - P(t)e_{-P}(t, t_0)z^{\sigma}(t) \leq Q(t)e_{-P}(t, t_0)z^{\gamma}(t),$$

and so

$$(e_{-P}(t, t_0)z(t))^{\Delta} \leq Q(t)e_{-P}(t, t_0)z^{\gamma}(t).$$

Therefore, for $t \in [t_0, \infty)_{\mathbb{T}}$,

$$u^{\Delta}(t) \leq Q(t)e_{-P}^{1-\gamma}(t, t_0)u^{\gamma}(t),$$

where $u(t) := e_{-P}(t, t_0)z(t) > 0$. It follows that

$$\frac{u^{\Delta}(t)}{u^{\gamma}(t)} \leq Q(t)e_{-P}^{1-\gamma}(t, t_0), \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.$$

Integrating the above inequality from t_0 to t, we obtain

$$\int_{t_0}^{t} \frac{u^{\Delta}(s)}{u^{\gamma}(s)}\Delta s \leq \int_{t_0}^{t} Q(s)e_{-P}^{1-\gamma}(s, t_0)\Delta s. \tag{2.2}$$

We claim that

$$\int_{t_0}^{t} \frac{u^{\Delta}(s)}{u^{\gamma}(s)}\Delta s \geq \frac{1}{1-\gamma}[u^{1-\gamma}(t) - u^{1-\gamma}(t_0)],$$

for all $t \geq t_0$. Define

$$F(u(t)) := \int_{u(t_0)}^{u(t)} \frac{d\tau}{\tau^{\gamma}} = \frac{1}{1-\gamma}[u^{1-\gamma}(t) - u^{1-\gamma}(t_0)].$$

By Pötzsche chain rule ([6, Theorem 1.90]), we have

$$(F(u(t)))^{\Delta} = \int_0^1 F'(u_h(t))\, dh\, u^{\Delta}(t) = \int_0^1 \frac{u^{\Delta}(t)}{(u_h(t))^{\gamma}} dh. \tag{2.3}$$

For a fixed point t, $t \geq t_0$, we have

$$u_h(t) = (1-h)u(t) + hu^{\sigma}(t) \begin{cases} \leq u(t), & u^{\Delta}(t) \leq 0 \\ \geq u(t), & u^{\Delta}(t) \geq 0, \end{cases}$$

and so

$$\frac{u^{\Delta}(t)}{(u_h(t))^{\gamma}} \leq \begin{cases} \dfrac{u^{\Delta}(t)}{(u(t))^{\gamma}}, & u^{\Delta}(t) \leq 0 \\ \dfrac{u^{\Delta}(t)}{(u(t))^{\gamma}}, & u^{\Delta}(t) \geq 0. \end{cases}$$

Then

$$\frac{u^{\Delta}(t)}{(u_h(t))^{\gamma}} \leq \frac{u^{\Delta}(t)}{u^{\gamma}(t)}, \quad \text{for } t \geq t_0,$$

and so, from (2.3), we have

$$(F(u(t)))^{\Delta} \leq \frac{u^{\Delta}(t)}{u^{\gamma}(t)}, \quad \text{for } t \geq t_0.$$

Hence it follows that, for $t \geq t_0$,

$$\int_{t_0}^{t} \frac{u^{\Delta}(s)}{u^{\gamma}(s)} \Delta s \geq F(u(t)) = \int_{u(t_0)}^{u(t)} \frac{d\tau}{\tau^{\gamma}} = \frac{1}{1-\gamma}[u^{1-\gamma}(t) - u^{1-\gamma}(t_0)]. \tag{2.4}$$

Then, from (2.2) and (2.4), we get the disired result

$$\frac{u^{1-\gamma}(t)}{1-\gamma} \leq \frac{u^{1-\gamma}(t_0)}{1-\gamma} + \int_{t_0}^{t} Q(s) e_{-P}^{1-\gamma}(s, t_0) \Delta s,$$

and consequently

$$\frac{u^{1-\gamma}(t)}{1-\gamma} \leq \frac{y_0^{1-\gamma}}{1-\gamma} + \int_{t_0}^{t} Q(s) e_{-P}^{1-\gamma}(s, t_0) \Delta s.$$

Therefore

$$\begin{aligned} \frac{y^{1-\gamma}(t)}{1-\gamma} \leq \frac{z^{1-\gamma}(t)}{1-\gamma} &= [e_P(t, t_0)]^{1-\gamma} \frac{u^{1-\gamma}(t)}{1-\gamma} \\ &\leq [e_P(t, t_0)]^{1-\gamma} \left[\frac{y_0^{1-\gamma}}{1-\gamma} + \int_{t_0}^{t} Q(s) e_{-P}^{1-\gamma}(s, t_0) \Delta s \right]. \end{aligned}$$

This completes the proof. □

Now, we will state and prove the main results.

Theorem 2.4. *Assume that there exists a sufficiently large $T \geq t_0$ such that $R(t, t_0) \geq 1$, for $t \geq T$ and*

$$\int_{t_0}^{\infty} R^{\gamma}(t, t_0) \left| p(t) \right| \Delta t < \infty, \tag{2.5}$$

(and, if $\gamma = 1$, suppose $\lim_{t\to\infty} R(t, t_0) \left| p(t) \right| \mu(t) = 0$), where $R(t, t_0) := \int_{t_0}^{t} \frac{\Delta s}{r(s)}$. Then Eq. (1.1) *has a solution x satisfying that*

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} r(s) x^{\Delta}(s) \Delta s = A \neq 0. \tag{2.6}$$

Proof. Let $x(t)$ is a solution of Eq. (1.1) with $x(t_0) \neq 0$. By integrating by parts, we obtain

$$\begin{aligned} \int_{t_0}^{t} R(t, \sigma(s))(r(s)x^{\Delta}(s))^{\Delta} \Delta s &= R(t, s) r(s) x^{\Delta}(s) \Big|_{t_0}^{t} - \int_{t_0}^{t} [R(t, s)]^{\Delta_s} r(s) x^{\Delta}(s) \Delta s \\ &= -R(t, t_0)(r x^{\Delta})(t_0) - \int_{t_0}^{t} \frac{-1}{r(s)} r(s) x^{\Delta}(s) \Delta s \\ &= -R(t, t_0)(r x^{\Delta})(t_0) + x(t) - x(t_0). \end{aligned}$$

Therefore, from (1.1), we have

$$x(t) = x(t_0) + R(t, t_0)(r x^{\Delta})(t_0) - \int_{t_0}^{t} R(t, \sigma(s)) p(s) \phi_{\gamma}(x(s)) \Delta s,$$

which implies

$$|x(t)| \le |x(t_0)| + R(t,t_0)\ \left|(rx^{\Delta})(t_0)\right| + \int_{t_0}^{t} R(t,\sigma(s))\,|p(s)|\,|x(s)|^{\gamma}\,\Delta s$$

$$\le |x(t_0)| + R(t,t_0)\ \left|(rx^{\Delta})(t_0)\right| + R(t,t_0)\int_{t_0}^{t} |p(s)|\,|x(s)|^{\gamma}\,\Delta s. \tag{2.7}$$

Since, for sufficiently large $T \ge t_0$ such that $R(t,t_0) \ge 1$, for $t \ge T$. Then (2.7) can be written as

$$|x(t)| \le KR(t,t_0) + R(t,t_0)\int_{t_0}^{t} R^{\gamma}(s,t_0)\,|p(s)|\left[\frac{|x(s)|}{R(s,t_0)}\right]^{\gamma}\Delta s,$$

where $K := |x(t_0)| + \left|(rx^{\Delta})(t_0)\right| > 0$. It follows that, for $t \ge T$,

$$y(t) \le K + \int_{t_0}^{t} R^{\gamma}(s,t_0)\,|p(s)|\,y^{\gamma}(s)\Delta s \tag{2.8}$$

where $y(t) := \dfrac{|x(t)|}{R(t,t_0)}$. Now, we want to prove that $y(t)$ is bounded above on $[T,\infty)_{\mathbb{T}}$. We consider the following two cases:
(a) $\gamma = 1$. Then by Lemma 2.2, (2.8) implies

$$y(t) \le Ke_{R|p|}(t,t_0), \quad \text{for } t \ge T.$$

Since

$$\int_{t_0}^{\infty} R(t,t_0)\,|p(t)|\,\Delta t < \infty \text{ and } \lim_{t\to\infty} R(t,t_0)\,|p(t)|\,\mu(t) = 0,$$

we obtain, by Lemma 2.1 that $e_{R|p|}(t,t_0)$ is bounded above and therefore $y(t)$ is bounded above on $[T,\infty)_{\mathbb{T}}$, for $\gamma = 1$.
(b) $\gamma \ne 1$. Because of (2.5), without loss of generality, we can assume

$$K^{1-\gamma} + (1-\gamma)\int_{t_0}^{\infty} R^{\gamma}(s,t_0)\,|p(s)|\,\Delta s > 0.$$

Then, by Lemma 2.3, (2.8) implies

$$\frac{y^{1-\gamma}(t)}{1-\gamma} \le \left[\frac{K^{1-\gamma}}{1-\gamma} + \int_{t_0}^{t} R^{\gamma}(s,t_0)\,|p(s)|\,\Delta s\right].$$

and so

$$y(t) \le \left[K^{1-\gamma} + (1-\gamma)\int_{t_0}^{t} R^{\gamma}(s,t_0)\,|p(s)|\,\Delta s\right]^{\frac{1}{1-\gamma}}$$

$$\le \left[K^{1-\gamma} + (1-\gamma)\int_{t_0}^{\infty} R^{\gamma}(s,t_0)\,|p(s)|\,\Delta s\right]^{\frac{1}{1-\gamma}} \overset{(2.5)}{<} \infty.$$

Therefore $y(t)$ is bounded above on $[T,\infty)_{\mathbb{T}}$, for $\gamma \ne 1$. Then, from Cases (a) and (b), we have $y(t)$ is bounded above on $[T,\infty)_{\mathbb{T}}$, for all γ. It means that there exists $K_1 > 0$ such that $y(t) < K_1$, for $t \in [T,\infty)_{\mathbb{T}}$ and so

$$|x(t)| \le K_1 R(t,t_0), \quad \text{for } t \in [T,\infty)_{\mathbb{T}}. \tag{2.9}$$

By Eq. (1.1), we have

$$r(t)x^{\Delta}(t)=r(t_0)x^{\Delta}(t_0)-\int_{t_0}^{t}p(s)\phi_\gamma(x(s))\Delta s. \tag{2.10}$$

Since

$$\left|\int_{t_0}^{t}p(s)\phi_\gamma(x(s))\Delta s\right|\le K_1^{\gamma}\int_{t_0}^{\infty}R^{\gamma}(s,t_0)\left|p(s)\right|\Delta s<\infty,$$

we get $\lim_{t\to\infty} r(t)x^{\Delta}(t)=L$ exists. Then, for given $\epsilon>0$ there exits a $T_1\in[T,\infty)_{\mathbb{T}}$ such that

$$A-\epsilon<r(t)x^{\Delta}(t)<A+\epsilon,\quad \text{for } t\in[T_1,\infty)_{\mathbb{T}}\,.$$

It is easy to show that

$$(A-\epsilon)\left(1-\frac{T_1}{t}\right)<\frac{1}{t}\int_{T_1}^{t}r(s)x^{\Delta}(s)\Delta s<(A+\epsilon)\left(1-\frac{T_1}{t}\right),$$

which implies

$$\lim_{t\to\infty}\frac{1}{t}\int_{T_1}^{t}r(s)x^{\Delta}(s)\Delta s=A.$$

Now, we want to show that A is nonzero. Let $x(t)$ is a solution of Eq. (1.1) on $[T_2,\infty)_{\mathbb{T}}$, $T_2\ge T_1$, whose initial conditions satisfy

$$\left|x(T_2)\right|=\alpha>0,\quad \left|r(T_2)x^{\Delta}(T_2)\right|=\beta>0,\quad T_2\in[T_1,\infty)_{\mathbb{T}}\,.$$

Using (2.10), we have

$$\begin{aligned}\left|r(t)x^{\Delta}(t)\right|&=\left|r(T_2)x^{\Delta}(T_2)-\int_{T_2}^{t}p(s)\phi_\gamma(x(s))\Delta s\right|\\&\ge\left|r(T_2)x^{\Delta}(T_2)\right|-\left|\int_{T_2}^{t}p(s)\phi_\gamma(x(s))\Delta s\right|\\&\ge\beta-\int_{T_2}^{\infty}\left|p(s)\right|\left|x(s)\right|^{\gamma}\Delta s\\&\ge\beta-K_1^{\gamma}\int_{T_2}^{\infty}\left|p(s)\right|R^{\gamma}(s,t_0)\Delta s. \qquad (2.11)\end{aligned}$$

Since

$$\lim_{T_2\to\infty}\int_{T_2}^{\infty}\left|p(s)\right|R^{\gamma}(s,t_0)\Delta s=0. \tag{2.12}$$

From (2.11) and (2.12). then there is a $T_3\in[T_2,\infty)_{\mathbb{T}}$, sufficiently large, such that

$$\left|r(t)x^{\Delta}(t)\right|\ge\frac{\beta}{2}>0,\quad \text{for } t\in[T_3,\infty)_{\mathbb{T}},$$

which implies

$$\lim_{t\to\infty}\frac{1}{t}\int_{T}^{t}r(s)x^{\Delta}(s)\Delta s=A\neq 0.$$

This completes the proof. □

Example 2.5. Let $t_0 > 0$ and $\mathbb{T} = \{t_n : n \in \mathbb{N}_0\}$ is a discrete time scale and consider the dynamic equation

$$\Delta\left(r\left(t_n\right)\Delta x(t_n)\right) + p(t_n)\phi_\gamma(x(t_n)) = 0, \tag{2.13}$$

Define

$$r\left(t_n\right) := \frac{1}{t_n^{\beta_1 - 1}}, \quad p\left(t_n\right) := \frac{(-1)^n}{t_n^{\beta_2}},$$

with $\beta_i \in \mathbb{R}$, $i = 1, 2$ such that $\beta_1 \geq 1$ and $\beta_2 > 1 + \gamma\beta_1$. The Eq. (2.13) has a solution x satisfying (2.6), since

$$t^{\beta_1} \geq R(t, t_0) = \int_{t_0}^{t} \frac{\Delta s}{r(s)} = \int_{t_0}^{t} s^{\beta_1 - 1}\Delta s \geq t_0^{\beta_1 - 1}\left(t - t_0\right) > 1,$$

for $t \geq T \geq t_0^{1-\beta_1}\left(1 + t_0^{\beta_1}\right)$, and

$$\int_{t_0}^{\infty} R^\gamma(t, t_0)\left|p(t)\right|\Delta t \leq \sum_{n=0}^{\infty} \frac{\Delta t_n}{t_n^{\beta_2 - \gamma\beta_1}} < \infty,$$

for those time scales $[t_0, \infty)_\mathbb{T}$, where $\int_{t_0}^{\infty} \frac{1}{t^p}\Delta t < \infty$ when $p > 1$. This holds for many time scales (see Theorems 5.64 and 5.65 in [7] and see Example 5.63 where this result does not hold).

Example 2.6. Let $\mathbb{T} = \mathbb{R}$ and consider the differential equation

$$\left(r\left(t\right)x'(t)\right)' + p(t)\phi_\gamma(x(t)) = 0, \tag{2.14}$$

with $r\left(t\right)$ is a positive nonincreasing function on $[t_0, \infty)$ and

$$p\left(t\right) := r^\gamma\left(t\right)\left[\frac{\lambda_1}{t^{\delta_1}} + \frac{\lambda_2 \sin t}{t^{\delta_2}}\right],$$

where $\lambda_i, \delta_i \in \mathbb{R}$, $i = 1, 2$ such that $\delta_i > 1 + \gamma$, $i = 1, 2$. It is easy to see that

$$t \geq R(t, t_0) \geq \frac{t - t_0}{r(t_0)} > 1, \quad \text{for } t \geq T \geq r(t_0) + t_0,$$

and

$$\int_{t_0}^{\infty} R^\gamma(t, t_0)\left|p(t)\right| dt \leq \int_{t_0}^{\infty}\left[\frac{|\lambda_1|}{t^{\delta_1 - \gamma}} + \frac{|\lambda_2|}{t^{\delta_2 - \gamma}}\right] dt < \infty.$$

The Eq. (2.14) has a solution x satisfying

$$\lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} r(s)x^\Delta(s)\Delta s = A \neq 0.$$

We leave to the interested reader the construction of additional examples on other time scales.

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