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POSITIVITY OF THE GREEN FUNCTION FOR A FOUR POINT FOURTH ORDER FOCAL BOUNDARY VALUE PROBLEM

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. We consider a four point fourth order boundary value problem of focal type. A sufficient and necessary condition for the positivity of the Green function for the problem is established.

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1. INTRODUCTION

Boundary value problems are important both from a theoretical perspective as well as for their many applications in the physical and engineering sciences. The study of positive solutions for boundary value problems has been very active for the last two decades. In a recent paper [1], Anderson and Avery considered the fourth order four-point right focal boundary value problem

$$u''''(t) + f(u(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = u'(p) = u''(q) = u'''(1) = 0, \tag{1.2}$$

under the assumption that

$$1/2 < p < (1 + p)/2 < q < 1. \tag{1.3}$$

With this motivation, we in this paper consider the boundary value problem that consists of the fourth order equation

$$u''''(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1.4}$$

and the boundary conditions (1.2). Throughout the paper we assume that

(H1) $f : [0, 1] \times (-\infty, \infty) \to (-\infty, \infty)$ is a continuous function, and $p, q \in [0, 1]$ are constants.

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Now we define $G : [0,1] \times [0,1] \to (-\infty, \infty)$ by

$$\begin{aligned} G(t,s) &= -t(p^2/2 + p(q-s)H(q-s) - pq - ((p-s)^2/2)H(p-s)) \\ &\quad - t^2(q/2 - ((q-s)/2)H(q-s)) + t^3/6 - ((t-s)^3/6)H(t-s). \end{aligned} \tag{1.5}$$

Here $H : R \to R$ is the unit step function given by

$$H(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0. \end{cases}$$

Then, $G(t,s)$ is the Green function for the boundary value problem that consists of (1.4) and (1.2). And, the problem (1.4), (1.2) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) f(s, u(s))\, ds, \quad 0 \leq t \leq 1. \tag{1.6}$$

Our goal is to establish a necessary and sufficient condition for the positivity of the Green function in (1.5), and show that the new condition improves (1.3). In this paper, when we say the Green function $G(t,s)$ is positive, we mean $G(t,s) \geq 0$ for $0 \leq t, s \leq 1$.

2. POSITIVITY OF THE GREEN FUNCTION

Anderson and Avery [1] proved the following result.

Theorem 2.1. *If $p, q \in [0,1]$ satisfy the condition (1.3), then*

$$G(t,s) \geq 0, \quad 0 \leq t, s \leq 1.$$

Next, we will improve the condition (1.3). To this end, we begin with a technical lemma.

Lemma 2.2. *If a pair (x,y) in $[0,1] \times [0,1]$ satisfies*

$$\left(1 - \frac{\sqrt{3y}}{3}\right) y \leq x \leq \left(1 + \frac{\sqrt{3y}}{3}\right) y,$$

then $6xy - 3x^2 - 3y^2 + y^3 \geq 0$.

Proof. For any fixed y in $[0,1]$, define

$$h(x) = 6xy - 3x^2 - 3y^2 + y^3 = (-3)x^2 + (6y)x - 3y^2 + y^3.$$

The quadratic function $h(x)$ has two real zeros:

$$x_1 = \left(1 - \frac{\sqrt{3y}}{3}\right) y \ \text{ and } \ x_2 = \left(1 + \frac{\sqrt{3y}}{3}\right) y.$$

Since $h(x)$ opens downwards, we have $h(x) \geq 0$ for $x_1 \leq x \leq x_2$. □

Lemma 2.3. *Let $k(x) = (1 - \frac{\sqrt{3x}}{3})x$. Then $k(x)$ is increasing on $[0, 4/3]$ and decreasing on $[4/3, \infty)$.*

Proof. The proof is simple and is omitted here. □

The next lemma provides some information about the sign property of $G(t,s)$.

Theorem 2.4. *If $p, q \in [0,1]$ are such that*

$$2p \le 3q, \quad p \ge \left(1 - \frac{\sqrt{3q}}{3}\right) q, \tag{2.1}$$

then

$$G(t,s) \ge 0, \quad 0 \le t, s \le 1.$$

Proof. The expression of the Green function $G(t,s)$ involves three unit step functions $H(p-s), H(q-s)$, and $H(t-s)$. Each one takes the value of 0 or 1. Overall, we will have eight cases to discuss.

Case 1: $H(p-s) = 1, H(q-s) = 1$, and $H(t-s) = 1$.

In this case, we have

$$0 \le s \le p \le 1, \quad 0 \le s \le q \le 1, \quad \text{and} \quad 0 \le s \le t \le 1$$

which lead to $G(t,s) = s^3/6 \ge 0$.

Case 2: $H(p-s) = 1, H(q-s) = 1$, and $H(t-s) = 0$.

In this case, we have

$$0 \le s \le p \le 1, \quad 0 \le s \le q \le 1, \quad \text{and} \quad 0 \le t < s \le 1$$

which implies

$$G(t,s) = \frac{1}{6} t(3s(s-t) + t^2) \ge 0.$$

Case 3: $H(p-s) = 1, H(q-s) = 0$, and $H(t-s) = 1$.

In this case, we have

$$0 \le q < s \le p \le 1 \quad \text{and} \quad 0 \le s \le t \le 1$$

which, together with $2p \le 3q$, leads to

$$G(t,s) = \frac{1}{2}(t-p)^2(s-q) + \frac{1}{6}(s-p)^2(s+2p) + \frac{1}{6}p^2(3q-2p) \ge 0.$$

Case 4: $H(p-s) = 1, H(q-s) = 0$, and $H(t-s) = 0$.

In this case, we have

$$0 \le q < s \le p \le 1 \quad \text{and} \quad 0 \le t < s \le 1$$

which, together with $2p \le 3q$, implies

$$G(t,s) = \frac{t}{6}\left(3(s-p)^2 + (p-t)^2 + (2p-t)(3q-2p)\right) \ge 0$$

due to the fact that $2p - t \ge 2s - t > 0$.

Case 5: $H(p-s) = 0, H(q-s) = 1$, and $H(t-s) = 1$.

In this case, we have

$$0 \le p < s \le q \le 1 \quad \text{and} \quad 0 \le p < s \le t \le 1$$

and

$$G(t,s) = \frac{1}{6}(s^3 + 6pts - 3tp^2 - 3ts^2) = \frac{1}{6}s^3(1-t) + \frac{t}{6}(6ps - 3p^2 - 3s^2 + s^3). \tag{2.2}$$

In this case, we have $1 \ge q \ge s > p \ge 0$. It is seen from Lemma 2.3 that $k(q) \ge k(s)$ since $1 \ge q \ge s \ge 0$. Thus, we have $s > p \ge k(q) \ge k(s)$. Therefore, the pair (p,s) satisfies the conditions of Lemma 2.2 from which, we have $6ps - 3p^2 - 3s^2 + s^3 \ge 0$ and thus, $G(t,s) \ge 0$ in view of (2.2).

Case 6: $H(p-s) = 0$, $H(q-s) = 1$, and $H(t-s) = 0$.

In this case, we have

$$0 \le t < s \le q \le 1 \quad \text{and} \quad 0 \le p < s \le q \le 1$$

and

$$\begin{aligned} G(t,s) &= \frac{t}{6}\left(t^2 + 6ps - 3p^2 - 3ts\right) \\ &= \frac{t}{6}\left(6ps - 3p^2 - 3s^2 + s^3\right) + \frac{t}{6}\left((s-t)^2 + s(s-t) + s^2(1-s)\right). \end{aligned} \tag{2.3}$$

As we have shown in Case 5, the inequalities $1 \ge q \ge s > p \ge 0$ implies that $6ps - 3p^2 - 3s^2 + s^3 \ge 0$. It follows that $G(t,s) \ge 0$ in this case.

Case 7: $H(p-s) = 0, H(q-s) = 0$, and $H(t-s) = 1$.

In this case, we have

$$0 \le p < s \le t \le 1 \quad \text{and} \quad 0 \le q < s \le t \le 1$$

and

$$G(t,s) = \frac{1}{6}\left(s^3 - 3tp^2 + 6tpq - 3t^2q + 3t^2s - 3ts^2\right). \tag{2.4}$$

This case can be divided into two subcases: (1) $0 \le q \le p < s \le t \le 1$ or (2) $0 \le p \le q < s \le t \le 1$.

For the first subcase where $0 \le q \le p < s \le t \le 1$, the function $G(t,s)$ in (2.4) can be expressed as

$$G(t,s) = \frac{1}{24}(s-p)^3 + \frac{1}{8}(s-p)(s+p-2t)^2 + \frac{1}{2}(p-q)(t-p)^2 + \frac{p^2}{6}(3q-2p) \ge 0.$$

For the second subcase where $0 \le p \le q < s \le t \le 1$, the function $G(t,s)$ in (2.4) can be expressed as

$$G(t,s) = \frac{t}{6}(6pq - 3p^2 - 3q^2 + q^3) + \frac{q^3(1-t)}{6} + \frac{(s-q)^3}{24} + \frac{1}{8}(s-q)(s+q-2t)^2. \tag{2.5}$$

In this second subcase, we have $q \ge p \ge k(q)$. Therefore, the pair (p,q) satisfies the conditions of Lemma 2.2. Thus, we have $6pq - 3p^2 - 3q^2 + q^3 \ge 0$ which implies $G(t,s) \ge 0$ in view of (2.5).

Case 8: $H(p-s)=0, H(q-s)=0$, and $H(t-s)=0$.

In this case, we have $p<s$, $q<s$, and $t<s$. And, we have

$$G(t,s)=\frac{t}{6}(t^2-3qt+6pq-3p^2)\equiv\frac{t}{6}g_1(t), \tag{2.6}$$

where

$$g_1(t)=t^2-3qt+6pq-3p^2.$$

First, we notice that

$$g_1(t)=(t-3q/2)^2+3(2p-q)(3q-2p)/4.$$

Hence, if $p>q$, then $g_1(t)\geq 0$. It follows that, if $p>q$, then $G(t,s)\geq 0$ for $t,s\in[0,1]$.

Next, we assume that $p\leq q$. Under this assumption, the pair (p,q) satisfies the conditions of Lemma 2.2 and thus

$$6pq-3p^2-3q^2+q^3\geq 0, \tag{2.7}$$

which leads to

$$\begin{aligned} g_1(t)&=(6pq-3p^2-3q^2+q^3)+(t^2-3qt+3q^2-q^3)\\ &\geq t^2-3qt+3q^2-q^3\equiv g_2(t). \end{aligned}$$

Observe that

$$g_2(t)=(3q/2-t)^2+q^2(3/4-q)\geq 0 \quad \text{for} \quad 0\leq q\leq 3/4$$

and

$$g_2(t)=(3q-1-t)(1-t)+(1-q)^3\geq 0 \quad \text{for} \quad 2/3\leq q\leq 1$$

due to the fact that $3q-1-t\geq 1-t\geq 0$ for $q\geq 2/3$. Thus, we have $g_1(t)\geq g_2(t)\geq 0$ for all q in $[0,1]$ when $p\leq q$. This concludes the proof of $G(t,s)\geq 0$ for Case 8.

The proof of the theorem is now complete. □

We would like to point out that Theorem 2.4 is established under a very general assumption. Notice that we have never assumed $p\leq q$ in its proof. The next theorem shows that condition (2.1) is not only sufficient but also necessary for the positivity of the Green function $G(t,s)$.

Theorem 2.5. *Let $p,q\in[0,1]$. If*

$$\textit{either}\ \ 2p>3q\ \ \textit{or}\ \ p<\left(1-\frac{\sqrt{3q}}{3}\right)q, \tag{2.8}$$

then there exists a point $(t_0,s_0)\in[0,1]\times[0,1]$ such that $G(t_0,s_0)<0$.

Proof. If $2p > 3q$, then $q < 2p/3 \leq 2/3 < 1$ which implies $H(q-1) = 0$. It is easily seen that

$$G(p,1) = (3q-2p)p^2/6 < 0.$$

If $p < \left(1 - \frac{\sqrt{3q}}{3}\right) q$, then $p - q < 0$ implying $H(p-q) = 0$. Thus, we have

$$\begin{aligned} G(1,q) &= -\frac{1}{2}\left(p^2 - 2pq + q^2 - \frac{1}{3}q^3\right) \\ &= -\frac{1}{2}\left(p - \left(1 - \frac{\sqrt{3q}}{3}\right) q - \frac{2q\sqrt{3q}}{3}\right)\left(p - \left(1 - \frac{\sqrt{3q}}{3}\right) q\right) < 0. \end{aligned}$$

The proof is complete. □

3. A COMPARISON

Let

$$R_1 = \{(q,p) \in [0,1] \times [0,1] : 1/2 < p < (1+p)/2 < q < 1\}$$

and

$$R_2 = \left\{(q,p) \in [0,1] \times [0,1] : 2p \leq 3q,\ p \geq \left(1 - \frac{\sqrt{3q}}{3}\right) q\right\}$$

be two subsets of the square $[0,1] \times [0,1]$ in the qp-plane. Obviously, we have $R_1 \subset R_2$. Theorems 2.4 and 2.5 indicate that R_2 contains all the points (q,p) that define the problem by (1.4) and (1.2) with positive Green function $G(t,s)$.

The region of R_1 is a right triangle whose legs are $\frac{1}{2}$ and $\frac{1}{4}$, respectively. So the area of R_1 is obviously 0.0625. The area of R_2, denoted by $A(R_2)$, can also be easily computed:

$$\begin{aligned} A(R_2) &= \int_0^{\frac{2}{3}} \left(\frac{3}{2}q - \left(1 - \frac{\sqrt{3}}{3}q^{\frac{1}{2}}\right) q\right) dq + \int_{\frac{2}{3}}^1 \left(1 - \left(1 - \frac{\sqrt{3}}{3}q^{\frac{1}{2}}\right) q\right) dq \\ &= \frac{1}{6} + \frac{2\sqrt{3}}{15} \approx 0.3976. \end{aligned}$$

Therefore, the region of R_2 is more than six times that of R_1. In this sense, the condition (2.1) improves the condition (1.3) significantly.

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