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NONNEGATIVE SOLUTIONS OF A CLASS OF SYSTEMS OF ALGEBRAIC EQUATIONS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. In this paper, we study the existence of nonnegative solutions of the nonlinear system of n equations

$$x = \lambda AF(x),$$

where the parameter $\lambda > 0$, A is an $n \times n$ real matrix A, $F(x)$ maps $\mathbf{R}_+^n$ to $\mathbf{R}^n$ with $AF(x) \geq 0$ for all $x \in \mathbf{R}_+^n$. Our results may allow negative elements in A and significantly extend and improve those in the literature. Sharp bounds for the parameter λ are also given.

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1. INTRODUCTION

Solving algebraic equations as one of the oldest problems in mathematics still confronts mathematicians and other scientists alike. Numerous mathematical problems such as numerical solutions of differential equations, discrete boundary value problems, and steady states of a complex dynamic system can be reduced to the study of the existence of positive solutions of algebraic equations (e.g., [1, 6, 9]). For examples, the number of solutions of algebraic equations plays a central role in determining the number of relative equilibria of the N-body problem [5]; in many biological systems, one would like to know the number of positive steady states of a differential equation model and how the number varies by changing the biological parameters [8].

In this paper, we shall study the existence of positive solutions of a class of nonlinear algebraic systems which is closely related to positive solutions of differential equations (see [7, 10]),

$$x = \lambda AF(x) \tag{1.1}$$

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in $\mathbf{R}_+^n$, where $\mathbf{R}_+ = [0, \infty)$, $\lambda > 0$ is a parameter, $A = (a_{ij})$ is an $n \times n$ real matrix, the variables $x = (x_1, \dots, x_n)^t \in \mathbf{R}_+^n$, and the function $F(x) = (f_1(x), f_2(x), \dots, f_n(x))^t : \mathbf{R}_+^n \to \mathbf{R}^n$ is continuous in $\mathbf{R}_+^n$. Due to the non-negativeness of x, it's natural to assume that $\lambda AF(x) \in \mathbf{R}_+^n$ for all $x \in \mathbf{R}_+^n$. Note that in our setting, some elements of A may be negative and some functions $f_i(x)$ may also be negative. By a *nonnegative solution* of the system (1.1), we mean a nonzero vector $x = (x_1, \dots, x_n)^t \in \mathbf{R}_+^n$ which satisfies the equations in the system (1.1). By a *positive solution* of the system (1.1), we mean a vector $x \in \mathbf{R}_+^n$ which satisfies the equations in the system and $x_i > 0$ for all $i = 1, \dots, n$.

A recent series of papers studied the positive solutions of the system (1.1) under various conditions on A and $F(x)$ such as [7, 10]. When $\lambda > 0$, A is nonnegative (i.e., $a_{ij} \geq 0$ for $i, j = 1, \dots, n$), and $F(x) \geq 0$ for all $x \in \mathbf{R}_+^n$, it is the case discussed in [7]. If A is positive and $f_j(x) \equiv f_j(x_j)$, that is, $f_j(x)$ depends only on the variable x_j, it is the case discussed in [10].

In this paper, some elements of A may take negative values and we assume that the condition (C1) below holds throughout this paper.

(C1) $AF(x) \geq 0$ for all $x \in \mathbf{R}_+^n$ and $F(x)$ is continuous on $\mathbf{R}_+^n$.

We also need another condition for some results.

(C2) For each nonzero $x \in \mathbf{R}_+^n$, $AF(x) \geq 0$ is nonzero.

Note that the conditions are weaker than similar conditions in [7, 10]. For example, we can take $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ and $F = (x_1 + 1, \sin x_2, x_2 + x_3)^t$. With these weaker conditions on the matrix A and the function $F(x)$, in Section 2, we present new and improved results on existence of nonnegative solutions and positive solutions of the system (1.1). We also give sharp bounds of the parameter λ for the existence of nonnegative solutions. Note also that our method presents a uniform treatment of the results on the system (1.1) given in [7, 10].

2. EXISTENCE OF NONNEGATIVE AND POSITIVE SOLUTIONS

Let $\|\cdot\|_p$, $(p \geq 1)$ be any l_p norm on $\mathbf{R}^n$ and $\|A\|_p$ be the matrix norm induced by the vector norm. We impose an additional condition (C3) throughout the paper,

(C3) there are two positive constants $m > 0$ and $M > 0$ such that

$$m\|F(x)\|_p \leq \|AF(x)\|_p \leq M\|F(x)\|_p \tag{2.1}$$

for all $x \in \mathbf{R}_+^n \backslash \{0\}$.

Note that one of the choices for M is $M = \|A\|_p$. When A is invertible, one of the choices for m is $1/\|A^{-1}\|_p$.

In [7], it is assumed that $A \geq 0$, $F(x) \geq 0$, and each column of A contains a nonzero element. In [10], it is assumed that $A > 0$ and $F(x) \geq 0$. In both cases, every $f_i(x)$ is present in the vector $AF(x)$ with a nonzero coefficient and thus

$$m = \min\{a_{ij} \neq 0 \mid i, j = 1, \ldots, n\}$$

satisfies the inequality (2.1) since

$$\begin{aligned}\|AF(x)\|_p &= \left(\sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} f_j(x)\right)^p\right)^{1/p} \geq \left(\sum_{i=1}^n \sum_{j=1}^n a_{ij}^p \left(f_j(x)\right)^p\right)^{1/p} \\ &\geq m \left(\sum_{i=1}^n \left(f_i(x)\right)^p\right)^{1/p} = m\|F(x)\|_p.\end{aligned}$$

Therefore, our treatment of the system (1.1) covers both cases in [7, 10].

Let

$$F_0 = \lim_{x \in \mathbf{R}_+^n,\, \|x\|_p \to 0} \frac{\|F(x)\|_p}{\|x\|_p} \quad \text{and} \quad F_\infty = \lim_{x \in \mathbf{R}_+^n,\, \|x\|_p \to \infty} \frac{\|F(x)\|_p}{\|x\|_p}.$$

For a positive number r, let

$$\Omega_r = \{x \in \mathbf{R}_+^n : \|x\|_p \leq r\} \quad \text{and} \quad \partial\Omega_r = \{x \in \mathbf{R}_+^n : \|x\|_p = r\}.$$

Lemma 2.1. *For the system* (1.1) *with* $T(x) = \lambda AF(x)$,

(i) *if* $F_0 = 0$, *for any value of* λ, *there is a positive constant* r *such that for all* $x \in \partial\Omega_r$,

$$\|T(x)\|_p \leq \|x\|_p;$$

(ii) *if* $F_0 = \infty$, *for any value of* λ, *there is a positive constant* r *such that for all* $x \in \partial\Omega_r$,

$$\|T(x)\|_p \geq \|x\|_p;$$

(iii) *if* $0 < F_0 < \infty$,

(a) *there is a positive constant* r_1 *such that for all* $x \in \partial\Omega_{r_1}$,

$$\|T(x)\|_p \leq \|x\|_p$$

for any $\lambda < \frac{1}{MF_0}$;

(b) *there is a positive constant* r_2 *such that for all* $x \in \partial\Omega_{r_2}$,

$$\|T(x)\|_p \geq \|x\|_p$$

for $\lambda > \frac{1}{mF_0}$.

Proof. Note that

$$\lambda m\|F(x)\|_p \leq \|T(x)\|_p = \|\lambda AF(x)\|_p = \lambda\|AF(x)\|_p \leq \lambda M\|F(x)\|_p.$$

Part (i). Since $F_0 = 0$, for any fixed $\lambda > 0$ and any $\epsilon \leq \frac{1}{\lambda M}$, there is an $r > 0$ such that for any $x \in \partial\Omega_r$,

$$\frac{\|F(x)\|_p}{\|x\|_p} \leq \epsilon$$

and thus $\|F(x)\|_p \leq \epsilon\|x\|_p$ for $x \in \partial\Omega_r$. Therefore, for $x \in \partial\Omega_r$,

$$\|T(x)\|_p = \|\lambda AF(x)\|_p \leq \lambda M\|F(x)\|_p \leq \lambda M\epsilon\|x\|_p \leq \|x\|_p.$$

Part (ii). Since $F_0 = \infty$, for any fixed $\lambda > 0$ and $N \geq \frac{1}{\lambda m}$, there is an $r > 0$ such that for any $x \in \partial\Omega_r$,

$$\frac{\|F(x)\|_p}{\|x\|_p} \geq N$$

and thus $\|F(x)\|_p \geq N\|x\|_p$ for $x \in \partial\Omega_r$. Therefore, for $x \in \partial\Omega_r$,

$$\|T(x)\|_p = \|\lambda AF(x)\|_p \geq \lambda m\|F(x)\|_p \geq \lambda mN\|x\|_p \geq \|x\|_p.$$

Part (iii). Since $F_0 > 0$, for any small $0 < \epsilon < F_0$ there is an $r_1 > 0$ such that for any $x \in \partial\Omega_{r_1}$,

$$\frac{\|F(x)\|_p}{\|x\|_p} \leq F_0 + \epsilon$$

and thus $\|F(x)\|_p \leq (F_0 + \epsilon)\|x\|_p$. Therefore, for $x \in \partial\Omega_{r_1}$,

$$\|T(x)\|_p = \|\lambda AF(x)\|_p \leq \lambda M\|F(x)\|_p \leq \lambda M(F_0 + \epsilon)\|x\|_p \leq \|x\|_p$$

if $\lambda \leq \frac{1}{M(F_0+\epsilon)} < \frac{1}{MF_0}$. Similarly, there is an $r_2 > 0$ such that for any $x \in \partial\Omega_{r_2}$,

$$\frac{\|F(x)\|_p}{\|x\|_p} \geq F_0 - \epsilon$$

and thus $\|F(x)\|_p \geq (F_0 - \epsilon)\|x\|_p$. Therefore, for $x \in \partial\Omega_{r_2}$,

$$\|T(x)\|_p = \|\lambda AF(x)\|_p \geq \lambda m\|F(x)\|_p \geq \lambda m(F_0 - \epsilon)\|x\|_p \geq \|x\|_p$$

if $\lambda \geq \frac{1}{m(F_0-\epsilon)} > \frac{1}{mF_0}$. □

Lemma 2.2. *For the system* (1.1) *with* $T(x) = \lambda AF(x)$*,*

(i) *if* $F_\infty = 0$*, for any value of* λ*, there is a positive constant* r *such that for all* $x \in \partial\Omega_r$*,*

$$\|T(x)\|_p \leq \|x\|_p;$$

(ii) *if* $F_\infty = \infty$*, for any value of* λ*, there is a positive constant* r *such that for all* $x \in \partial\Omega_r$*,*

$$\|T(x)\|_p \geq \|x\|_p;$$

(iii) *if* $0 < F_\infty < \infty$*,*

(a) *there is a positive constant r_1 such that for all $x \in \partial\Omega_{r_1}$,*

$$\|T(x)\|_p \leq \|x\|_p$$

for any $\lambda < \frac{1}{MF_\infty}$;

(b) *there is a positive constant r_2 such that for all $x \in \partial\Omega_{r_2}$,*

$$\|T(x)\|_p \geq \|x\|_p$$

for $\lambda > \frac{1}{mF_\infty}$.

Proof. Note again that

$$\lambda m\|F(x)\|_p \leq \|T(x)\|_p = \|\lambda AF(x)\|_p = \lambda\|AF(x)\|_p \leq \lambda M\|F(x)\|_p.$$

Part (i). Since $F_\infty = 0$, for any fixed $\lambda > 0$ and any $\epsilon \leq \frac{1}{\lambda M}$, there is an $r > 0$ such that for any $x \in \partial\Omega_r$,

$$\frac{\|F(x)\|_p}{\|x\|_p} \leq \epsilon$$

and thus $\|F(x)\|_p \leq \epsilon\|x\|_p$ for $x \in \partial\Omega_r$. Therefore, for $x \in \partial\Omega_r$,

$$\|T(x)\|_p = \|\lambda AF(x)\|_p \leq \lambda M\|F(x)\|_p \leq \lambda M\epsilon\|x\|_p \leq \|x\|_p.$$

Part (ii). Since $F_\infty = \infty$, for any fixed $\lambda > 0$ and $N \geq \frac{1}{\lambda m}$, there is an $r > 0$ such that for any $x \in \partial\Omega_r$,

$$\frac{\|F(x)\|_p}{\|x\|_p} \geq N$$

and thus $\|F(x)\|_p \geq N\|x\|_p$ for $x \in \partial\Omega_r$. Therefore, for $x \in \partial\Omega_r$,

$$\|T(x)\|_p = \|\lambda AF(x)\|_p \geq \lambda m\|F(x)\|_p \geq \lambda mN\|x\|_p \geq \|x\|_p.$$

Part (iii). Since $F_\infty > 0$, for any small $0 < \epsilon < F_\infty$ there is an $r_1 > 0$ such that for any $x \in \partial\Omega_{r_1}$,

$$\frac{\|F(x)\|_p}{\|x\|_p} \leq F_\infty + \epsilon$$

and thus $\|F(x)\|_p \leq (F_\infty + \epsilon)\|x\|_p$. Therefore, for $x \in \partial\Omega_{r_1}$,

$$\|T(x)\|_p = \|\lambda AF(x)\|_p \leq \lambda M\|F(x)\|_p \leq \lambda M(F_\infty + \epsilon)\|x\|_p \leq \|x\|_p$$

if $\lambda \leq \frac{1}{M(F_\infty+\epsilon)} < \frac{1}{MF_\infty}$. Similarly, there is an $r_2 > 0$ such that for any $x \in \partial\Omega_{r_2}$,

$$\frac{\|F(x)\|_p}{\|x\|_p} \geq F_\infty - \epsilon$$

and thus $\|F(x)\|_p \geq (F_\infty - \epsilon)\|x\|_p$. Therefore, for $x \in \partial\Omega_{r_2}$,

$$\|T(x)\|_p = \|\lambda AF(x)\|_p \geq \lambda m\|F(x)\|_p \geq \lambda m(F_\infty - \epsilon)\|x\| \geq \|x\|_p$$

if $\lambda \geq \frac{1}{m(F_\infty-\epsilon)} > \frac{1}{mF_\infty}$. $\square$

Lemma 2.3. *When the condition* (C2) *also holds, for any $r > 0$, let*

$$k_r = \min_{x \in \partial\Omega_r} \|F(x)\|_p \quad \text{and} \quad K_r = \max_{x \in \partial\Omega_r} \|F(x)\|_p.$$

(a) When $\lambda \geq \lambda_1 = \frac{r}{mk_r}$,

$$\|T(x)\|_p \geq \|x\|_p$$

for $x \in \partial\Omega_r$.

(b) When $\lambda \leq \lambda_2 = \frac{r}{MK_r}$,

$$\|T(x)\|_p \leq \|x\|_p$$

for $x \in \partial\Omega_r$.

Proof. Note that $k_r \neq 0$ under the condition (C2). For $x \in \partial\Omega_r$, the results directly follow from

$$\|T(x)\|_p = \|\lambda AF(x)\|_p \geq \lambda m \|F(x)\|_p \geq \lambda m k_r = \lambda m k_r \frac{\|x\|_p}{r} \geq \|x\|_p$$

and

$$\|T(x)\|_p = \|\lambda AF(x)\|_p \leq \lambda M \|F(x)\|_p \leq \lambda M K_r = \lambda M K_r \frac{\|x\|_p}{r} \leq \|x\|_p.$$

□

The following well-known Krasnselskii's fixed point theorem in conical shells K of a Banach space X was applied to obtain the main results in [7] (see Lemma 1 in [7]). The proofs of our main results are also based on this theorem with $K = \mathbf{R}_+^n$. It is trivial that with the condition (C1), $T(x) = \lambda AF(x)$ in the system (1.1) maps $\mathbf{R}_+^n$ to $\mathbf{R}_+^n$.

Theorem 2.4 (See [2, 3, 4])**.** *Let X be a Banach space and K ($\subset X$) be a cone. Assume that Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let*

$$T : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$$

be completely continuous operator such that either

(i) $\|Tu\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$ *or*
(ii) $\|Tu\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_2$,

then T has a fixed point in $\bar{\Omega}_2 \backslash \Omega_1$.

Remark 2.5. For the system (1.1), T having a fixed point in $\bar{\Omega}_2 \backslash \Omega_1$ implies that the fixed point is nonzero in $\mathbf{R}_+^n$. When $A > 0$ and $F(x) \geq 0$ for $x \in \mathbf{R}_+^n$, all solutions will be positive since there is at least one nonzero component in $F(x)$. Thus the results in Theorems 4.1–4.8 of the paper [10] are special cases of our results.

With the three lemmas above and Theorem 2.4, similar results to those in Theorems 1, 2 and 3 of the paper [7] and those in the paper [10] are still true under our weaker conditions. We summarize the results in Theorem 2.6 below. Its proof is also similar and thus omitted.

Theorem 2.6. *Assume that condition* (C1) *and* (C3) *hold.*

(a) *If* $F_0 = 0$ *and* $F_\infty = \infty$, *the system* (1.1) *has at least one nonnegative solution for all* $\lambda > 0$.
(b) *If* $F_0 = \infty$ *and* $F_\infty = 0$, *the system* (1.1) *has at least one nonnegative solution for all* $\lambda > 0$.

Furthermore, if the condition (C2) *also holds*

(c) *If* $F_0 = 0$ *or* $F_\infty = 0$, *there exists a* $\lambda_0 > 0$ *such that the system* (1.1) *has at least one nonnegative solution for* $\lambda > \lambda_0$.
(d) *If* $F_0 = \infty$ *or* $F_\infty = \infty$, *there exists a* $\lambda_0 > 0$ *such that the system* (1.1) *has at least one nonnegative solution for* $0 < \lambda < \lambda_0$.
(e) *If* $F_0 = F_\infty = 0$, *there exists a* $\lambda_0 > 0$ *such that the system* (1.1) *has at least two nonnegative solutions for* $\lambda > \lambda_0$.
(f) *If* $F_0 = F_\infty = \infty$, *there exists a* $\lambda_0 > 0$ *such that the system* (1.1) *has at least two nonnegative solutions for* $0 < \lambda < \lambda_0$.
(g) *If* $F_0 > 0$ *and* $F_\infty > 0$, *there exists a* $\lambda_0 > 0$ *such that the system* (1.1) *has no nonnegative solution for* $\lambda > \lambda_0$.
(h) *If* $F_0 < \infty$ *and* $F_\infty < \infty$, *there exists a* $\lambda_0 > 0$ *such that the system* (1.1) *has no nonnegative solutions for* $0 < \lambda < \lambda_0$.

Theorem 2.7. *Assume that condition* (C1) *and* (C3) *hold.*

(a) *If* $F_0 = 0$ *and* $0 < F_\infty < \infty$, *the system* (1.1) *has at least one nonnegative solution for* $\lambda > \frac{1}{mF_\infty}$.
(b) *If* $F_0 = \infty$ *and* $0 < F_\infty < \infty$, *the system* (1.1) *has at least one nonnegative solution for* $\lambda < \frac{1}{MF_\infty}$.
(c) *If* $0 < F_0 < \infty$ *and* $F_\infty = 0$, *the system* (1.1) *has at least one nonnegative solution for* $\lambda > \frac{1}{mF_0}$.
(d) *If* $0 < F_0 < \infty$ *and* $F_\infty = \infty$, *the system* (1.1) *has at least one nonnegative solution for* $\lambda < \frac{1}{MF_0}$.

Proof. From Theorem 2.4, Part (a) can be obtained from combining Part (i) of Lemma 2.1 and Part (iii)-(b) of Lemma 2.2. Other results can be similarly obtained from Lemma 2.1 and 2.2. □

Theorem 2.8. *The bounds given for the parameter* λ *in Theorem* 2.7 *are sharp.*

Proof. The sharpness of the bounds can be obtained from the following examples with A an identity matrix (thus $m = M = 1$) and the 1-norm $\|\cdot\|_1$.

Let $(f_1(x), f_2(x)) = (x_1 - |\sin x_1 \cos x_2|, x_2 - |\cos x_1 \sin x_2|)$. Note that $\sin a \geq 0$ and $\cos a \geq 0$ when $a \geq 0$ is near 0. It is clear that $F_0 = 0$ and $F_\infty = 1$. It can be verified that for $\lambda < \frac{1}{mF_\infty} = 1$, the system $x = \lambda F(x)$ has only the trivial zero $(0,0)^t$. Thus the bound in Part (a) is sharp.

Let $(f_1(x), f_2(x)) = (x_1 + 1, x_2 + 1)$. Then $F_0 = \infty$ and $F_\infty = 1$. It can be verified that for $\lambda > \frac{1}{MF_0} = 1$, the system $x = \lambda F(x)$ has no solution in $\mathbf{R}^2_+$. Thus the bound in Part (b) is sharp.

Let $(f_1(x), f_2(x)) = (|\sin x_1 \cos x_2|, |\cos x_1 \sin x_2|)$. Then $F_0 = 1$ and $F_\infty = 0$. It can be verified that for $\lambda < \frac{1}{mF_0} = 1$, the system $x = \lambda F(x)$ has only the trivial zero $(0,0)^t$. Thus the bound in Part (c) is sharp.

Let $(f_1(x), f_2(x)) = (x_1 + x_1^2, x_2 + x_2^2)$. Then $F_0 = 1$ and $F_\infty = \infty$. It can be verified that for $\lambda > \frac{1}{MF_0} = 1$, the system $x = \lambda F(x)$ has only the trivial zero $(0,0)^t$. Thus the bound in Part (d) is sharp. □

The proof of next theorem follows directly from Part (iii) of Lemmas 2.1 and 2.2 as well as Theorem 2.4.

Theorem 2.9. *Assume that the condition* (C1) *and* (C3) *hold. If* $0 < F_0 < \infty$ *and* $0 < F_\infty < \infty$, *the system* (1.1) *has at least one nonnegative solution if there is a* λ *satisfying either* $\frac{1}{mF_\infty} < \lambda < \frac{1}{MF_0}$ *or* $\frac{1}{mF_0} < \lambda < \frac{1}{MF_\infty}$.

Remark 2.10. More results similar to Theorem 2.9 can be obtained from combining the results in Part (iii) of Lemmas 2.1 and 2.2 with those in Part (a) and (b) of Lemma 2.3.

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