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ON POSITIVE SOLUTIONS OF STURM-LIOUVILLE BOUNDARY-VALUE PROBLEM FOR FOURTH-ORDER IMPULSIVE DIFFERENTIAL EQUATIONS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT: In this paper, we study Sturm-Liouville boundary-value problem for fourth-order impulsive differential equations. Applying a three critical points theory, new existence results of positive solutions are obtained.

Keywords: Sturm-Liouville boundary-value problem; Positive solutions; Three critical points theory; Variational methods; Impulsive effects.

AMS (MOS) Subject Classification. 58E30, 34B18

1. Introduction

In this paper, we study the Sturm-Liouville boundary-value problem for the fourth-order impulsive differential equations with a positive parameter λ

$$(1.1)\qquad \begin{cases} u^{(iv)}(t) - u''(t) + u(t) = \lambda f(t, u(t)),\ t \in [0,T] \setminus \{t_1, t_2, \dots, t_l\}, \\ -\Delta u'''(t_i) = \lambda I_{1i}(u(t_i)), \quad i = 1, 2, \dots, l, \\ -\Delta u''(t_i) = \lambda I_{2i}(u'(t_i)), \quad i = 1, 2, \dots, l, \\ au(0) - bu'(0) = 0,\ cu(T) + du'(T) = 0, \\ au''(0) - bu'''(0) = 0,\ cu''(T) + du'''(T) = 0, \end{cases}$$

where a, b, c and d are positive real constants, and $0 = t_0 < t_1 < \cdots < t_l < t_{l+1} = T$, $\Delta u'''(t_i) = u'''(t_i^+) - u'''(t_i^-)$, $\Delta u''(t_i) = u''(t_i^+) - u''(t_i^-)$, where $u'''(t_i^+)$, $u''(t_i^+)$ and $u'''(t_i^-)$, $u''(t_i^-)$ denote the right and the left limits, respectively, of $u'''(t_i)$, $u''(t_i)$ at $t = t_i$ $(i = 1, 2, \dots, l)$.

Recently, there has been much application of variational methods in the study of the existence of solutions for impulsive boundary-value problems [3, 5, 8, 9, 12, 14,

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15, 16, 17]. Related basic information is available in [4, 6]. Some papers for a fourth-order equation are about Neumann boundary condition and Dirichlet condition, the others are mostly for a second-order equation. In [8], the authors studied the following problem

$$\begin{cases} u^{(iv)}(t) + Au''(t) + Bu(t) = f(t, u(t)), \text{ a.e. } t \in [0, T], \\ -\Delta u''(t_j) = I_{1j}(u'(t_j)), \quad j = 1, 2, \ldots, l, \\ -\Delta u'''(t_j) = I_{2j}(u(t_j)), \quad j = 1, 2, \ldots, l, \\ u(0) = u(T) = u''(0^+) = u''(T^-) = 0. \end{cases} \tag{1.2}$$

It was proved that when f, I_{1j} and I_{2j} satisfy some conditions, (1.2) has at least one solution or infinitely many classical solutions by methods of variational methods. Moreover, as far as we know, besides [3, 9, 10, 12] for second-order equations, little study is about the Sturm-Liouville boundary-value problem for second-order equations or higher order. In [9], the following problem with impulsive effects is studied

$$\begin{cases} -(\rho(t)\Phi_p(x'(t)))' + s(t)\Phi_p(x(t)) = f(t, x(t)), \ t \neq t_i, \text{ a.e. } t \in [a, b], \\ -\Delta(\rho(t_i)\Phi_p(x'(t_i))) = I_i(x(t_i)), \quad i = 1, 2, \ldots, l, \\ \alpha x'(a) - \beta x(a) = A, \ \gamma x'(b) + \sigma x(b) = B. \end{cases} \tag{1.3}$$

It is proved that (1.3) has at least two positive solutions by means of variational methods, when f and I_i satisfy some conditions.

On the other hand, in the work about application of variational methods, more and more three critical points theorems are used and generalized [2, 7, 11], we choose one of them, that is, Theorem 2.1 of [2] (see Theorem 2.1). In [2], the authors studied the following equations without impulsive effects

$$\begin{cases} u^{iv} + Au'' + Bu = \lambda f(t, u) \ in[0, 1], \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0. \end{cases} \tag{1.4}$$

They essentially get multiplicity results when f satisfy some conditions.

It is aimed to apply the three critical points theorem used in [2] to problem (1.1), which is with impulsive effects and about Sturm-Liouville boundary conditions. What's more, to prove the existence of positive solutions is also our aim. Taking the impulse effects, the Sturm-Liouville boundary conditions and the transformation for the problem to get nonnegative solutions into account, it will be difficult to get the corresponding variational functional J. Moreover, we must get over the difficulties such as how to make the transformation of problem (1.1) to get the positive solutions and how to prove J (or its subitems Φ, Ψ) and the assumptions that we have chosen to satisfy the conditions of the three critical points theorem.

We assume the following conditions are fulfilled:

(H0) For I_{1i}, I_{2i} and f,

(c1) $f \in C([0,T] \times [0,+\infty);[0,+\infty))$, $I_{1i} \in C([0,+\infty);(-\infty,0])$, $I_{2i} \in C(R;R)$, $i = 1,2,\ldots,l$;

(c2) $f(t,0) = I_{1i}(0) = I_{2i}(0) = 0$ for almost every $t \in [0,T]$ and $I_{2i}(x)x \geq 0$ for all $x \in R$.

This paper is organized as follows. In Section 2, we present some preliminaries and establish the variational structure. In Section 3, we discuss the existence of solutions for problem (1.1), and some examples are given in this section.

2. **Preliminaries and Variational Structure**

In this section, the following three critical points theorem will be needed in our discussion. Let X be a nonempty set and $\Phi, \Psi : X \to R$ be two functions. For all $r,\ r_1,\ r_2 > \inf_X \Phi,\ r_2 > r_1,\ r_3 > 0$, we define

$$\varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty,r[)} \frac{(\sup_{u \in \Phi^{-1}(]-\infty,r[)} \Psi(u)) - \Psi(u)}{r - \Phi(u)},$$

$$\beta(r_1,r_2) := \inf_{u \in \Phi^{-1}(]-\infty,r_1[)} \sup_{v \in \Phi^{-1}([r_1,r_2[)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)},$$

$$\gamma(r_2,r_3) := \frac{\sup_{u \in \Phi^{-1}(]-\infty,r_2+r_3[)} \Psi(u)}{r_3},$$

$$\alpha(r_1,r_2,r_3) := \max\big\{\varphi(r_1), \varphi(r_2), \gamma(r_2,r_3)\big\}.$$

Lemma 2.1 (Theorem 2.1 [2])**.** *Let X be a reflexive real Banach space, $\Phi : X \to R$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative addmits a continuous inverse on X^*, $\Psi : X \to R$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

(1) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$*;*

(2) *for every u_1, u_2 such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$ one has*

$$\inf_{t \in [0,1]} \Psi\big(tu_1 + (1-t)u_2\big) \geq 0.$$

Assume that there are three positive constants r_1, r_2, r_3 with $r_1 < r_2$, such that

(i) $\varphi(r_1) < \beta(r_1,r_2)$*;*

(ii) $\varphi(r_2) < \beta(r_1,r_2)$*;*

(iii) $\gamma(r_2,r_3) < \beta(r_1,r_2)$*.*

Then, for each $\lambda \in]\frac{1}{\beta(r_1,r_2)},\ \frac{1}{\alpha(r_1,r_2,r_3)}[$, the functional $\Phi - \lambda\Psi$ admits three distinct critical points u_1, u_2, u_3 such that $u_1 \in \Phi^{-1}(]-\infty,r_1[)$, $u_2 \in \Phi^{-1}([r_1,r_2[)$ and $u_3 \in \Phi^{-1}(]-\infty,r_2+r_3[)$.

Let $X := \{u \in H^2(0,T) | au(0) - bu'(0) = 0,\ cu(T) + du'(T) = 0\}$ be equipped with the inner product

$$(u,v) = \int_0^T u(t)v(t) + u'(t)v(t) + u''(t)v''(t)dt, \quad \forall u, v \in X,$$

which induces the usual norm

$$\|u\|_X = \Big(\int_0^T |u(t)|^2 + |u'(t)|^2 + |u''(t)|^2 dt\Big)^{\frac{1}{2}}.$$

Since X is sequentially closed in $Y = H^2(0;T)$, being compactly embedded in $C^1([0;T]$ (see the Rellich-Kondrachov theorem in Chapter 6 of [1]), it turns out that $(X; \|\cdot\|_X)$ is a Banach space.

Define the usual norm of $C^1([0,T]),\ L^2(0,T)$, respectively, they are

$$\|u\| = \max\Big\{\max_{t\in[0,T]} |u(t)|,\ \max_{t\in[0.T]} |u'(t)|\Big\},\ \|u\|_{L^2} = \Big(\int_0^T u^2(t)dt\Big)^{\frac{1}{2}}.$$

Lemma 2.2 (Lemma 2.2 [9])**.** *For $u \in X$, let $u^{\pm} = \max\{\pm u, 0\}$. Then the following five properties hold:*

(i) $u \in X \Rightarrow u^+,\ u^- \in X$;
(ii) $u = u^+ - u^-$;
(iii) $\|u^+\|_X \le \|u\|_X$;
(iv) *If (u_n) uniformly converges to u in $C([0,T])$, then (u_n^+) uniformly converges to u^+ in $C([0,T])$;*
(v) $u^+(t)u^-(t) = 0,\ (u^+)'(t)(u^-)'(t) = 0$ *for a.e.* $t \in [0,T]$.

Definition 2.1. A function $u \in X$ is said to be a classical solution of problem (1.1), if u satisfies the equation in (1.1) for a.e. $t \in [0,T] \setminus \{t_1, t_2, \dots, t_l\}$ and the impulsive condition and boundary condition of (1.1). Moreover, u is said to be a nonnegative classical solution of problem (1.1) if $u(t) \ge 0$ for $t \in [0,T]$ and positive classical solution of problem (1.1) if $u(t) \ge 0$ and $u(t) \not\equiv 0$ for $t \in [0,T]$.

Lemma 2.3. *If $u \in C([0,T])$ is a classical solution of problem*

$$\begin{cases} u^{(iv)}(t) - u''(t) + u(t) = \lambda f(t, u^+(t)),\ t \in [0,T] \setminus \{t_1, t_2, \dots, t_l\}, \\ -\Delta u'''(t_i) = \lambda I_{1i}(u^+(t_i)), \quad i = 1, 2, \dots, l, \\ -\Delta u''(t_i) = \lambda I_{2i}((u^+)'(t_i)), \quad i = 1, 2, \dots, l, \\ au(0) - bu'(0) = 0,\ cu(T) + du'(T) = 0, \\ au''(0) - bu'''(0) = 0,\ cu''(T) + du'''(T) = 0, \end{cases} \tag{2.1}$$

then $u(t) \ge 0$ for $t \in [0,T]$, and hence it is a nonnegative classical solution of problem (1.1).

Proof. Since $u \in C[0,T]$ and $f \in C([0,T] \times [0,+\infty);[0,+\infty))$, we have $u^{(iv)} \in C[0,T]\backslash\{t_1,t_2,\dots,t_i\}$. If $u \in C([0,T])$ is a classical solution of problem (2.1), by Lemma 2.2, (H0) and boundary conditions, we have

$$\begin{aligned}
0 &= \int_0^T (u^{(iv)}(t) - u''(t) + u(t) - \lambda f(t,u^+(t)))u^-(t)dt \\
&= \sum_{i=0}^{l} u'''(t)u^-(t)\Big|_{t=t_i^+}^{t_{i+1}} - \sum_{i=0}^{l} u''(t)(u^-)'(t)\Big|_{t=t_i^+}^{t_{i+1}} + \int_0^T u''(t)(u^-)''(t)dt \\
&\quad - u'(t)u^-(t)\Big|_0^T + \int_0^T u'(t)(u^-)'(t)dt + \int_0^T u(t)u^-(t)dt - \lambda\int_0^T f(t,u^+(t))u^-(t)dt \\
&= \lambda\sum_{i=1}^{l} I_{1i}(u^+(t_i))u^-(t_i) - \lambda\sum_{i=1}^{l} I_{2i}((u^+)'(t_i))(u^-)'(t_i) + u'''(T)u^-(T) \\
&\quad - u'''(0)u^-(0) - u''(T)(u^-)'(T) + u''(0)(u^-)'(0) - u'(T)u^-(T) + u'(0)u^-(0) \\
&\quad - \lambda\int_0^T f(t,u^+(t))u^-(t)dt - \|u^-\|_X^2 \\
&= -\frac{c}{d}u''(T)u^-(T) - \frac{a}{b}u''(0)u^-(0) - u''(T)(u^-)'(T) + u''(0)(u^-)'(0) \\
&\quad + \frac{c}{d}u(T)u^-(T) + \frac{a}{b}u(0)u^-(0) - \|u^-\|_X^2 \\
&= -\frac{c}{d}u''(T)u^-(T) - \frac{a}{b}u''(0)u^-(0) - u''(T)(u^-)'(T) + u''(0)(u^-)'(0) \\
&\quad - \frac{c}{d}\big(u^-(T)\big)^2 - \frac{a}{b}\big(u^-(0)\big)^2 - \|u^-\|_X^2.
\end{aligned}$$

If $u(T) \geq 0$, $u^-(T) = 0$, $(u^-)'(T) = 0$,

$$-\frac{c}{d}u''(T)u^-(T) - u''(T)(u^-)'(T) = 0;$$

If $u(T) < 0$, $u^-(T) = -u(T)$, $(u^-)'(T) = -u'(T)$,

$$\begin{aligned}
-\frac{c}{d}u''(T)u^-(T) - u''(T)(u^-)'(T) &= \frac{c}{d}u''(T)u(T) + u''(T)u'(T) \\
&= \frac{c}{d}u''(T)u(T) + u''(T)(-\frac{c}{d}u(T)) = 0.
\end{aligned}$$

Similarly, $-\frac{a}{b}u''(0)u^-(0) + u''(0)(u^-)'(0) = 0$. One has

$$0 = -\frac{c}{d}\big(u^-(T)\big)^2 - \frac{a}{b}\big(u^-(0)\big)^2 - \|u^-\|_X^2 \leq -\|u^-\|_X^2,$$

so $u^-(t) = 0$ for $t \in [0,T]$, that is, $u(t) \geq 0$. The proof is complete. □

Remark 2.1. By Lemma 2.3, it suffices to obtain classical solutions of (2.1) in order to find the nonnegative classical solutions of problem (1.1). Moreover, the nonnegative classical solutions would be positive classical solutions when verifying they are not equivalent to 0.

For each $u \in X$, set

$$\Phi(u) = \frac{1}{2}\|u\|_X^2 + \frac{c}{2d}\big(u(T)\big)^2 + \frac{a}{2b}\big(u(0)\big)^2, \tag{2.2}$$

$$\Psi(u) = -\sum_{i=1}^{l}\int_0^{u^+(t_i)} I_{1i}(s)ds + \sum_{i=1}^{l}\int_0^{(u^+)'(t_i)} I_{2i}(s)ds + \int_0^T F(t, u^+(t))dt, \tag{2.3}$$

$$J(u) = \Phi(u) - \lambda\Psi(u), \tag{2.4}$$

where $F(t,u) = \int_0^u f(t,s)ds$.

It is clear that Φ, Ψ and J are differentiable at any $u \in X$. If $u \geq 0$, in some intervals, $u^+ = u$, $(u^+)' = u'$,

$$\Psi'(u)(v) = -\sum_{i=1}^{l} I_{1i}(u^+(t_i))v(t_i) + \sum_{i=1}^{l} I_{2i}((u^+)'(t_i))v'(t_i) + \int_0^T f(t,u^+(t))v(t)dt;$$

If $u < 0$, in some intervals, $u^+ = 0$, $(u^+)' = 0$, by (H0)

$$\begin{aligned} 0 &= \Psi'(u)(v) \\ &= -\sum_{i=1}^{l} I_{1i}(u^+(t_i))v(t_i) + \sum_{i=1}^{l} I_{2i}((u^+)'(t_i))v'(t_i) + \int_0^T f(t,u^+(t))v(t)dt. \end{aligned}$$

So we have

$$\Phi'(u)(v) = \int_0^T (u''v'' + u'v' + uv)dt + \frac{c}{d}u(T)v(T) + \frac{a}{b}u(0)v(0), \tag{2.5}$$

$$\begin{aligned} \Psi'(u)(v) = &-\sum_{i=1}^{l} I_{1i}(u^+(t_i))v(t_i) + \sum_{i=1}^{l} I_{2i}((u^+)'(t_i))v'(t_i) \\ &+ \int_0^T f(t,u^+(t))v(t)dt, \end{aligned} \tag{2.6}$$

$$\begin{aligned} J'(u)(v) = &\int_0^T (u''v'' + u'v' + uv)dt + \lambda\sum_{i=1}^{l} I_{1i}(u^+(t_i))v(t_i) \\ &- \lambda\sum_{i=1}^{l} I_{2i}((u^+)'(t_i))v'(t_i) - \lambda\int_0^T f(t,u^+(t))v(t)dt \\ &+ \frac{c}{d}u(T)v(T) + \frac{a}{b}u(0)v(0). \end{aligned} \tag{2.7}$$

Definition 2.2. A function $u \in X$ is said to be a weak solution of (2.1), if u satisfies $J'(u)(v) = 0$ for all $v \in X$.

Lemma 2.4. *If $u \in X$ is a weak solution of* (2.1)*, then u is a classical solution of* (2.1).

Proof. It is similar to the proof of Lemma 2.4 in [13], so we omit it here. ☐

Lemma 2.5 (Lemma 2.5 of [13]). *Let $u \in X$. Then $\|u\| \le M\|u\|_X$, where*

$$M = (\frac{1}{\sqrt{T}} + \sqrt{T}).$$

We need the following lemmas for applying Lemma 2.1 in Theorem 3.1.

Lemma 2.6. *The functional Φ in (2.2) is convex.*

Proof. Let $\varepsilon \in (0,1)$ and $u, v \in X$,

$$\begin{aligned}
&\Phi(\varepsilon u + (1-\varepsilon)v)\\
&= \frac{1}{2}\|\varepsilon u + (1-\varepsilon)v\|_X^2 + \frac{c}{2d}\big(\varepsilon u(T) + (1-\varepsilon)v(T)\big)^2 + \frac{a}{2b}\big(\varepsilon u(0) + (1-\varepsilon)v(0)\big)^2\\
&= \frac{1}{2}\int_0^T \big|\varepsilon u''(t) + (1-\varepsilon)v''(t)\big|^2 + \big|\varepsilon u'(t) + (1-\varepsilon)v'(t)\big|^2 + \big|\varepsilon u(t)\\
&\quad + (1-\varepsilon)v(t)\big|^2 dt + \frac{c}{2d}\big(\varepsilon u(T) + (1-\varepsilon)v(T)\big)^2 + \frac{a}{2b}\big(\varepsilon u(0) + (1-\varepsilon)v(0)\big)^2\\
&\le \frac{1}{2}\int_0^T \varepsilon|u''(t)|^2 + (1-\varepsilon)|v''(t)|^2 + \varepsilon|u'(t)|^2 + (1-\varepsilon)|v'(t)|^2 + \varepsilon|u(t)|^2\\
&\quad + (1-\varepsilon)|v(t)|^2 dt + \frac{c}{2d}\big(\varepsilon u^2(T) + (1-\varepsilon)v^2(T)\big) + \frac{a}{2b}\big(\varepsilon u^2(0) + (1-\varepsilon)v^2(0)\big)\\
&= \varepsilon\Phi(u) + (1-\varepsilon)\Phi(v),
\end{aligned}$$

which implies that Φ is convex. The proof is complete. □

Lemma 2.7. *$\Phi' : X \to X^*$ admits a continuous inverse on X^*.*

Proof. Firstly, for every $u \in X\backslash\{0\}$, it follows from (2.5) that

$$\begin{aligned}
\lim_{\|u\|_X \to +\infty} \frac{\Phi'(u)(u)}{\|u\|_X} &= \lim_{\|u\|_X \to +\infty} \frac{\|u\|_X^2 + \frac{c}{d}u^2(T) + \frac{a}{b}u^2(0)}{\|u\|_X}\\
&= \lim_{\|u\|_X \to +\infty} \left(\|u\|_X + \frac{\frac{c}{d}u^2(T) + \frac{a}{b}u^2(0)}{\|u\|_X}\right)\\
&= +\infty,
\end{aligned}$$

which means Φ' is coercive.

Moreover, given $u, v \in X$, one has

$$\begin{aligned}
&(\Phi'(u) - \Phi'(v))(u - v) = \|u - v\|_X^2 + \frac{c}{d}\big(u(T) - v(t)\big)^2 + \frac{a}{b}\big(u(0) - v(0)\big)^2\\
&\ge \|u - v\|_X^2,
\end{aligned}$$

so Φ' is uniformly monotone. By Theorem 26.A(d) of [18], we have that $(\Phi')^{-1}$ exists and is continuous on X^*. Thus, $\Phi' : X \to X^*$ admits a continuous inverse on X^*. The proof is complete. □

Lemma 2.8. *$\Psi' : X \to X^*$ is a continuous and compact operator.*

Proof. First we will show that Ψ' is strongly continuous on X. Let $u_n \rightharpoonup u$ as $n \to \infty$ on X; by [18] we have u_n converges uniformly to u on $[0, T]$ as $n \to \infty$. Since f is continuous, one has $f(t, u_n) \to f(t, u)$ as $n \to \infty$. Moreover, $I_{1i},\ I_{2i}$ are continuous. So $\Psi'(u_n) \to \Psi'(u)$, which implies that Ψ' is a compact operator by Proposition 26.2 of [18] and that Ψ' is continuous. The proof is compete. □

3. **Main results**

In this section, we shall show our main results and prove them. We need the following conditions:

(H1) There exists a constant $\mu \in (0, \frac{1}{2})$ and put

$$k := \Big\{2M^2\big[\frac{16}{\mu^3T^3} + \frac{4}{3\mu T} - \frac{37}{60}\mu T + \frac{T}{2}\big]\Big\}^{-1}, \tag{3.1}$$

$$I(s) := \sum_{i=1}^{l} |I_{1i}(s)| + \sum_{i=1}^{l} |I_{2i}(s)|. \tag{3.2}$$

Moreover, there exist four positive constants m, n, p, q, with $\sqrt{k}m < n < \sqrt{k}p < \sqrt{k}q$ such that

(c3)

$$\frac{\int_0^m I(s)ds + \int_0^T F(t,m)dt}{m^2} < k\frac{-\int_0^m I(s)ds + \int_{\mu T}^{(1-\mu)T} F(t,n)dt - \int_0^T F(t,m)dt}{n^2},$$

(c4)

$$\frac{\int_0^p I(s)ds + \int_0^T F(t,p)dt}{p^2} < k\frac{-\int_0^m I(s)ds + \int_{\mu T}^{(1-\mu)T} F(t,n)dt - \int_0^T F(t,m)dt}{n^2},$$

(c5)

$$\frac{\int_0^q I(s)ds + \int_0^T F(t,q)dt}{q^2 - p^2} < k\frac{-\int_0^m I(s)ds + \int_{\mu T}^{(1-\mu)T} F(t,n)dt - \int_0^T F(t,m)dt}{n^2}.$$

Theorem 3.1. *Suppose (H0) and (H1) hold. Then, for every*

$$\lambda \in \Big]\frac{n^2}{2M^2k}\Big(-\int_0^m I(s)ds + \int_{\mu T}^{(1-\mu)T} F(t,n)dt - \int_0^T F(t,m)dt\Big)^{-1},$$
$$\min\Big\{\frac{m^2}{2M^2}\Big(\int_0^m I(s)ds + \int_0^T F(t,m)dt\Big)^{-1},$$
$$\frac{p^2}{2M^2}\Big(\int_0^p I(s)ds + \int_0^T F(t,p)dt\Big)^{-1},$$
$$\frac{q^2-p^2}{2M^2}\Big(\int_0^q I(s)ds + \int_0^T F(t,q)dt\Big)^{-1}\Big\}\Big[,$$

the problem (1.1) *has at least three distinct nonnegative classical solutions* u_i $(i = 1, 2, 3)$*, such that* $\|u_i\| < q$*, for* $i = 1, 2, 3$*, which means the problem* (1.1) *has at least two distinct positive classical solutions.*

Proof. Owing to Lemma 2.4, our aim is to apply Lemma 2.1 to Φ in (2.2) and Ψ in (2.3).

Φ is coercive, obviously, convex by Lemma 2.6 and its Gâteaux derivative admits a continuous inverse by Lemma 2.7. Ψ's Gâteaux derivative is continuous and compact by Lemma 2.8. Clearly, $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. By (H0), we have

$$-\int_0^s I_{1i}(s)ds \geq 0, \quad \int_0^s I_{2i}(s)ds \geq 0, \quad F(t,s) = \int_0^s f(t,s)ds \geq 0,$$

which deduces $\Psi(u) \geq 0$ for all $u \in X$. So Φ and Ψ satisfy the hypotheses of Lemma 2.1.

Let $\bar{v} \in X$ be defined by

$$\bar{v}(t) = \begin{cases} \frac{2n}{\mu^2 T^2} t^2, & t \in [0, \frac{\mu T}{2}], \\ -\frac{2n}{\mu^2 T^2}(t - \mu T)^2 + n, & t \in]\frac{\mu T}{2}, \mu T], \\ n & t \in]\mu T, (1-\mu)T], \\ -\frac{2n}{\mu^2 T^2}[t - (1-\mu)T]^2 + n, & t \in](1-\mu)T, (1-\frac{\mu}{2})T], \\ \frac{2n}{\mu^2 T^2}(t - T)^2, & t \in](1-\frac{\mu}{2})T, T], \end{cases}$$

so

$$\bar{v}'(t) = \begin{cases} \frac{4n}{\mu^2 T^2} t, & t \in [0, \frac{\mu T}{2}], \\ -\frac{4n}{\mu^2 T^2}(t - \mu T), & t \in]\frac{\mu T}{2}, \mu T], \\ 0 & t \in]\mu T, (1-\mu)T], \\ -\frac{4n}{\mu^2 T^2}[t - (1-\mu)T], & t \in](1-\mu)T, (1-\frac{\mu}{2})T], \\ \frac{4n}{\mu^2 T^2}(t - T), & t \in](1-\frac{\mu}{2})T, T]. \end{cases}$$

It is easy to verify that

$$\bar{v}^+ = \bar{v}, \ \bar{v}^- = 0, \tag{3.3}$$

and

$$\Phi(\bar{v}) = \left[\frac{16}{\mu^3 T^3} + \frac{4}{3\mu T} - \frac{37}{60}\mu T + \frac{T}{2}\right] n^2 = \frac{n^2}{2M^2 k}. \tag{3.4}$$

Put $r_1 = \frac{m^2}{2M^2}$, $r_2 = \frac{p^2}{2M^2}$, and $r_3 = \frac{q^2 - p^2}{2M^2}$. From $\sqrt{k}m < n < \sqrt{k}p < \sqrt{k}q$, one has $r_1 < \Phi(\bar{v}) < r_2$, which means $\bar{v} \in \Phi^{-1}([r_1, r_2[)$, and $r_3 > 0$. When $\Phi(u) < r_1$, by Lemma 2.5 and (2.2), we have

$$\max\left\{\max_{t\in[0,T]} |u^+(t)|, \ \max_{t\in[0.T]} |(u^+)'(t)|\right\} \leq M\|u^+\|_X \leq M\|u\|_X \leq \sqrt{2M^2\Phi(u)} < m,$$

hence, with (2.2), (3.2) and (H0) we have

$$
\begin{aligned}
&\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u) \\
&\le \max_{|\xi|\le m}\sum_{i=1}^{l}\int_0^{\xi}(-I_{1i}(s))\,ds + \max_{|\xi|\le m}\sum_{i=1}^{l}\int_0^{\xi}|I_{2i}(s)|ds + \int_0^T \max_{|\xi|\le m} F(t,\xi)dt \qquad (3.5)\\
&= \int_0^m I(s)ds + \int_0^T F(t,m)dt.
\end{aligned}
$$

Similarly, we obtain

$$
\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) \le \int_0^p I(s)ds + \int_0^T F(t,p)dt, \qquad (3.6)
$$

and

$$
\sup_{u\in\Phi^{-1}(]-\infty,r_2+r_3[)} \Psi(u) \le \int_0^q I(s)ds + \int_0^T F(t,q)dt. \qquad (3.7)
$$

Therefore, taking into consideration that $0 \in \Phi^{-1}(]-\infty, r_1[)$ and $0 \in \Phi^{-1}(]-\infty, r_2[)$, from (3.5), (3.6) and (3.7), we have

$$
\begin{aligned}
\varphi(r_1) &\le \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} \qquad (3.8)\\
&\le \frac{2M^2}{m^2}\Big(\int_0^m I(s)ds + \int_0^T F(t,m)dt\Big),\\
\varphi(r_2) &\le \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} \qquad (3.9)\\
&\le \frac{2M^2}{p^2}\Big(\int_0^p I(s)ds + \int_0^T F(t,p)dt\Big),\\
\gamma(r_2,r_3) &= \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2+r_3[)}\Psi(u)}{r_3} \qquad (3.10)\\
&\le \frac{2M^2}{q^2-p^2}\Big(\int_0^q I(s)ds + \int_0^T F(t,q)dt\Big).
\end{aligned}
$$

On the other hand, by (3.3) and the definition of $\bar v$,

$$
\begin{aligned}
\Psi(\bar v) &= -\sum_{i=1}^{l}\int_0^{\bar v(t_i)} I_{1i}(s)ds + \sum_{i=1}^{l}\int_0^{\bar v'(t_i)} I_{2i}(s)ds + \int_0^T F(t,\bar v(t))dt\\
&\ge -\sum_{i=1}^{l}\int_0^{\min_{t\in[0,T]}\bar v(t)} I_{1i}(s)ds + \sum_{i=1}^{l}\int_0^{\min_{t\in[0,T]}|\bar v'(t)|} |I_{2i}(s)|ds \qquad (3.11)\\
&\quad + \int_{\mu T}^{(1-\mu)T} F(t,\bar v(t))dt\\
&= \int_{\mu T}^{(1-\mu)T} F(t,n)dt.
\end{aligned}
$$

Taking into consideration that $\bar{v} \in \Phi^{-1}([r_1, r_2[)$, by (3.4), (3.5), (3.11), (c3) and $\Phi(u) \geq 0$, one has

$$\begin{aligned} \beta(r_1, r_2) &\geq \inf_{u \in \Phi^{-1}(]-\infty, r_1[)} \frac{\Psi(\bar{v}) - \Psi(u)}{\Phi(\bar{v}) - \Phi(u)} \\ &\geq \frac{\int_{\mu T}^{(1-\mu)T} F(t,n)dt - \Big(\int_0^m I(s)ds + \int_0^T F(t,m)dt\Big)}{\frac{n^2}{2M^2k}} \qquad (3.12) \\ &= \frac{2M^2k}{n^2}\Big(-\int_0^m I(s)ds + \int_{\mu T}^{(1-\mu)T} F(t,n)dt - \int_0^T F(t,m)dt\Big). \end{aligned}$$

By (3.8), (3.9), (3.10), (3.12) and (c3)-(c5) of (H1), we have

$$\alpha(r_1, r_2, r_3) < \beta(r_1, r_2).$$

Up to this point, the conditions of Lemma 2.1 are all fulfilled. By Lemma 2.1, it follows that, for each

$$\begin{aligned} \lambda \in \Bigg]& \frac{n^2}{2M^2k}\Big(-\int_0^m I(s)ds + \int_{\mu T}^{(1-\mu)T} F(t,n)dt - \int_0^T F(t,m)dt\Big)^{-1}, \\ &\min\Big\{\frac{m^2}{2M^2}\Big(\int_0^m I(s)ds + \int_0^T F(t,m)dt\Big)^{-1}, \\ &\frac{p^2}{2M^2}\Big(\int_0^p I(s)ds + \int_0^T F(t,p)dt\Big)^{-1}, \\ &\frac{q^2-p^2}{2M^2}\Big(\int_0^q I(s)ds + \int_0^T F(t,q)dt\Big)^{-1}\Big\}\Bigg[, \end{aligned}$$

the functional $J = \Phi - \lambda\Psi$ has three distinct points u_i $(i = 1, 2, 3)$ in X with $\Phi(u_i) < r_2 + r_3$, which by Lemma 2.5 and (2.2) deduces

$$\begin{aligned} \|u_i^+\| &= \max\Big\{\max_{t\in[0,T]} |u_i^+(t)|,\ \max_{t\in[0.T]} |(u_i^+)'(t)|\Big\} \\ &\leq M\|u_i^+\|_X \leq M\|u_i\|_X \leq \sqrt{2M^2\Phi(u_i)} < q. \end{aligned}$$

The proof is complete. □

Remark 3.1. If we choose different $\overline{v}$, the constrictions on F, I_i are different.

Example 3.1. Let $T = 1$, $t_i \in (0,1)$, $a, b, c, d > 0$, $i = 1, 2, \ldots, l$. Consider Sturm-Liouville boundary-value problem with impulse

$$\begin{cases} u^{(iv)}(t) - u''(t) + u(t) = \lambda f(t, u(t)),\ t \in [0,1] \setminus \{t_1, t_2, \ldots, t_l\}, \\ -\Delta u'''(t_i) = \lambda I_{1i}(u(t_i)), \quad i = 1, 2, \ldots, l, \\ -\Delta u''(t_i) = \lambda I_{2i}(u'(t_i)), \quad i = 1, 2, \ldots, l, \\ au(0) - bu'(0) = 0,\ cu(1) + du'(1) = 0, \\ au''(0) - bu'''(0) = 0,\ cu''(1) + du'''(1) = 0, \end{cases} \qquad (3.13)$$

where

$$f(t,s) = -I_{1i}(s) = I_{2i}(s) = \begin{cases} 0, & 0 < s \le 1, \\ 16483t(s-1), & 1 < s \le 2, \\ -16483t(s-3), & 2 < s \le 3, \\ 0, & s > 3. \end{cases}$$

For every $\lambda \in [\frac{1977944}{1977960}, +\infty)$, problem (3.13) has at least two distinct positive classical solutions.

In fact, compared with (1.1), $M = 2$, (H0) is fulfilled. Let $\mu = \frac{1}{4}$, so $k = \frac{30}{247243}$. Considering with $\sqrt{k}m < n < \sqrt{k}p < \sqrt{k}q$, we can choose $m = \frac{1}{2}$, $n = 2$, and sufficiently large p, q, while $q^2 - p^2$ is also sufficiently large. We have

$$(3.14)\qquad \begin{cases} \frac{n^2}{2M^2k}\Big(-\int_0^m I(s)ds + \int_{\mu T}^{(1-\mu)T} F(t,n)dt - \int_0^T F(t,m)dt \Big)^{-1} = \frac{1977944}{1977960}, \\ \frac{m^2}{2M^2}\Big(\int_0^m I(s)ds + \int_0^T F(t,m)dt \Big)^{-1} = \infty, \\ \frac{p^2}{2M^2}\Big(\int_0^p I(s)ds + \int_0^T F(t,p)dt \Big)^{-1} \text{is sufficiently large}, \\ \frac{q^2-p^2}{2M^2}\Big(\int_0^q I(s)ds + \int_0^T F(t,q)dt \Big)^{-1} \text{is sufficiently large}, \end{cases}$$

and that (H1) is satisfied. Applying Theorem 3.1, problem (3.13) has at least two distinct positive classical solutions for every $\lambda \in [\frac{1977944}{1977960}, +\infty)$.

REFERENCES

[1] Robert A. Adams, *Sobolev Spaces*, Academic Press, New York, San Francisco and London, 1975.

[2] G. Bonanno and Beatrice Di Bella, A boundary value problem for fourth-order elastic beam equations, *J. Math. Anal. Appl.*, 343:1166–1176, 2008.

[3] G. Bonanno and G. Riccobono, Multiplicity results for SturmCLiouville boundary value problems, *Appl. Math. Comput.*, 210:294–297, 2009.

[4] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer Verlag, New York, 1989.

[5] J.J. Nieto and D. ORegan, Variational approach to impulsive differential equations, *Nonlinear Anal. Real World Appl.*, 10:680–690, 2009.

[6] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applicatins to Differential Equations*, CBMS Regional Conference Series in Mathematics, Volume 65 (American Mathematical Society, Providence, RI), 1986.

[7] B. Ricceri, On a three critical points theorem, *Arch. Math. (Basel)*, 75:220–226, 2000.

[8] J. Sun, H. Chen and L. Yang, Variational methods to fourth-order impulsive differential equations, *J. Appl. Math. Comput.*, 35:323–340, 2011.

[9] Y. Tian and W. Ge, Applications of variational methods to boundary-value problem for impulsive differential equations, *Proc. Edinb. Math. Soc.*, 51:509–527, 2008.

[10] Y. Tian and W. Ge, Multiple solutions of impulsive Sturm-Liouville boundary value problem via lower and upper solutions and variational methods, *J. Math. Anal. Appl.*, 387:475–489, 2012.

[11] Y. Tian and W. Ge, Second-order Sturm-Liouville boundary value problem involving the one-dimensional p-Laplacian, *Rocky Mountain J. Math.* Volume 38, Number 1, 309–327, 2008.

[12] Y. Tian and W. Ge, Variational methods to Sturm-Liouville boundary value problem for impulsive differential equations, *Nonlinear Anal.*, 72:277–287, 2010.

[13] Y. Tian and X. Liu, Applications of variational methods to Sturm-Liouville boundary-value problem for fourth-order impulsive differential equations, *Math. Methods Appl. Sci.*, DOI: 10.1002/mma.2787.

[14] Y. Tian, J. Wang and W. Ge, Variational methods to mixed boundary value problem for impulsive differential equations with a parameter, *Taiwanese J. Math.* 13 No. 4:1353–1370, 2009.

[15] H. Zhang and Z. Li, Variational approach to impulsive differential equations with periodic boundary conditions, *Nonlinear Anal. Real World Appl.*, 11:67–78, 2010.

[16] Z. Zhang and R. Yuan, An application of variational methods to Dirichlet boundary value problem with impulses, *Nonlinear Anal. Real World Appl.*, 11:155–162, 2010.

[17] J. Zhou and Y. Li, Existence and multiplicity of solutions for some Dirichlet problems with impulsive effects, *Nonlinear Anal.*, 71:2856–2865, 2009.

[18] E. Zeidler, *Nonlinear functional analysis and its applications*, Vol. 2 Springer, Berlin, 1990.