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COMPARISON THEOREMS FOR EVEN ORDER DYNAMIC EQUATIONS ON TIME SCALES

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. Consider the following pair of even order linear dynamic equations on a time scale

$$x^{\Delta^n}(t) + p(t)x(t) = 0, \tag{0.1}$$

$$x^{\Delta^n}(t) + q(t)x(t) = 0, \tag{0.2}$$

where $p, q \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, n is even, $\mathbb{T}$ is a time scale. In this paper, we obtain some point-wise and integral comparison theorems for the above equations. These will be shown to be "sharp" by means of specific examples.

Theorem 0.1. *Suppose that $p(t) \geq q(t)$ for all large t. Then if the equation (0.2) is oscillatory, it follows that the equation (0.1) is oscillatory.*

Theorem 0.2. *Suppose that $\int_t^\infty p(s)\Delta s \geq \int_t^\infty q(s)\Delta s$ for all large t. Then if the equation (0.2) is oscillatory, it follows that the equation (0.1) is oscillatory.*

As applications, we get that

Theorem 0.3. *Let n be even. If*

$$\liminf_{k\to\infty} k^{n-1} \sum_{i=k}^{\infty} p(i) > \frac{|M_{n0}|}{n-1}, \tag{0.3}$$

then the difference equation $\Delta^n x(k) + p(k)x(k) = 0$ is oscillatory, where M_{n0} is the minimum of $P_n(\lambda) = \lambda(\lambda-1)\cdots(\lambda-n+1)$, $\lambda \in [0,1]$.

In particular, we get that the fourth order difference equation

$$\Delta^4 x(k) + p(k)x(k) = 0$$

is oscillatory if

$$\liminf_{k\to\infty} k^4 p(k) > 1.$$

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1. Introduction

Consider the following even order linear dynamic equations on a time scale:

$$x^{\Delta^n}(t) + p(t)x(t) = 0, \tag{1.1}$$

$$x^{\Delta^n}(t) + q(t)x(t) = 0, \tag{1.2}$$

where $p, q \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, n is even, $\mathbb{T}$ is a time scale. We recall that a solution of (1.1) (or (1.2)) is said to be oscillatory on $[a, \infty)_{\mathbb{T}}$ in case it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory on $[a, \infty)_{\mathbb{T}}$. Equation (1.1) (or (1.2)) is said to be oscillatory on $[a, \infty)_{\mathbb{T}}$ in case all of its solutions are oscillatory.

When $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$, respectively, there are many comparison theorems for differential equations and difference equations for (1.1) and (1.2), respectively (see e.g., [1], [9]–[8]).

A very detailed discussion of comparison theorems for higher order differential equations may be found in [8].

In this paper, we derive both point-wise and integral comparison theorems for (1.1) and (1.2). As applications, we get that

Theorem 1.1. *Let n be even. If*

$$\liminf_{k\to\infty} k^{n-1} \sum_{i=k}^{\infty} p(i) > \frac{|M_{n0}|}{n-1}, \tag{1.3}$$

then the difference equation $\Delta^n x(k) + p(k)x(k) = 0$ is oscillatory, where M_{n0} is the minimum of $P_n(\lambda) = \lambda(\lambda - 1)\cdots(\lambda - n + 1)$, $\lambda \in [0, 1]$.

In particular, we get that the fourth order difference equation

$$\Delta^4 x(k) + p(k)x(k) = 0$$

is oscillatory if

$$\liminf_{k\to\infty} k^4 p(k) > 1.$$

For completeness (see [3] and [4] for elementary results for the time scale calculus), we recall some basic results for dynamic equations and the calculus on time scales. Let $\mathbb{T}$ be a time scale (i.e., a nonempty closed subset of $\mathbb{R}$) with $\sup \mathbb{T} = \infty$. The forward jump operator is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf \varnothing := \sup \mathbb{T}$ and $\sup \varnothing := \inf \mathbb{T}$, where $\varnothing$ denotes the empty set. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$, we say t is left-scattered. If $\sigma(t) = t$ and $t \neq \sup \mathbb{T}$, we say that t is right dense, while if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$, we say t is left-dense. Given a time scale interval $[c, d]_{\mathbb{T}} := \{t \in \mathbb{T} : c \leq t \leq d\}$ in $\mathbb{T}$, the notation $[c, d]^{\kappa}{}_{\mathbb{T}}$ denotes the interval $[c, d]_{\mathbb{T}}$ in case $\rho(d) = d$ and denotes the interval $[c, d)_{\mathbb{T}}$ in case $\rho(d) < d$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. We also recall that the notation C_{rd} denotes the set of all functions which are continuous at all right dense points and have finite left-sided limits at left dense points. We say that $x : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}$ provided

$$x^{\Delta}(t) := \lim_{s \to t} \frac{x(t) - x(s)}{t - s},$$

exists when $\sigma(t) = t$ (here, by $s \to t$ it is understood that s approaches t in the time scale) and when x is continuous at t and $\sigma(t) > t$,

$$x^{\Delta}(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}.$$

Note that if $\mathbb{T} = \mathbb{R}$, then the delta derivative is just the standard derivative, and when $\mathbb{T} = \mathbb{Z}$, the delta derivative is just the forward difference operator. Hence, our results contain the discrete and continuous cases as special cases and generalizes these results to arbitrary time scale.

2. **Point-wise Comparison Theorems**

In this section we consider the case when a point-wise inequality holds between the coefficients $p(t)$ and $q(t)$. We first introduce the Taylor monomials as follows:

Let k be a nonnegative integer and $s, t \in \mathbb{T}$, we define a sequence of functions $h_k(t, s)$ as follows:

$$h_k(t, s) := \begin{cases} 1, & \text{if } k = 0, \\ \int_s^t h_{k-1}(\tau, s)\Delta\tau, & \text{if } k \geq 1. \end{cases}$$

By Theorem 1.60 of [3] and Theorem 4.1 of [5], we have

$$h_k^{\Delta_t}(t, s) = \begin{cases} 0, & \text{if } k = 0, \\ h_{k-1}(t, s), & \text{if } k \geq 1, \end{cases}$$

and

$$h_k(t, s) = \int_s^t h_{k-1}(t, \sigma(\tau))\Delta\tau \quad \text{for } k \geq 1, \tag{2.1}$$

where $h_k^{\Delta_t}(t, s)$ denotes for each fixed s, the derivative of $h_k(t, s)$ with respect to t.

Lemma 2.1 ([3])**.** *Let $m \in \mathbb{N}$ and suppose that $f : \mathbb{T} \to \mathbb{C}$ is m times Δ-differentiable on $\mathbb{T}^{\kappa^m}$. Let $\alpha \in \mathbb{T}^{\kappa^{m-1}}$, $t \in \mathbb{T}$. Then we have the formula (Taylor's formula)*

$$f(t) = \sum_{k=0}^{m-1} h_k(t,\alpha) f^{\Delta^k}(\alpha) + \int_\alpha^t h_{m-1}(t,\sigma(\tau)) f^{\Delta^m}(\tau)\Delta\tau. \tag{2.2}$$

Lemma 2.2 ([7])**.** *Let $x(t) > 0$ for $t \geq t_0$ with $x^{\Delta^n}(t)$ of constant sign on $[t_0,\infty)_{\mathbb{T}}$ and not eventually zero, then there exists an integer $m \in [0,n]$ with $m+n$ odd for $x^{\Delta^n}(t) \leq 0$ or $m+n$ even for $x^{\Delta^n}(t) \geq 0$ such that*

(1) *If $m \leq n-1$, then $(-1)^{m+i} x^{\Delta^i}(t) > 0$ for $t \geq t_0$ and $m \leq i \leq n-1$.*
(2) *If $m > 1$, then there exists $T \geq t_0$ such that $x^{\Delta^i}(t) > 0$ for $1 \leq i \leq m-1$ and $t \geq T$.*

The following lemma is from [4, Theorem 5.37 (i)].

Lemma 2.3 (Leibniz Formula)**.** *If $f(t,s), f^{\Delta_t}(t,s)$ are rd-continuous, then*

$$\left[\int_a^t f(t,s)\Delta s\right]^{\Delta_t} = f(\sigma(t),t) + \int_a^t f^{\Delta_t}(t,s)\Delta s.$$

Lemma 2.4 ([9] Knaster's fixed-point theorem)**.** *Assume that $(X,\leq)$ is an ordered set. Let Ω be a subset of X with the following properties: The infimum of Ω belongs to Ω and every nonempty subset of Ω has a supremum which belongs to Ω. If $S : \Omega \to \Omega$ is an increasing mapping, i.e., $x \leq y$ implies $Sx \leq Sy$, then S has a fixed point in Ω.*

Suppose that u is a solution of (1.1) satisfying the condition

$$u(t) > 0 \quad \text{for} \quad t \geq t_0 \geq 0.$$

Then from Lemma 2.2, there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ and an integer $l \in \{0,1,\cdots,n\}$ with $n+l$ odd such that

$$u^{\Delta^i}(t) > 0 \quad \text{for} \quad t \geq t_1, (i = 0,1,\cdots,l-1), \tag{2.3}$$

$$(-1)^{i+l} u^{\Delta^i}(t) > 0 \quad \text{for} \quad t \geq t_1, (i = l, l+1,\cdots,n-1). \tag{2.4}$$

From (2.2), it is easy to obtain the following:

Lemma 2.5. *Let the function u satisfy the conditions* (2.3) *and* (2.4)*. Then*

$$\int_{t_1}^\infty h_{n-l-1}(t,\sigma(\tau))|u^{\Delta^n}(\tau)|\Delta\tau < \infty$$

and for $t \geq t_1$,

$$u^{\Delta^l}(t) \geq \int_t^\infty h_{n-l-1}(t,\sigma(\tau))|u^{\Delta^n}(\tau)|\Delta\tau,$$

$$u(t) \geq u(t_1) + \int_{t_1}^t h_{l-1}(t,\sigma(s)) \int_s^\infty h_{n-l-1}(s,\sigma(\tau))|u^{\Delta^n}(\tau)|\Delta\tau\Delta s.$$

Proof. From (2.2), we have

$$u^{\Delta^l}(t) = \sum_{j=l}^{n-1} h_{j-l}(t,s)u^{\Delta^j}(s) + \int_s^t h_{n-l-1}(t,\sigma(\tau))u^{\Delta^n}(\tau)\Delta\tau. \tag{2.5}$$

Define the functions g_k by

$$g_0(r,s) \equiv 1 \quad \text{and} \quad g_{k+1}(r,s) = \int_s^r g_k(\sigma(\tau),s)\Delta\tau \quad \text{for } k \in \mathbb{N}_0.$$

It is easy to show that for $k \in \mathbb{N}_0$ and $r \geq s$, we have $g_k(r,s) \geq 0$. By Theorem 1.112 of [3], $h_i(t,s) = (-1)^i g_i(s,t)$, $i \in \mathbb{N}_0$. So we have

$$\sum_{j=l}^{n-1} h_{j-l}(t,s)u^{\Delta^j}(s) = \sum_{j=l}^{n-1} g_{j-l}(s,t)(-1)^{j-l}u^{\Delta^j}(s) \geq 0 \quad \text{for} \quad s \geq t \geq t_1.$$

Therefore, from (2.5), for $s \geq t \geq t_1$, we get that

$$\begin{aligned} u^{\Delta^l}(t) &\geq -\int_t^s h_{n-l-1}(t,\sigma(\tau))u^{\Delta^n}(\tau)\Delta\tau \\ &= \int_t^s h_{n-l-1}(t,\sigma(\tau))|u^{\Delta^n}(\tau)|\Delta\tau. \end{aligned}$$

Letting $s \to \infty$, it follows that

$$u^{\Delta^l}(t) \geq \int_t^\infty h_{n-l-1}(t,\sigma(\tau))|u^{\Delta^n}(\tau)|\Delta\tau \quad \text{for} \quad t \geq t_1. \tag{2.6}$$

Now, in (2.6), replacing t by s first and then multiplying both sides by $h_{l-1}(t,\sigma(s))$ and integrating from t_1 to t, we obtain

$$\begin{aligned} &\int_{t_1}^t h_{l-1}(t,\sigma(s))u^{\Delta^l}(s)\Delta s \\ &\quad \geq \int_{t_1}^t h_{l-1}(t,\sigma(s)) \int_s^\infty h_{n-l-1}(s,\sigma(\tau))|u^{\Delta^n}(\tau)|\Delta\tau\Delta s \quad \text{for} \quad t \geq t_1. \end{aligned} \tag{2.7}$$

From (2.1), we see that $h_k^{\Delta_s}(t,s) = -h_{k-1}(t,\sigma(s)), k \in \mathbb{N}$. So integrating by parts, we have

$$\begin{aligned} &\int_{t_1}^t h_{l-1}(t,\sigma(s))u^{\Delta^l}(s)\Delta s \\ &\quad = h_{l-1}(t,s)u^{\Delta^{l-1}}(s)|_{s=t_1}^t + \int_{t_1}^t h_{l-2}(t,\sigma(s))u^{\Delta^{l-1}}(s)\Delta s \\ &\quad \leq \int_{t_1}^t h_{l-2}(t,\sigma(s))u^{\Delta^{l-1}}(s)\Delta s \quad \text{for} \quad t \geq t_1. \end{aligned}$$

Repeating the above procedure leads to

$$\begin{aligned} \int_{t_1}^t h_{l-1}(t,\sigma(s))u^{\Delta^l}(s)\Delta s &\leq \int_{t_1}^t h_0(t,\sigma(s))u^{\Delta}(s)\Delta s \\ &= u(t) - u(t_1) \quad \text{for} \quad t \geq t_1. \end{aligned} \tag{2.8}$$

From (2.7) and (2.8), we get

$$u(t) \geq u(t_1) + \int_{t_1}^{t} h_{l-1}(t,\sigma(s)) \int_{s}^{\infty} h_{n-l-1}(s,\sigma(\tau))|u^{\Delta^n}(\tau)|\Delta\tau\Delta s \quad \text{for} \quad t \geq t_1.$$

□

Lemma 2.6. *Let n be even. Suppose that*

$$p(t) \geq q(t) \quad \textit{for large} \quad t, \tag{2.9}$$

and the equation (1.1) *has a solution satisfying conditions* (2.3) *and* (2.4)*. Then the equation* (1.2) *also has such a solution.*

Proof. Let u be a solution of the equation (1.1) satisfying conditions (2.3) and (2.4). From Lemma 2.5, for $t \geq t_1$, we have

$$\begin{aligned} u(t) &\geq u(t_1) + \int_{t_1}^{t} h_{l-1}(t,\sigma(s)) \int_{s}^{\infty} h_{n-l-1}(s,\sigma(\tau))|u^{\Delta^n}(\tau)|\Delta\tau\Delta s \\ &= u(t_1) + \int_{t_1}^{t} h_{l-1}(t,\sigma(s)) \int_{s}^{\infty} h_{n-l-1}(s,\sigma(\tau))p(\tau)u(\tau)\Delta\tau\Delta s \\ &\geq u(t_1) + \int_{t_1}^{t} h_{l-1}(t,\sigma(s)) \int_{s}^{\infty} h_{n-l-1}(s,\sigma(\tau))q(\tau)u(\tau)\Delta\tau\Delta s. \end{aligned} \tag{2.10}$$

Let X be the Banach space of all bounded rd-continuous functions on $[t_1,\infty)_{\mathbb{T}}$ with sup norm $\|x\| = \sup_{t\geq t_1}|x(t)|$.

Let

$$\Omega := \{\omega \in X : 0 \leq \omega(t) \leq 1, \quad \text{for } t \geq t_1\},$$

which is endowed with the usual point-wise ordering $\leq$: $\omega_1 \leq \omega_2 \Leftrightarrow \omega_1(t) \leq \omega_2(t)$ for all $t \geq t_1$. It is easy to see that, for any nonempty $A \subset \Omega$, $\sup A \in \Omega$. Define a mapping S on Ω by

$$\begin{aligned} (S\omega)(t) := \frac{1}{u(t)}[u(t_1) + \int_{t_1}^{t} h_{l-1}(t,\sigma(s)) \int_{s}^{\infty} h_{n-l-1}(s,\sigma(\tau)) \\ \times q(\tau)\omega(\tau)u(\tau)\Delta\tau\Delta s] \quad \text{for} \quad t \geq t_1. \end{aligned}$$

By (2.10), it follows that $S\Omega \subset \Omega$ and S is nondecreasing. Therefore, by Lemma 2.4 (Knaster's fixed-point theorem), there exists an $\omega \in \Omega$ such that $S\omega = \omega$. Hence,

$$\begin{aligned} \omega(t) = \frac{1}{u(t)}[u(t_1) + \int_{t_1}^{t} h_{l-1}(t,\sigma(s)) \int_{s}^{\infty} h_{n-l-1}(s,\sigma(\tau)) \\ \times q(\tau)\omega(\tau)u(\tau)\Delta\tau\Delta s] \quad \text{for} \quad t \geq t_1. \end{aligned}$$

Set $z(t) := \omega(t)u(t)$, then z is rd-continuous (in fact continuous) and $z(t) > 0$ for $t \geq t_1$ and

$$z(t) = u(t_1) + \int_{t_1}^{t} h_{l-1}(t, \sigma(s)) \int_{s}^{\infty} h_{n-l-1}(s, \sigma(\tau)) \\ \times q(\tau) z(\tau) \Delta\tau \Delta s \quad \text{for} \quad t \geq t_1.$$

It is easy to see that

$$z^{\Delta^n}(t) + q(t)z(t) = 0,$$

and z satisfies (2.3) and (2.4). □

From Lemma 2.6, we can now obtain the main theorem:

Theorem 2.7. *Let n be even. Suppose that $p(t) \geq q(t)$ for all large t. Then if the equation* (1.2) *is oscillatory, it follows that the equation* (1.1) *is oscillatory.*

By using the above ideas, we can also consider the following pair of even order nonlinear dynamic equations on a time scale:

$$x^{\Delta^n}(t) + p(t)f(x(t)) = 0, \tag{2.11}$$

$$x^{\Delta^n}(t) + q(t)f(x(t)) = 0, \tag{2.12}$$

where $p, q \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, n is even, $\mathbb{T}$ is a time scale, $f : \mathbb{R} \to \mathbb{R}$ is continuous, nondecreasing, $f(-u) = -f(u)$ for $u \in \mathbb{R}$, and $uf(u) > 0$ for $u \neq 0$.

Under the above assumptions, the following theorem can now be obtained quite easily.

Theorem 2.8. *Let n be even. Suppose that $p(t) \geq q(t)$ for all large t. Then if the equation* (2.12) *is oscillatory, it follows that the equation* (2.11) *is also oscillatory.*

3. **Integral Comparison Theorem**

We next wish to replace the point-wise comparison between the coefficients by an integral comparison. We will need the following second mean value theorem (see [4, Theorem 5.45]).

Lemma 3.1. *Let h be a bounded function that is integrable on $[a, b]_{\mathbb{T}}$. Let m_H and M_H be the infimum and supremum, respectively, of the function $H(t) := \int_t^b h(s)\Delta s$ on $[a, b]_{\mathbb{T}}$. Suppose that g is nondecreasing with $g(t) \geq 0$ on $[a, b]_{\mathbb{T}}$. Then there is some number Λ with $m_H \leq \Lambda \leq M_H$ such that*

$$\int_a^b h(t)g(t)\Delta t = g(b)\Lambda.$$

In Lemma 2.6, replacing the assumption (2.9) by the following integral condition (3.2), we have:

Lemma 3.2. *Let n be even. Suppose that*

$$\int_t^\infty p(s)\Delta s \geq \int_t^\infty q(s)\Delta s \quad \textit{for large} \quad t \geq t_1,$$

and the equation (1.1) *has a solution satisfying the conditions* (2.3) *and* (2.4)*. Then the equation* (1.2) *also has such a solution.*

To prove Lemma 3.2, we need the following lemma:

Lemma 3.3. *Suppose that $R \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ is positive and nondecreasing and*

$$\int_t^\infty p(s)\Delta s \geq \int_t^\infty q(s)\Delta s \quad \textit{for large} \quad t \geq t_1. \tag{3.1}$$

Then

$$\int_t^\infty p(s)R(s)\Delta s \geq \int_t^\infty q(s)R(s)\Delta s \quad \textit{for large} \quad t \geq t_1. \tag{3.2}$$

Proof. Indeed, otherwise, there would exist $s_2 > s_1 \geq t_1$ such that

$$\int_{s_1}^{s_2} q(s)R(s)\Delta s > \int_{s_1}^\infty p(s)R(s)\Delta s.$$

Thus, by Lemma 3.1, we have

$$\begin{aligned} R(s_2)\int_{s_2}^\infty p(s)\Delta s &\leq \int_{s_2}^\infty p(s)R(s)\Delta s \\ &< \int_{s_1}^{s_2} [q(s) - p(s)]R(s)\Delta s \\ &= R(s_2)\Lambda, \end{aligned} \tag{3.3}$$

where $m_\Phi \leq \Lambda \leq M_\Phi$, m_Φ and M_Φ are the infimum and supremum, respectively, of the function $\Phi(t) := \int_t^{s_2}[q(s) - p(s)]\Delta s$ on $[s_1, s_2]_\mathbb{T}$.

By the definition of supremum, for any $\epsilon > 0$, there exists a $\tau \in [s_1, s_2]_\mathbb{T}$ such that

$$M_\Phi - \epsilon < \int_\tau^{s_2} [q(s) - p(s)]\Delta s.$$

Take $\epsilon = \int_{s_2}^\infty q(s)\Delta s$. We get that

$$\begin{aligned} R(s_2)\Lambda &\leq R(s_2)[M_\Phi - \epsilon + \epsilon] \\ &\leq R(s_2)\left[\int_\tau^{s_2} [q(s) - p(s)]\Delta s + \int_{s_2}^\infty q(s)\Delta s\right]. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), it follows that

$$\int_\tau^\infty p(s)\Delta s < \int_\tau^\infty q(s)\Delta s,$$

which contradicts the inequality (3.1). Hence, (3.2) is proved.

We may now complete the proof of the Lemma as follows. Let u be a solution of the equation (1.1) satisfying (2.3) and (2.4). Let $R(\tau) := h_{n-l-1}(s,\sigma(\tau))u(\tau)$. Since n is even and l is odd, we have

$$\begin{aligned} h_{n-l-1}(s,\sigma(\tau)) &= (-1)^{n-l-1} g_{n-l-1}(\sigma(\tau),s) \\ &= g_{n-l-1}(\sigma(\tau),s) \geq 0 \quad \text{for} \quad \tau \geq s. \end{aligned}$$

Since $g_{n-l-1}(t,s)$ is nondecreasing with respect to t for $t \geq s$, it follows that $h_{n-l-1}(s,\sigma(\tau))$ is nondecreasing with respect to τ for $\tau \geq s$. Therefore, R is nondecreasing with respect to τ for $\tau \geq s$.

From Lemma 2.5, using Lemma 3.3, we get that

$$\begin{aligned} u(t) &\geq u(t_1) + \int_{t_1}^{t} h_{l-1}(t,\sigma(s)) \int_{s}^{\infty} h_{n-l-1}(s,\sigma(\tau))|u^{\Delta^n}(\tau)|\Delta\tau\Delta s \\ &= u(t_1) + \int_{t_1}^{t} h_{l-1}(t,\sigma(s)) \int_{s}^{\infty} h_{n-l-1}(s,\sigma(\tau))p(\tau)u(\tau)\Delta\tau\Delta s \\ &\geq u(t_1) + \int_{t_1}^{t} h_{l-1}(t,\sigma(s)) \int_{s}^{\infty} h_{n-l-1}(s,\sigma(\tau))q(\tau)u(\tau)\Delta\tau\Delta s \quad \text{for} \quad t \geq t_1. \end{aligned}$$

The remainder of the proof is the same as the proof of Lemma 2.6, hence it is omitted. □

From Lemma 3.3, we can now obtain the following theorem:

Theorem 3.4. *Let n be even. Suppose that*

$$\int_{t}^{\infty} p(s)\Delta s \geq \int_{t}^{\infty} q(s)\Delta s \quad \textit{for all large} \quad t.$$

Then if the equation (1.2) *is oscillatory, it follows that the equation* (1.1) *is oscillatory.*

To get the main theorem of [6], we need the following lemma:

Lemma 3.5. *Let n be even, and let $P_n(\lambda) := \lambda(\lambda-1)\cdots(\lambda-n+1)$. Let $b_i \in [i,i+1]$ be the point at which the local maximum of $|P_n|$ on $[i,i+1]$ is attained and put $M_{ni} := P_n(b_i)$, $i = 0,1,\ldots,n-2$. Then*

$$(-1)^{i+1}M_{ni} > 0, \quad M_{ni} = M_{n,n-2-i}, \quad i = 0,1,\ldots,n-2,$$

and

$$|M_{n0}| > |M_{n1}| > \cdots > |M_{n,[\frac{n}{2}]-1}|,$$

where $[\frac{n}{2}]$ is the greatest integer less than or equal to $\frac{n}{2}$.

Using Theorem 3.4 and Lemma 3.5, we can obtain the following corollary, which is the main theorem of [6]. It should be noted that the proof of the following corollary is substantially different from Chanturiya's proof.

Corollary 3.6. *Let n be even. If*

$$\liminf_{t\to\infty} t^{n-1}\int_t^\infty p(s)ds > \frac{|M_{n0}|}{n-1}, \tag{3.5}$$

then the differential equation $x^{(n)}(t)+p(t)x(t)=0$ is oscillatory.

Proof. From (3.5), there exists a γ such that

$$\liminf_{t\to\infty} t^{n-1}\int_t^\infty p(s)ds > \frac{\gamma}{n-1} > \frac{|M_{n0}|}{n-1}.$$

So

$$\int_t^\infty p(s)ds > \int_t^\infty \frac{\gamma}{s^n}ds \quad \text{for large} \quad t. \tag{3.6}$$

The Euler equation

$$x^{(n)}(t)+\frac{\gamma}{t^n}x(t)=0 \tag{3.7}$$

has a solution of the form $x(t)=t^\alpha$ if α is a root of the polynomial equation

$$\lambda(\lambda-1)\cdots(\lambda-n+1)+\gamma=0. \tag{3.8}$$

From Lemma 3.5 and $\gamma > |M_{n0}|$, it is easy to see that the equation (3.8) has only complex roots. So the Euler equation (3.7) is oscillatory.

Therefore, if we take $q(t)=\frac{\gamma}{t^n}$, then using Theorem 3.4 and (3.6), we get that the differential equation $x^{(n)}(t)+p(t)x(t)=0$ is oscillatory. $\square$

4. **Example**

Example 4.1. Consider the fourth order difference equation

$$\Delta^4 x(k)+p(k)x(k)=0, \tag{4.1}$$

where $p(k)\geq 0$. We will prove the following theorem.

Theorem 4.2. *Assume that*

$$\liminf_{k\to\infty} k^4 p(k) > 1. \tag{4.2}$$

Then the equation (4.1) is oscillatory.

Consider the fourth order Euler difference equation

$$(k+3)(k+2)(k+1)k\Delta^4 x(k)+\gamma x(k)=0. \tag{4.3}$$

By [1], the characteristic polynomial of the equation (4.3) is

$$\lambda(\lambda-1)(\lambda-2)(\lambda-3)+\gamma=0. \tag{4.4}$$

Let us set $\beta := \lambda-\frac{3}{2}$. Then from (4.4), we get that

$$\left(\beta+\frac{3}{2}\right)\left(\beta+\frac{1}{2}\right)\left(\beta-\frac{1}{2}\right)\left(\beta-\frac{3}{2}\right)+\gamma=0.$$

So

$$\beta^4 - \frac{5}{2}\beta^2 + \frac{9}{16} + \gamma = 0. \tag{4.5}$$

From (4.5), we get that when $\gamma > 1$, the equation (4.4) has complex valued roots λ_i, $i = 0, 1, 2, 3$, where

$$\lambda_0 = \bar{\lambda_1} = \frac{3}{2} - \left[\frac{5}{4} \pm i(\gamma - 1)^{\frac{1}{2}}\right]^{\frac{1}{2}},$$

$$\lambda_2 = \bar{\lambda_3} = \frac{3}{2} + \left[\frac{5}{4} \pm i(\gamma - 1)^{\frac{1}{2}}\right]^{\frac{1}{2}}.$$

Assume that λ is a characteristic root of (4.4). From [1], $x(k) = \frac{\Gamma(k+\lambda)}{\Gamma(k)}$ is a solution of (4.3).

In the following, we will prove that for any complex root $a + bi$ of (4.4), the real part and the imaginary part of $\Gamma(k + a + bi)$ are oscillatory for large k. Therefore, the real part and the imaginary part of $x(k) = \frac{\Gamma(k+a+bi)}{\Gamma(k)}$ are also oscillatory.

We shall make use of the following lemma (see [14]).

Lemma 4.3.

$$\lim_{k\to\infty} \frac{k^z \Gamma(k)}{\Gamma(k + z)} = 1, \quad \textit{where} \quad Re(z) > 0.$$

Take m sufficiently large such that $Re(m + a + bi) > 0$. Then from Lemma 4.3, we have:

Lemma 4.4.

$$\lim_{k\to\infty} \frac{k^{m+a+bi}\Gamma(k)}{\Gamma(k + m + a + bi)} = 1.$$

From Lemma 4.4, we have

$$\frac{k^{m+a+bi}\Gamma(k)}{\Gamma(k + m + a + bi)} = 1 + \alpha(k) + i\beta(k),$$

where $\alpha(k) \to 0$, $\beta(k) \to 0$ as $k \to \infty$. So

$$\begin{aligned}
&\Gamma(k + m + a + bi) \\
&\quad = \frac{k^{m+a}\Gamma(k)e^{bi\ln k}[1 + \alpha(k) - i\beta(k)]}{[1 + \alpha(k)]^2 + [\beta(k)]^2} \\
&\quad = \frac{k^{m+a}\Gamma(k)}{[1 + \alpha(k)]^2 + [\beta(k)]^2} \\
&\quad \times \{(1 + \alpha(k))\cos(b\ln k) + \beta(k)\sin(b\ln k) \\
&\quad\quad - i[\beta(k)\cos(b\ln k) - \sin(b\ln k)(1 + \alpha(k))]\}.
\end{aligned}$$

Note that $\alpha(k) \to 0$, $\beta(k) \to 0$ as $k \to \infty$ and $b \neq 0$. So

$$(1 + \alpha(k))\cos(b\ln k) + \beta(k)\sin(b\ln k)$$

and

$$-[\beta(k)\cos(b\ln k) - \sin(b\ln k)(1+\alpha(k))]$$

will change sign for large k. Therefore, the real part and the imaginary part of $x(k) = \frac{\Gamma(k+a+bi)}{\Gamma(k)}$ are oscillatory. So when $\gamma > 1$, the equation (4.3) is oscillatory.

From (4.2), we get that there exists a $\gamma > 1$ such that

$$(k+3)(k+2)(k+1)kp(k) \geq k^4 p(k) > \gamma > 1 \quad \text{for large} \quad k.$$

Using Theorem 2.7, we may complete the proof of Theorem 4.2.

Remark 4.5. When $\gamma = 1$, the characteristic polynomial (4.4) has real roots $\lambda_{1,2} = \frac{3\pm\sqrt{5}}{2}$. That is, the Euler equation (4.3) has non-oscillatory solutions. Therefore, the result of Theorem 4.2 is "sharp."

Example 4.6. Consider the linear difference equation

$$\Delta^4 x(k) + \frac{\gamma}{k^4} x(k) = 0, \tag{4.6}$$

where $\gamma > 1$. From the proof of Theorem 4.2, we deduce that the equation (4.6) is oscillatory.

5. Comparison Theorems for Difference Equations

Theorem 5.1. *Let n be even. If*

$$\liminf_{k\to\infty} k^{n-1} \sum_{i=k}^{\infty} p(i) > \frac{|M_{n0}|}{n-1}, \tag{5.1}$$

then the difference equation $\Delta^n x(k) + p(k)x(k) = 0$ is oscillatory, where M_{n0} is the minimum of $P_n(\lambda) = \lambda(\lambda-1)\cdots(\lambda-n+1)$, $\lambda \in [0,1]$.

Proof. From (5.1), there exists a γ such that

$$\liminf_{k\to\infty} k^{n-1} \sum_{i=k}^{\infty} p(i) > \frac{\gamma}{n-1} > \frac{|M_{n0}|}{n-1}. \tag{5.2}$$

From [1], we know that the Euler difference equation

$$\Delta^n x(k) + \frac{\gamma}{k(k+1)\cdots(k+n-1)} x(t) = 0 \tag{5.3}$$

has a solution of the form $x(k) = \frac{\Gamma(k+\alpha)}{\Gamma(k)}$ if α is a root of the polynomial equation

$$\lambda(\lambda-1)\cdots(\lambda-n+1) + \gamma = 0. \tag{5.4}$$

From Lemma 3.5 and $\gamma > |M_{n0}|$, it is easy to show that the equation (5.4) only has complex roots. So from Example 4.1, the Euler equation (5.3) is oscillatory.

Take $q(k) = \frac{\gamma}{k(k+1)\cdots(k+n-1)}$. From the properties of the factorial function $t^{\underline{r}}$ of [3], we have

$$q(k) = \gamma(k-1)^{\underline{-n}} = \frac{\gamma}{-n+1}\Delta(k-1)^{\underline{-n+1}}.$$

So

$$\sum_{i=k}^{\infty} q(i) = \frac{\gamma}{-n+1} \sum_{i=k}^{\infty} \Delta(i-1)^{\underline{-n+1}}$$
$$= \frac{\gamma}{n-1}(k-1)^{\underline{-n+1}}$$
$$= \frac{\gamma}{n-1} \cdot \frac{1}{k(k+1)\cdots(k+n-2)}.$$

From (5.2), we get that for large k,

$$\sum_{i=k}^{\infty} p(i) > \frac{\gamma}{(n-1)k^{n-1}} \tag{5.5}$$
$$\geq \frac{\gamma}{n-1} \cdot \frac{1}{k(k+1)\cdots(k+n-2)}$$
$$= \sum_{i=k}^{\infty} q(i).$$

Using (5.5) and Theorem 3.4, we get that the difference equation $\Delta^n x(k)+p(k)x(k) = 0$ is oscillatory. □

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