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EXISTENCE OF NODAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS WITH TWO MULTI-POINT BOUNDARY CONDITIONS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. In this paper, we study the nonlinear boundary value problem consisting of the equation $y'' + w(t)f(y) = 0$ on $[a, b]$ and the multi-point boundary condition

$$y'(a) - \sum_{j=1}^{l} h_j y'(\xi_j) = 0, \quad y'(b) - \sum_{i=1}^{m} k_i y'(\eta_i) = 0.$$

We establish the existence of various nodal solutions by matching the solutions of two boundary value problems at some point in (a, b), each of which involves one separated boundary condition and one multi-point boundary condition. We also obtain conditions under which this problem does not have certain types of nodal solutions.

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1. INTRODUCTION

We study the nonlinear boundary value problem (BVP) consisting of the equation

$$y'' + w(t)f(y) = 0, \quad t \in (a, b), \tag{1.1}$$

where $a, b \in \mathbb{R}$ with $a < b$; and the multi-point boundary condition (BC)

$$y'(a) - \sum_{j=1}^{l} h_j y'(\xi_j) = 0, \quad y'(b) - \sum_{i=1}^{m} k_i y'(\eta_i) = 0. \tag{1.2}$$

Throughout this paper and without further mention, we assume the following:

(H1) $w \in C^1[a, b]$ such that $w(t) > 0$ on $[a, b]$;
(H2) $f \in C(\mathbb{R})$ such that $yf(y) > 0$ for $y \neq 0$, $f(-y) = -f(y)$, and f is locally Lipschitz on $(-\infty, 0) \cup (0, \infty)$;
(H3) there exist extended real numbers $f_0, f_\infty \in [0, \infty]$ such that

$$f_0 = \lim_{y\to 0} f(y)/y \quad \text{and} \quad f_\infty = \lim_{|y|\to\infty} f(y)/y;$$

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(H4) $a < \eta_1 < \cdots < \eta_m < b$ and $k_i \in \mathbb{R}$ for $i = 1, \ldots, m$;
(H5) $a < \xi_1 < \cdots < \xi_l < b$ and $h_j \in \mathbb{R}$ for $j = 1, \ldots, l$.

The existence of solutions, especially positive solutions, of BVPs with multi-point BCs have been studied extensively, see [2, 5, 8, 9, 10, 20, 21, 33] and the references therein. In this paper, we study the existence of nodal solutions, i.e., solutions with a specific zero-counting property in (a, b), of the multi-point BVP (1.1), (1.2). Great progress has been made to the study of such solutions for nonlinear BVPs consisting of Eq. (1.1) (and more general forms of equations) and two-point separated BCs, see [13, 15, 23, 26, 27, 28]. The existence of nodal solutions of BVPs with nonlocal BCs has also received a lot of attention in research. We refer the reader to [1, 3, 4, 6, 12, 14, 16, 22, 24, 25, 30, 31, 32] for some recent work on this topic. In particular, Kong and Kong [12] made progress on the existence of nodal solutions of the BVP consisting of Eq. (1.1) and the separated-multi-point BC

$$\begin{aligned} &\cos\alpha\; y(a) - \sin\alpha\; y'(a) = 0, \quad \alpha \in [0, \pi), \\ &y'(b) - \textstyle\sum_{i=1}^{m} k_i y'(\eta_i) = 0. \end{aligned} \tag{1.3}$$

In fact, they obtained conditions for the existence of various nodal solutions of BVP (1.1), (1.3) by comparing f_0 and f_∞ with the eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ of the Sturm-Liouville Problem (SLP) consisting of the equation

$$-y'' = \lambda w(t) y, \quad t \in (a, b), \tag{1.4}$$

and the two-point BC

$$\begin{aligned} &\cos\alpha\; y(a) - \sin\alpha\; y'(a) = 0, \quad \alpha \in [0, \pi), \\ &y(b) = 0. \end{aligned} \tag{1.5}$$

Note that the results in [12] work with a variable w and a general separated BC at a, and f_0, f_∞ are allowed to be 0 and ∞. Moreover, the eigenvalues of SLP (1.4), (1.5) are guaranteed to exist, easy to compute numerically, and are algebraically simple. The ideas in [12] have been applied in [3, 4, 14] to deal with other BVPs with one separated BC and one multi-point or integral BC. However, we note that the shooting method, which was used to deal with one multi-point BC, fails to work on BVPs with double multi-point BC (1.2). Recently in [16], the authors further developed the methods used in [12] for BVPs with separated-multi-point BCs to BVPs with double multi-point BCs. More specifically, we studied the BVP consisting of Eq. (1.1) and the double multi-point BC

$$y(a) - \sum_{j=1}^{l} h_j y(\xi_j) = 0, \quad y(b) - \sum_{i=1}^{m} k_i y(\eta_i) = 0. \tag{1.6}$$

By matching the nodal solutions of BVPs with the separated-multi-point BCs

$$y'(c) = 0, \quad y(b) - \sum_{i=1}^{m} k_i y(\eta_i) = 0$$

and

$$y(a) - \sum_{j=1}^{l} h_j y(\xi_j) = 0, \quad y'(d) = 0,$$

we established the existence of various nodal solutions.

In this paper, we will use the matching method to study BVP (1.1), (1.2). To do this, we first extend the results in [12] for BVP (1.1), (1.3) to BVPs consisting of Eq. (1.1) and one of the following BCs

$$y'(c) = 0, \quad y'(b) - \sum_{i=1}^{m} k_i y'(\eta_i) = 0 \tag{1.7}$$

and

$$y'(a) - \sum_{j=1}^{l} h_j y'(\xi_j) = 0, \quad y'(d) = 0, \tag{1.8}$$

respectively, where $c \in [a, b)$ and $d \in (a, b]$. We then show that the solutions of these problems will meet at some $c = d \in (a, b)$ and hence produce nodal solutions for BVP (1.1), (1.2). To prove the existence of nodal solutions of BVP (1.1), (1.2) with one zero, we also establish the existence of positive and negative solutions of BVPs consisting of Eq. (1.1) and one of the BCs

$$y(c) = 0, \quad y'(b) - \sum_{i=1}^{m} k_i y'(\eta_i) = 0 \tag{1.9}$$

and

$$y'(a) - \sum_{j=1}^{l} h_j y'(\xi_j) = 0, \quad y(d) = 0, \tag{1.10}$$

respectively, and then show that these solutions meet at some $c = d \in (a, b)$, which produce nodal solutions for BVP (1.1), (1.2) with one zero. Our results are under explicit conditions and f is allowed to be superlinear and sublinear. We will also obtain conditions for the non-existence of solutions.

This paper is structured as follows: we present the main results in Section 2, and then give the proofs in Section 3 after several technical lemmas are established.

2. MAIN RESULTS

We aim to study solutions of BVP (1.1), (1.2) which fall into certain classes defined as follows.

Definition 2.1. Let $n \in \mathbb{N}_0 := \{0, 1, \ldots\}$. Then a solution y of BVP (1.1), (1.2) is said to belong to a class $\mathcal{S}_n^\gamma$ for $\gamma \in \{+, -\}$ if

(i) y has exactly n zeros in (a, b),
(ii) $\gamma y(t) \geq 0$ in a right-neighborhood of a.

To establish criteria for BVP (1.1), (1.2) to have various nodal solutions, we consider the following SLP consisting of Eq. (1.4) and one of the two-point BCs

$$y'(a) = 0, \quad y(b) = 0, \tag{2.1}$$

and

$$y(a) = 0, \quad y'(b) = 0. \tag{2.2}$$

It is well known that the spectrum of SLP (1.4), (2.1) consists of an infinite number of real simple eigenvalues, $\{\mu_n^{[1]}\}_{n=0}^\infty$, that satisfy

$$0 < \mu_0^{[1]} < \mu_1^{[1]} < \cdots < \mu_n^{[1]} < \cdots, \text{ and } \mu_n^{[1]} \to \infty;$$

and the spectrum of SLP (1.4), (2.2) consists of an infinite number of real eigenvalues, $\{\mu_n^{[2]}\}_{n=0}^\infty$, that satisfy

$$0 < \mu_0^{[2]} < \mu_1^{[2]} < \cdots < \mu_n^{[2]} < \cdots, \text{ and } \mu_n^{[2]} \to \infty.$$

Moreover, for $n \in \mathbb{N}_0$, any eigenfunction associated with $\mu_n^{[1]}$ or $\mu_n^{[2]}$ has exactly n simple zeros in (a, b), see [34, Theorem 4.3.2].

We denote $w'_\pm(t) := \max\{\pm w'(t), 0\}$ along with

$$\gamma_j^+ = \int_a^{\xi_j} \frac{w'_+(t)}{w(t)} dt, \; j = 1, \ldots, l, \text{ and } \gamma_i^- = \int_{\eta_i}^b \frac{w'_-(t)}{w(t)} dt, \; i = 1, \ldots, m.$$

The following two theorems are the main results on the existence of nodal solutions of BVP (1.1), (1.2) in $\mathcal{S}_n^\gamma$ for $n \geq 2$ and $n = 1$, respectively.

Theorem 2.2. *Let $n \in \mathbb{N}_0$. Assume either*

(i) $f_0 \leq \min\{\mu_{\lfloor n/2 \rfloor}^{[1]}, \mu_{\lfloor n/2 \rfloor}^{[2]}\}$ *and* $f_\infty = \infty$ *, or*
(ii) $f_\infty \leq \min\{\mu_{\lfloor n/2 \rfloor}^{[1]}, \mu_{\lfloor n/2 \rfloor}^{[2]}\}$ *and* $f_0 = \infty$,

where $\lfloor n/2 \rfloor$ is the integer part of $n/2$. Suppose that

$$\sum_{i=1}^m |k_i| e^{\gamma_i^-/2} < 1 \quad \text{and} \quad \sum_{j=1}^l |h_j| e^{\gamma_j^+/2} < 1. \tag{2.3}$$

Then BVP (1.1), (1.2) *has a solution $y \in \mathcal{S}_{n+2}^\gamma$ for $\gamma \in \{+, -\}$.*

Theorem 2.3. *Assume either (i) $f_0 = 0$ and $f_\infty = \infty$, or (ii) $f_\infty = 0$ and $f_0 = \infty$. Suppose that the inequalities in* (2.3) *hold. Then BVP* (1.1), (1.2) *has a solution $y \in \mathcal{S}_1^\gamma$ for $\gamma \in \{+, -\}$.*

To establish criteria for the non-existence of nodal solutions of BVP (1.1), (1.2), let $\{\nu_n\}_{n=0}^{\infty}$ be the eigenvalues for the SLP consisting of Eq. (1.4) and the two-point Dirichlet BC

$$y(a) = 0, \quad y(b) = 0. \tag{2.4}$$

Again, $\{\nu_n\}_{n=0}^{\infty}$ satisfy

$$0 < \nu_0 < \nu_1 < \cdots \nu_n < \cdots, \text{ and } \nu_n \to \infty;$$

and any eigenfunction associated with ν_n has exactly n simple zeros in (a, b) for $n \in \mathbb{N}_0$, see [34, Theorem 4.3.2].

Theorem 2.4. *(i) Assume $f(y)/y < \nu_n$ for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP* (1.1), (1.2) *has no solution in $\mathcal{S}_i$ for all $i \geq n + 2$;*

(ii) Assume $f(y)/y > \nu_n$ for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP (1.1), (1.2) *has no solution in $\mathcal{S}_i$ for all $i \leq n$.*

3. PROOFS OF THE MAIN RESULTS

We first classify the solutions of BVPs (1.1), (1.7) and (1.1), (1.9) with $c \in [a, b)$ and BVPs (1.1), (1.8) and (1.1), (1.10) with $d \in (a, b]$ into the following classes, as extensions of the class defined in Definition 2.1.

Definition 3.1. Let $n \in \mathbb{N}_0 := \{0, 1, \ldots\}$.

(a) For any $c \in [a, b)$, a solution y of BVP (1.1), (1.7) or (1.1), (1.9) is said to belong to a class $\mathcal{S}_n^{\gamma}[c, b]$ for $\{+, -\}$ if
  (i) y has exactly n zeros in (c, b),
  (ii) $\gamma y(t) > 0$ in a right-neighborhood of c.

(b) For any $d \in (a, b]$, a solution y of BVP (1.1), (1.8) or (1.1), (1.10) is said to belong to class $\mathcal{S}_n^{\gamma}[a, d]$ for $\{+, -\}$ if
  (i) y has exactly n zeros in (a, d),
  (ii) $\gamma y(t) > 0$ in a left-neighborhood of d.

In order to prove Theorem 2.2, we begin by considering BVP (1.1), (1.7) with $c \in [a, b)$. Let $\{\mu_n^{[1]}(c)\}_{n=0}^{\infty}$ be the eigenvalues for the SLP consisting of Eq. (1.4) and the two point BC

$$y'(c) = 0, \quad y(b) = 0. \tag{3.1}$$

Again, $\{\mu_n^{[1]}(c)\}_{n=0}^{\infty}$ satisfy

$$0 < \mu_0^{[1]}(c) < \mu_1^{[1]}(c) < \cdots < \mu_n^{[1]}(c) < \cdots, \text{ and } \mu_n^{[1]}(c) \to \infty;$$

and any eigenfunction associated with $\mu_n^{[1]}(c)$ has exactly n simple zeros in (c,b) for $n \in \mathbb{N}_0$ and $c \in [a,b)$. Note that $\mu_n^{[1]}(a) = \mu_n^{[1]}$ with $\mu_n^{[1]}$ being the n-th eigenvalue of SLP (1.4), (2.1).

From [29], we have that any initial value problem (IVP) associated with Eq. (1.1) has a unique solution which exists on the whole interval $[a,b]$. As a result, the solution depends continuously on the initial condition (IC) and parameters. Let $c \in [a,b)$ and let $y(t,\rho)$ be the solution of the IVP consisting of the Eq. (1.1) and the initial conditions

$$y(c) = \rho \quad \text{and} \quad y'(c) = 0, \tag{3.2}$$

where $\rho > 0$ is a parameter. Let $\theta(t,\rho)$ be the Prüfer angle of $y(t,\rho)$, ie, $\theta(\cdot,\rho)$ is a continuous function on $[c,b]$ such that

$$\tan\theta(t,\rho) = y(t,\rho)/y'(t,\rho) \quad \text{and} \quad \theta(c,\rho) = \pi/2.$$

By the continuous dependence of solutions on parameters, we have that $\theta(t,\rho)$ is continuous in ρ on $[0,\infty)$ for any $t \in [a,b]$. We note that the following two lemmas are minor extensions of Lemmas 4.1, 4.2, 4.4, and 4.5 in [15].

Lemma 3.2. *(i) Assume $f_0 \le \mu_n^{[1]}(c)$ for some $n \in \mathbb{N}_0$. Then for any $\epsilon > 0$, there exists $\rho_* > 0$ such that $\theta(b,\rho) \le (n+1)\pi + \epsilon$ for all $\rho \in (0,\rho_*]$.*

(ii) Assume $\mu_n^{[1]}(c) \le f_\infty$ for some $n \in \mathbb{N}_0$. Then for any $\epsilon > 0$, there exists $\rho^* > 0$ such that $\theta(b,\rho) \ge (n+1)\pi - \epsilon$ for all $\rho \in [\rho^*,\infty)$.

Lemma 3.3. *(i) Assume $f_\infty \le \mu_n^{[1]}(c)$ for some $n \in \mathbb{N}_0$. Then for any $\epsilon > 0$, there exists $\rho^* > 0$ such that $\theta(b,\rho) \le (n+1)\pi + \epsilon$ for all $\rho \in [\rho^*,\infty)$.*

(ii) Assume $f_0 \le \mu_n^{[1]}(c)$ for some $n \in \mathbb{N}_0$. Then for any $\epsilon > 0$, there exists $\rho_* > 0$ such that $\theta(b,\rho) \ge (n+1)\pi - \epsilon$ for all $\rho \in (0,\rho_*]$.

The following lemma is based on Lemmas 3.2 and 3.3.

Lemma 3.4. *Let $n \in \mathbb{N}_0$. Assume either*

(i) $f_0 \le \mu_n^{[1]}(c)$ *and* $\mu_{n+1}^{[1]}(c) < f_\infty$, *or*
(ii) $f_\infty \le \mu_n^{[1]}(c)$ *and* $\mu_{n+1}^{[1]}(c) < f_0$.

Suppose that the first inequality in (2.3) *holds. Then BVP* (1.1), (1.7) *has a solution $y \in \mathcal{S}_{n+1}^{\gamma}[c,b]$ for $\gamma \in \{+,-\}$.*

The proof is a modification of that of [12, Theorem 2.1]. For self-containedness, we still give the detail here.

Proof of Lemma 3.4. We first prove it under the assumption (i). Without loss of generality we assume $\gamma = +$. The case when $\gamma = -$ can be proved in the same way.

Let $y(t,\rho)$ be the solution of Eq. (1.1) satisfying (3.2) and $\theta(t,\rho)$ its Prüfer angle. By Lemma 3.2, for any small $\epsilon > 0$, there exist $0 < \rho_* < \rho^* < \infty$ such that

$$\theta(b,\rho) \le (n+1)\pi + \epsilon \quad \text{for all } \rho \in (0,\rho_*]$$

and

$$\theta(b,\rho) \ge (n+2)\pi - \epsilon \quad \text{for all } \rho \in [\rho^*,\infty).$$

By the continuity of $\theta(t,\rho)$ in ρ, there exist $\rho_* \le \rho_{n+1} < \rho_{n+2} \le \rho^*$ such that

$$\theta(b,\rho_{n+1}) = (n+1)\pi + \epsilon \quad \text{and} \quad \theta(b,\rho_{n+2}) = (n+2)\pi - \epsilon, \tag{3.3}$$

and

$$\theta(b,\rho_{n+1}) < \theta(b,\rho) < \theta(b,\rho_{n+2}) \quad \text{for } \rho_{n+1} < \rho < \rho_{n+2}. \tag{3.4}$$

Define an energy function for $y(t,\rho)$ by

$$E(t,\rho) = \frac{1}{2}[y'(t,\rho)]^2 + w(t)F(y(t,\rho)) \quad \text{for } t \in [a,b] \text{ and } \rho > 0, \tag{3.5}$$

where $F(y) = \int_0^y f(s)ds$. Then

$$E'(t,\rho) = w'(t)F(y(t,\rho)) \ge -\frac{w'_-(t)}{w(t)}E(t,\rho).$$

It follows that

$$\ln\frac{E(b,\rho)}{E(\eta_i,\rho)} = \int_{\eta_i}^b \frac{E'(t,\rho)}{E(t,\rho)}dt \ge -\int_{\eta_i}^b \frac{w'_-(t)}{w(t)}dt = -\gamma_i^-.$$

Thus

$$E(\eta_i,\rho) \le e^{\gamma_i^-} E(b,\rho), \quad i = 1,\dots,m. \tag{3.6}$$

We observe that for $\rho = \rho_{n+1}$ and $\rho = \rho_{n+2}$

$$E(\eta_i,\rho) \ge \frac{1}{2}[y'(\eta_i,\rho)]^2, \quad i = 1,\dots,m. \tag{3.7}$$

It is seen from (3.3) that as $\epsilon \to 0$

$$y(b,\rho) = o(1) \quad \text{and} \quad |y'(b,\rho)| = \rho + o(1)$$

and hence

$$E(b,\rho) = \frac{1}{2}[y'(b,\rho)]^2 + o(1) = \frac{1}{2}[y'(b,\rho)]^2[1+o(1)].$$

This implies that for $\rho = \rho_{n+1}$ and $\rho = \rho_{n+2}$

$$|y'(b,\rho)| = \sqrt{2E(b,\rho)}[1+o(1)] \text{ as } \epsilon \to 0; \tag{3.8}$$

and it follows from (3.7) that for $i = 1,\dots,m$,

$$|y'(\eta_i,\rho)| \le \sqrt{2E(\eta_i,\rho)}. \tag{3.9}$$

Define

$$\Gamma(\rho) = y'(b,\rho) - \sum_{i=1}^m k_i y'(\eta_i,\rho). \tag{3.10}$$

Assume $n = 2k - 1$ with $k \in \mathbb{N}_0$. Since $y'(b, \rho_{2k}) > 0$ and $y'(b, \rho_{2k+1}) < 0$, by (3.8), (3.9), (3.6), and (2.3), we have for $\epsilon > 0$ sufficiently small

$$\begin{aligned}
\Gamma(\rho_{2k}) &= y'(b, \rho_{2k}) - \sum_{i=1}^{m} k_i y'(\eta_i, \rho_{2k}) \\
&\geq y'(b, \rho_{2k}) - \sum_{i=1}^{m} |k_i||y'(\eta_i, \rho_{2k})| \\
&\geq \sqrt{2E(b, \rho_{2k})}[1 + o(1)] - \sum_{i=1}^{m} |k_i| \sqrt{2E(\eta_i, \rho_{2k})} \\
&\geq \sqrt{2E(b, \rho_{2k})} - \sum_{i=1}^{m} |k_i| \sqrt{2e^{\gamma_i^-} E(b, \rho_{2k})} + o(1) \\
&= \sqrt{2E(b, \rho_{2k})} \left(1 - \sum_{i=1}^{m} |k_i| e^{\gamma_i^-/2}\right) + o(1) > 0, \qquad (3.11)
\end{aligned}$$

and

$$\begin{aligned}
\Gamma(\rho_{2k+1}) &= y'(b, \rho_{2k+1}) - \sum_{i=1}^{m} k_i y'(\eta_i, \rho_{2k+1}) \\
&\leq y'(b, \rho_{2k+1}) + \sum_{i=1}^{m} |k_i||y'(\eta_i, \rho_{2k+1})| \\
&\leq -\sqrt{2E(b, \rho_{2k+1})}[1 + o(1)] + \sum_{i=1}^{m} |k_i| \sqrt{2E(\eta_i, \rho_{2k+1})} \\
&\leq -\sqrt{2E(b, \rho_{2k+1})} + \sum_{i=1}^{m} |k_i| \sqrt{2e^{\gamma_i^-} E(b, \rho_{2k+1})} + o(1) \\
&= \sqrt{2E(b, \rho_{2k+1})} \left(-1 + \sum_{i=1}^{m} |k_i| e^{\gamma_i^-/2}\right) + o(1) < 0. \qquad (3.12)
\end{aligned}$$

By the continuity of $\Gamma(\rho)$, there exists $\bar{\rho} \in (\rho_{2k}, \rho_{2k+1})$ such that $\Gamma(\bar{\rho}) = 0$. Similarly, for $n = 2k$ with $k \in \mathbb{N}_0$, there exists $\bar{\rho} \in (\rho_{2k+1}, \rho_{2k+2})$ such that $\Gamma(\bar{\rho}) = 0$. In both cases, since $\epsilon > 0$, we have from (3.4)

$$(n + 1)\pi < \theta(b, \bar{\rho}) < (n + 2)\pi.$$

Since for $t \in [a, b]$,

$$\theta'(t, \rho) = \cos^2 \theta(t, \rho) + w(t) \frac{f(y(t, \rho))y(t, \rho)}{r^2(t, \rho)}, \qquad (3.13)$$

where $r = (y^2 + y'^2)^{1/2}$, we have that $\theta(\cdot, \rho)$ is strictly increasing on $[c, b]$. We note that $y(t) = 0$ if and only if $\theta(t, \rho) = 0 \pmod{\pi}$. Thus y has exactly $n + 1$ zeros in (c, b). Initial condition (3.2) implies that $y(t, \bar{\rho}) > 0$ in a right-neighborhood of c. Therefore $y(t, \bar{\rho}) \in \mathcal{S}_{n+1}^+$.

The proof under the assumption (ii) is essentially the same as above except that it uses Lemma 3.3 instead of Lemma 3.2. □

By using a transformation, we obtain a parallel result to Lemma 3.4 on the existence of nodal solutions of BVP (1.1), (1.8) with $d \in (a, b]$. Let $\{\mu_n^{[2]}(d)\}_{n=0}^{\infty}$ be the eigenvalues for the SLP consisting of Eq. (1.4) and the two point BC

$$y(a) = 0, \; y'(d) = 0. \tag{3.14}$$

Again, $\{\mu_n^{[2]}(d)\}_{n=0}^{\infty}$ satisfy

$$0 < \mu_0^{[2]}(d) < \mu_1^{[2]}(d) < \cdots \mu_n^{[2]}(d) < \cdots, \text{ and } \mu_n^{[2]}(d) \to \infty;$$

and any eigenfunction associated with $\mu_n^{[2]}(d)$ has exactly n simple zeros in (a, d) for $n \in \mathbb{N}_0$ and $d \in (a, b]$. Note that $\mu_n^{[2]}(b) = \mu_n^{[2]}$ with $\mu_n^{[2]}$ being the n-the eigenvalue of SLP (1.4), (2.2).

Lemma 3.5. *Let $n \in \mathbb{N}_0$. Assume either*

(i) $f_0 \le \mu_n^{[2]}(d)$ *and* $\mu_{n+1}^{[2]}(d) < f_\infty$, *or*
(ii) $f_\infty \le \mu_n^{[2]}(d)$ *and* $\mu_{n+1}^{[2]}(d) < f_0$.

Suppose that the second inequality in (2.3) *holds. Then BVP* (1.1), (1.8) *has a solution $y \in \mathcal{S}_{n+1}^{\gamma}[a, d]$ for $\gamma \in \{+, -\}$.*

Proof. Consider the following transformation: $t = a+b-\tau$, $d = a+b-c$. Then BVP (1.1), (1.8) becomes the problem consisting of the equation

$$\frac{d^2y}{d\tau^2} + w(a+b-\tau)f(y) = 0, \quad \tau \in (a, b), \tag{3.15}$$

and BC

$$\frac{dy}{d\tau}(c) = 0, \quad \frac{dy}{d\tau}(b) - \sum_{j=1}^{l} h_j \frac{dy}{d\tau}(a+b-\xi_j) = 0. \tag{3.16}$$

Clearly, $c \in [a, b)$, $a \le a+b-\xi_j < b$ for $j = 1, 2, \ldots, l$ and

$$\begin{aligned}\int_{a+b-\xi_j}^{b} \frac{[w(a+b-\tau)]'_-}{w(a+b-\tau)} d\tau &= \int_{a+b-\xi_j}^{b} \frac{[-w'(a+b-\tau)]_-}{w(a+b-\tau)} d\tau \\ &= \int_{a+b-\xi_j}^{b} \frac{w'_+(a+b-\tau)}{w(a+b-\tau)} d\tau = \int_{a}^{\xi_j} \frac{w'_+(t)}{w(t)} dt = \gamma_j^+.\end{aligned}$$

Hence the second inequality in (2.3) implies that the first inequality in (2.3) holds for the transformed BVP (3.15), (3.16). Also note that $\{\mu_n^{[2]}(d)\}_{n=0}^{\infty}$ are eigenvalues of the SLP involving the equation

$$\frac{d^2y}{d\tau^2} + \lambda w(a+b-\tau)y = 0, \quad \tau \in (a, b),$$

and BC (3.1). Thus the conclusion follows from Lemma 3.4. □

We note that for each $n \in \mathbb{N}_0$, $\mu_n^{[1]}(c)$ strictly increasing and $\mu_n^{[1]}(c) \to \infty$ as $c \to b^-$, see [18, Theorems 2.2, 2.3] and [19, Theorem 4.1]. Thus, the second conditions in assumptions (i) and (ii) of Lemma 3.4 can be guaranteed by $f_\infty = \infty$ and $f_0 = \infty$, respectively. Similarly for Lemma 3.5. In this case, we are able to discuss the behavior of the nodal solutions to BVPs (1.1), (1.7) and (1.1), (1.8) as $c \to b^-$ and $d \to a^+$, respectively.

Lemma 3.6. *Assume the first inequality in* (2.3) *holds. Let $c \in [a, b)$ and $n \in \mathbb{N}_0$.*

(i) *Suppose that $f_0 \le \mu_n^{[1]}(c)$ and $f_\infty = \infty$ and let $y_n(t; c) \in \mathcal{S}_{n+1}^+[c, b]$ be the solution of BVP* (1.1), (1.7) *given by Lemma* 3.4. *Then $\lim_{c\to b^-} y_n(c; c) = \infty$.*

(ii) *Suppose that $f_\infty \le \mu_n^{[1]}(c)$ and $f_0 = \infty$ and let $y_n(t; c) \in \mathcal{S}_{n+1}^+[c, b]$ be the solution of BVP* (1.1), (1.7) *given by Lemma* 3.4. *Then $\lim_{c\to b^-} y_n(c; c) = 0$.*

Proof. (i) Assume the contrary. Then there exists an $n \in \mathbb{N}_0$ and a sequence $\{c_k\}_{k=1}^\infty \subset [a, b)$ such that $c_k \to b^-$ and $y_n(c_k; c_k) \to l \in [0, \infty)$ as $k \to \infty$. Since $y_n(c_k; c_k) \ne 0$, let $z_n(t; c_k) = y_n(t; c_k)/y_n(c_k; c_k)$. It follows that $z_n(t; c_k)$ is a solution of

$$z'' + w(t)g_k(z)z = 0,$$

where

$$g_k(z) := \begin{cases} \dfrac{f(y_n(c_k; c_k)z)}{y_n(c_k; c_k)z}, & \text{for } z \ne 0, \\ f_0, & \text{for } z = 0; \end{cases}$$

and $g_k(z)$ is a continuous function for $z \in \mathbb{R}$ since $f_0 < \infty$. Note that as $k \to \infty$,

$$g_k(z) \to \tilde{g}(z) := \begin{cases} f(lz)/lz, & l \in (0, \infty),\ z \ne 0, \\ f_0, & l = 0 \text{ or } z = 0, \end{cases}$$

which is also continuous. Note that $z_n(c_k; c_k) = 1$ and $z_n'(c_k; c_k) = 0$. Let $\bar{z}(t)$ be the solution of the IVP

$$z'' + w(t)\tilde{g}(z)z = 0, \quad \bar{z}(b) = 1, \quad \bar{z}'(b) = 0.$$

By the continuous dependence of solutions of IVPs on parameters, we have

$$\lim_{k\to\infty} z_n(t; c_k) = \bar{z}(t) \quad \text{uniformly for all } t \in [a, b].$$

Clearly, $\bar{z}(t) > 1/2$ in a left-neighborhood of b. Then for sufficiently large k, $z_n(t; c_k) > 1/2$ for $t \in [c_k, b]$. This shows that

$$y_n(t; c_k) > \frac{1}{2} y_n(c_k; c_k) > 0 \text{ for all } t \in [c_k, b].$$

This contradicts the assumption $y_n \in \mathcal{S}_{n+1}^+[c_k, b]$.

(ii) By replacing f_0 by f_∞, the proof follows similarly to that in part (i) above and is omitted. ∎

Lemma 3.7. *Assume the second inequality in* (2.3) *holds. Let* $d \in (a, b]$ *and* $n \in \mathbb{N}_0$.

(i) *Suppose that* $f_0 \le \mu_n^{[2]}(d)$ *and* $f_\infty = \infty$ *and let* $y_n(t; d) \in \mathcal{S}_{n+1}^+[a, d]$ *be the solution of BVP* (1.1), (1.8) *given by Lemma* 3.5. *Then* $\lim_{d\to a^+} y_n(d; d) = \infty$.
(ii) *Suppose that* $f_\infty \le \mu_n^{[2]}(d)$ *and* $f_0 = \infty$ *and let* $y_n(t; d) \in \mathcal{S}_{n+1}^+[a, d]$ *be the solution of BVP* (1.1), (1.8) *given by Lemma* 3.5. *Then* $\lim_{d\to a^+} y_n(d; d) = 0$.

Proof. This follows directly from Lemma 3.6 and the transformation used in the proof of Lemma 3.5. □

Remark 3.8. Lemmas 3.6 and 3.7 are for the existence of nodal solutions of BVPs (1.1), (1.7) and (1.1), (1.8) in the classes $\mathcal{S}_{n+1}^\gamma[c, b]$ and $\mathcal{S}_{n+1}^\gamma[a, d]$, respectively, for $\gamma = +$. Parallel results hold for $\gamma = -$.

Remark 3.9. (a) For $n \in \mathbb{N}_0$ and $c \in [a, b)$, Lemma 3.4 establishes the existence of a solution $y_n(t; c)$ of BVP (1.1), (1.7) in $\mathcal{S}_{n+1}^+[c, b]$. However, the uniqueness of such solutions is not guaranteed. We claim that for each $n \in \mathbb{N}_0$, there is at least one continuous curve Λ_n^c in the ρ-c plane, where $\rho = y_n(c; c)$ and $\rho \in (0, \infty)$, which satisfies that

(i) for each $(\rho, c) \in \Lambda_n^c$, $c \in [a, b)$;
(ii) for each $c \in [a, b)$, there is at least one point $(\rho, c) \in \Lambda_n^c$.

This is shown as follows:

Note that the solution y of Eq. (1.1) used to define the function Γ in (3.10) satisfies the IC $y(c) = \rho$, $y'(c) = 0$ and as a result, y' and Γ have continuous dependence on the initial point c. To emphasize such dependence, we rewrite Γ as

$$\Gamma(\rho, c) = y_c'(b, \rho) - \sum_{i=1}^{m} k_i y_c'(\eta_i, \rho).$$

Then Γ is a continuous function of (ρ, c). Since for each $n \in \mathbb{N}_0$ and $c \in [a, b)$, $y_n(c; c)$ is a root of $\Gamma(\rho, c)$, then $\rho = y_n(c; c)$ if and only if (ρ, c) is on the intersection set I of the continuous surfaces $z = \Gamma(\rho, c)$ and $z = 0$. From (3.11) and (3.12), we see that when $n = 2k$, $\Gamma(\rho_{2k}) > 0$ and $\Gamma(\rho_{2k+1}) < 0$ for all $c \in [a, b)$. Therefore, the intersection set I must contain a continuous curve in the ρ-c plane which starts at $c = a$ and ends up with $c = b$. Similarly for the case when $n = 2k + 1$.

(b) For $n \in \mathbb{N}_0$ and $d \in (a, b]$, Lemma 3.5 establishes the existence of a solution $y_n(t; d)$ of BVP (1.1), (1.8) in $\mathcal{S}_{n+1}^+[a, d]$. With the same argument as above, for each $n \in \mathbb{N}_0$, there is at least one continuous curve Λ_n^d in the ρ-d plane, where $\rho = y_n(d; d)$, which satisfies that

(i) for each $(\rho, d) \in \Lambda_n^d$, $d \in (a, b]$;
(ii) for each $d \in (a, b]$, there is at least one point $(\rho, d) \in \Lambda_n^d$.

Now we prove Theorem 2.2.

Proof of Theorem 2.2. Without loss of generality, we consider the case where $\gamma = +$, $f_0 \le \min\{\mu^{[1]}_{\lfloor n/2 \rfloor}, \mu^{[2]}_{\lfloor n/2 \rfloor}\} = \mu^{[1]}_{\lfloor n/2 \rfloor}$ and $f_\infty = \infty$. The other cases can be proved similarly. For $n \in \mathbb{N}_0$, let $i = \lfloor n/2 \rfloor$, $j = n - i$. Clearly $j \ge i$. For any $c \in [a, b)$ and $d \in (a, b]$, let $\mu^{[1]}_i(c)$ be the i-th eigenvalue of SLP (1.4), (3.1) and $\mu^{[2]}_j(d)$ the j-th eigenvalue of SLP (1.4), (3.14). By [18, Theorems 2.2, 2.3] and [19, Theorem 4.1], $\mu^{[1]}_i(c)$ is strictly increasing and $\mu^{[2]}_j(d)$ is strictly decreasing. It follows from the assumptions that

$$f_0 \le \mu^{[1]}_i = \mu^{[1]}_i(a) \le \mu^{[1]}_i(c) \text{ and } \mu^{[1]}_{i+1} = \mu^{[1]}_{i+1}(a) \le \mu^{[1]}_{i+1}(c) < f_\infty,$$

and

$$f_0 \le \mu^{[2]}_j = \mu^{[2]}_j(b) \le \mu^{[2]}_j(d) \text{ and } \mu^{[2]}_{j+1} = \mu^{[2]}_{j+1}(b) \le \mu^{[2]}_{j+1}(d) < f_\infty.$$

Since the inequalities in (2.3) hold, by Lemmas 3.4 and 3.5 we have that BVPs (1.1), (1.7) and (1.1), (1.8) have solutions $y^{[1]}_i \in \mathcal{S}^+_{i+1}[c, b]$ and $y^{[2]}_j \in \mathcal{S}^+_{j+1}[a, d]$, respectively. By Lemma 3.6, (i) and Lemma 3.7, (i)

$$\lim_{c \to b^-} y^{[1]}_i(c; c) = \infty \text{ and } \lim_{d \to a^+} y^{[2]}_j(d; d) = \infty.$$

Let $\rho^{[1]}_i(c) = y^{[1]}_i(c; c)$ such that $(\rho^{[1]}_i(c), c)$ is on the continuous curve Λ^c_i and $\rho^{[2]}_j(d) = y^{[2]}_j(d; d)$ such that $(\rho^{[2]}_j(d), d)$ is on the continuous curve Λ^d_j, both as defined in Remark 3.9. By the continuity of the curves Λ^c_i and Λ^d_j, there exists $c^* = d^* \in (a, b)$ such that $y^{[1]}_i(c^*; c^*) = y^{[2]}_j(d^*; d^*)$. Also note that $(y^{[1]}_i)'(c^*, c^*) = 0$ and $(y^{[2]}_j)'(d^*, d^*) = 0$. By the uniqueness of solutions of IVPs, we have $y^{[1]}_i(t, c^*) \equiv y^{[2]}_j(t, d^*)$ for $t \in [a, b]$. We denote $y_n(t) = y^{[1]}_i(t, c^*) = y^{[2]}_j(t, d^*)$ on $[a, b]$. Thus we have $y_n \in \mathcal{S}^+_{i+1}[c^*, b] \cap \mathcal{S}^+_{j+1}[a, d^*]$. This shows that y_n has $n + 2$ zeros on (a, b). It is easy to see $-y_n$ is also a solution of BVP (1.1), (1.2) since f is an odd function, and $-y_n$ has $n + 2$ zeros in (a, b). Clearly, condition (ii) in Definition 2.1 is satisfied by one of y_n and $-y_n$ for $\gamma = +$ and $\gamma = -$, respectively. Therefore, one of y_n and $-y_n$ is in $\mathcal{S}^+_{n+2}$ and the other is in $\mathcal{S}^-_{n+2}$. □

In order to prove Theorem 2.3, we introduce the following lemma on the existence and properties of positive and negative solutions of the BVP (1.1), (1.9) with $c \in [a, b)$. We denote by $\nu^{[1]}_0(c)$ the first eigenvalue of the SLP consisting of Eq. (1.4) and the BC

$$y(c) = 0, \ y(b) = 0.$$

for $c \in [a, b)$. Note that $\nu^{[1]}_0(c)$ is strictly increasing and $\nu^{[1]}_0(c) \to \infty$ as $c \to b^-$, see [18, Theorems 2.2, 2.3] and [19, Theorem 4.1].

Lemma 3.10. *Let $c \in [a, b)$ and the first inequality in* (2.3) *hold.*

(i) *Assume $f_0 = 0$ and $\nu^{[1]}_0(c) < f_\infty$. Then BVP* (1.1), (1.9) *has a solution $y(t; c) \in \mathcal{S}^+_0[c, b]$. Furthermore, if $f_\infty = \infty$, then $\lim_{c \to b^-} y'(c; c) = \infty$.*

(ii) *Assume* $f_\infty = 0$ *and* $\nu_0^{[1]}(c) < f_0$. *Then BVP* (1.1), (1.9) *has a solution* $y(t;c) \in \mathcal{S}_0^+[c,b]$. *Furthermore, if* $f_0 = \infty$, *then* $\lim_{c\to b^-} y'(c;c) = 0$.

Proof. (i) Assume $f_0 = 0$ and $\nu_0^{[1]}(c) < f_\infty$. We first show that BVP (1.1), (1.9) has a solution $y(t;c) \in \mathcal{S}_0^\gamma[c,b]$. Let $y(t,\rho)$ be the solution of Eq. (1.1) satisfying

$$y(c) = 0,\ y'(c) = \rho, \tag{3.17}$$

and $\theta(t,\rho)$ its Prüfer angle. Note that $f_0 = 0 < \nu_0^{[1]}(c) < f_\infty$. From [15, Lemmas 4.1, 4.4], there exist $0 < \rho_* < \rho^* < \infty$ such that

$$\theta(b,\rho) < \pi \quad \text{for all } \rho \in (0,\rho_*)$$

and

$$\theta(b,\rho) > \pi \quad \text{for all } \rho \in (\rho^*,\infty).$$

By the continuity of $\theta(t,\rho)$ in ρ, there exists $\rho_* \le \rho_1 \le \rho^*$ such that

$$\theta(b,\rho_1) = \pi \quad \text{and} \quad \theta(b,\rho) < \pi \text{ for all } \rho \in (0,\rho_1). \tag{3.18}$$

Define an energy function for $y(t,\rho_1)$ by (3.5). Then following similarly to that in the proof of Theorem 2.1 in [12], we have (3.6) holds and

$$|y'(b,\rho)| = \sqrt{2E(b,\rho)} \quad \text{and} \quad |y'(\eta_i,\rho)| \le \sqrt{2E(\eta_i,\rho)}, \quad i = 1,\ldots,m, \tag{3.19}$$

for $\rho = \rho_1$. Let $\Gamma(\rho)$ be defined as in (3.10). Since $y'(b,\rho_1) < 0$, by the first inequality in (2.3), it follows that $\Gamma(\rho_1) < 0$.

It follows from (3.17) that $y(t,0) = 0$ and $y'(t,0) = 0$ for all $t \in [a,b]$. By the continuous dependence of solutions of IVPs on parameters, we have $y'(t,\rho) = O(\rho)$ as $\rho \to 0$ uniformly for $t \in [a,b]$. Since $y(t,\rho) = \int_c^t y'(s,\rho)ds$, then as $\rho \to 0$, $y(t,\rho) = O(\rho)$ uniformly for $t \in [a,b]$, which follows that $f(y(t,\rho)) = o(\rho)$ uniformly for $t \in [a,b]$ since $f_0 = 0$. Thus from Eq. (1.1), we have $y''(t,\rho) = o(\rho)$ uniformly for $t \in [a,b]$. Consequently, for ρ sufficiently small,

$$y'(t,\rho) = \rho + \int_c^t y''(s,\rho)ds = \rho + o(\rho) > \rho/2, \text{ for all } t \in [a,b], \tag{3.20}$$

and then for all $i = 1,\ldots,m$,

$$y'(b,\rho) = y'(\eta_i,\rho) + \int_{\eta_i}^b y''(s,\rho)ds = y'(\eta_i,\rho) + o(\rho). \tag{3.21}$$

This implies that for ρ sufficiently small,

$$\frac{y'(\eta_i,\rho)}{y'(b,\rho)} = 1 + o(1). \tag{3.22}$$

Therefore, by the first inequality in (2.3) which implies $\sum_{i=1}^m |k_i| < 1$, we have that for ρ sufficiently small

$$\begin{aligned}\Gamma(\rho) &= y'(b,\rho) - \sum_{i=1}^m k_i y'(\eta_i,\rho) \\ &= y'(b,\rho)\left(1 - \sum_{i=1}^m k_i(1+o(1))\right) \\ &= y'(b,\rho)\left(1 - \sum_{i=1}^m k_i\right) + o(1) > 0. \end{aligned} \tag{3.23}$$

Let $\rho_2 \in (0,\rho_1)$ such that $\Gamma(\rho_2) > 0$. By the continuity of $\Gamma(\rho)$, there exists $\bar\rho \in (\rho_2,\rho_1)$ such that $\Gamma(\bar\rho) = 0$. From (3.18), $\theta(b,\bar\rho) < \pi$. This implies that $y(t,\bar\rho)$ is a positive solution on $[c,b]$, i.e., $y(t,\bar\rho) \in \mathcal{S}_0^+[c,b]$.

Suppose further that $f_\infty = \infty$. We show $\lim_{c\to b^-} y'(c;c) = \infty$. Assume the contrary. Then there exists a sequence $\{c_k\}_{k=0}^\infty \subset [a,b)$ such that $c_k \to b^-$ and $y'(c_k;c_k) \to l \in [0,\infty)$ as $k \to \infty$. Let $z(t;c_k) = y(t;c_k)/y'(c_k;c_k)$. Since $y'(c_k;c_k) > 0$, $z(t;c_k)$ is well-defined and is a solution of the equation

$$z'' + w(t)h_k(z)z = 0,$$

where

$$h_k(z) := \begin{cases} \dfrac{f(y'(c_k;c_k)z)}{y'(c_k;c_k)z}, & \text{for } z \neq 0, \\ 0, & \text{for } z = 0; \end{cases}$$

and $h_k(z)$ is a continuous function for $z \in \mathbb{R}$ since $f_0 = 0$. Note that as $k \to \infty$,

$$h_k(z) \to \tilde h(z) := \begin{cases} f(lz)/lz, & l \in (0,\infty),\ z \neq 0, \\ f_0, & l = 0 \text{ or } z = 0, \end{cases}$$

which is continuous. Also note that $z(c_k;c_k) = 0$, $z'(c_k;c_k) = 1$. Let $\bar z(t)$ be the solution of the IVP

$$z'' + w(t)\tilde h(z)z = 0, \quad \bar z(b) = 0, \quad \bar z'(b) = 1.$$

By the continuous dependence of solutions of IVPs on parameters, we have

$$\lim_{k\to\infty} z(t;c_k) = \bar z(t) \quad \text{uniformly for all } t \in [a,b].$$

We see from the initial condition that $\bar z(t) < 0$ in a left-neighborhood of b. It follows that $z(t;c_k) < 0$ for $t \in [c_k,b]$ and sufficiently large k. This shows that $y(t;c_k) < 0$ for all $t \in [c_k,b]$, which contradicts the fact that $y(t;c_k) \in \mathcal{S}_0^+[c_k,b]$.

(ii) Assume $f_\infty = 0$ and $\nu_0^{[1]}(c) < f_0$. We first show that BVP (1.1), (1.9) has a solution $y(t;c) \in \mathcal{S}_0^\gamma[c,b]$. Let $y(t,\rho)$ be the solution of IVP (1.1), (3.17) and $\theta(t,\rho)$

its Prüfer angle. By [15, Lemmas 4.2, 4.5] and the continuity of $\theta(t,\rho)$ in ρ, there exists $\rho_3 > 0$ such that

$$\theta(b,\rho_3) = \pi \quad \text{and} \quad \theta(b,\rho) < \pi \quad \text{for all } \rho \in (\rho_3,\infty). \tag{3.24}$$

Define an energy function for $y(t,\rho_3)$ by (3.5). Then following similarly to that in the proof of Theorem 2.1 in [12], we have (3.6) and (3.19) hold with $\rho = \rho_3$. Let $\Gamma(\rho)$ be defined as in (3.10). Since $y'(b,\rho_3) < 0$, by the first inequality in (2.3), it follows that $\Gamma(\rho_3) < 0$.

Now denote by $\nu_0^{[1]}(c,\beta)$ the first eigenvalue of the SLP consisting of Eq. (1.4) and the BC

$$y(c) = 0, \quad y(b)\cos\beta - y'(b)\sin\beta = 0.$$

Note that $\nu_0^{[1]}(c,\beta)$ is continuous and strictly increasing in β and $\nu_0^{[1]}(c,\pi/2) > 0$, see [17, Lemma 3.32] and [19, Theorem 4.2]. Hence there exists $\beta^* \in (0,\pi/2)$ such that $f_\infty = 0 < \nu_0^{[1]}(c,\beta^*)$. By [15, Lemma 4.5], there exists $\rho_1^{**} \geq \rho_3$ such that

$$0 < \theta(b,\rho) < \beta^* \quad \text{for all } \rho \in (\rho_1^{**},\infty).$$

Similarly, there exists $\rho_2^{**} \geq \rho_3$ such that

$$-\beta^* < \theta(a,\rho) \leq 0 \quad \text{for all } \rho \in (\rho_2^{**},\infty).$$

Let $\rho^{**} = \max\{\rho_1^{**},\rho_2^{**}\}$. Note that $y'(t,\rho) = \rho\cos\theta(t,\rho)$ for $t \in [a,b]$, we have

$$0 < \rho\cos\beta^* < y'(t,\rho) \leq \rho, \quad \text{for all } t \in [a,b] \text{ and } \rho \in (\rho^{**},\infty).$$

Let $\rho \in (\rho^{**},\infty)$. Since $y(t,\rho) = \int_c^t y'(s,\rho)ds$, for any fixed $\delta \in (0,b-c]$

$$\rho\delta\cos\beta^* < y(t,\rho) \leq \rho(b-c), \text{ for all } t \in [c+\delta,b].$$

If f is bounded, then $f(y(t,\rho)) = o(\rho)$ as $\rho \to \infty$ uniformly for $t \in [c,b]$. If f is unbounded, then for $y \geq 0$, define $f^*(y) = \max\{f(x) : 0 \leq x \leq y\}$. We note that $f_\infty = 0$ implies $f^*_\infty = 0$. Indeed, for $y \geq 0$, let $f^*(y) = f(y^*)$ for some $0 \leq y^* \leq y$. Note that $y^*/y \leq 1$ and f is unbounded, we have $y \to \infty$ implies $y^* \to \infty$. Then

$$0 \leq \frac{f^*(y)}{y} = \frac{f(y^*)}{y^*}\frac{y^*}{y} \leq \frac{f(y^*)}{y^*} \to 0, \text{ as } y \to \infty.$$

This implies $f^*(y(t,\rho)) = o(\rho)$ as $\rho \to \infty$ uniformly for $t \in [c+\delta,b]$. Since $f^*(y(t,\rho))$ is increasing in t,

$$0 \leq f^*(y(t,\rho)) \leq f^*(y(c+\delta,\rho)), \text{ for all } t \in [c,c+\delta].$$

It follows that $f^*(y(t,\rho)) = o(\rho)$ as $\rho \to \infty$ uniformly for $t \in [c,c+\delta]$. Hence, $f^*(y(t,\rho)) = o(\rho)$ as $\rho \to \infty$ uniformly for $t \in [c,b]$. We observe that

$$0 \leq f(y(t,\rho)) \leq f^*(y(t,\rho)) \text{ for all } t \in [c,b].$$

Then $f(y(t,\rho)) = o(\rho)$ as $\rho \to \infty$ uniformly for $t \in [c,b]$. It can be shown similarly that $f(y(t,\rho)) = o(\rho)$ as $\rho \to \infty$ uniformly for $t \in [a,c]$. As a result, $y''(t,\rho) = o(\rho)$

as $\rho \to \infty$ uniformly for $t \in [a,b]$. With essentially the same proof as in Part (i), we see that the BVP (1.1), (1.9) has a solution $y \in \mathcal{S}_0^+[c,b]$.

Suppose further that $f_0 = \infty$. Then with a similar proof as in Part (i) with f_0 and f_∞ interchanged, we can show that $\lim_{c\to b^-} y'(c;c) = 0$. We omit the details. $\square$

The subsequent lemma is a parallel result to Lemma 3.10 for BVP (1.1), (1.10) with $d \in (a,b]$. Here, we denote by $\nu_0^{[2]}(d)$ the first eigenvalue of the SLP consisting of Eq. (1.4) and the BC

$$y(a) = 0,\ y(d) = 0, \tag{3.25}$$

for $d \in (a,b]$. Note as $d \to a^+$, $\nu_0^{[2]}(d)$ is strictly decreasing and $\nu_0^{[2]}(d) \to \infty$, see [18, Theorems 2.2, 2.3] and [19, Theorem 4.1].

Lemma 3.11. *Let $d \in (a,b]$ and the second inequality in* (2.3) *hold.*

(i) *Assume $f_0 = 0$ and $\nu_0^{[2]}(d) < f_\infty$. Then BVP* (1.1), (1.10) *has a solution $y(t;d) \in \mathcal{S}_0^-[a,d]$. Furthermore, if $f_\infty = \infty$, then $\lim\limits_{d\to a^+} y'(d;d) = \infty$.*

(ii) *Assume $f_\infty = 0$ and $\nu_0^{[2]}(d) < f_0$. Then BVP* (1.1), (1.10) *has a solution $y(t;d) \in \mathcal{S}_0^-[a,d]$. Furthermore, if $f_0 = \infty$, then $\lim\limits_{d\to a^+} y'(d;d) = 0$.*

Proof. This follows directly from Lemma 3.10 and the transformation used in the proof of Lemma 3.5. $\square$

Remark 3.12. Lemmas 3.10 and 3.11 are results for nodal solutions of BVPs (1.1), (1.9) and (1.1), (1.10) in the classes $\mathcal{S}_0^+[c,b]$ and $\mathcal{S}_0^-[a,d]$, respectively. Parallel results hold for nodal solutions of BVPs (1.1), (1.9) and (1.1), (1.10) in the classes $\mathcal{S}_0^-[c,b]$ and $\mathcal{S}_0^+[a,d]$, respectively.

Proof of Theorem 2.3. Without loss of generality, we consider the case when $f_0 = 0$ and $f_\infty = \infty$. The other cases can be proved similarly. By Lemmas 3.10, (i) and 3.11, (i), BVPs (1.1), (1.9) and (1.1), (1.10) have solutions $y^{[1]}(t;c) \in \mathcal{S}_0^+[c,b]$ and $y^{[2]}(t;d) \in \mathcal{S}_0^-[a,d]$ satisfying

$$\lim_{c\to b^-} (y^{[1]})'(c;c) = \infty \quad \text{and} \quad \lim_{d\to a^+} (y^{[2]})'(d;d) = \infty,$$

respectively. It easy to see that the results in Remark 3.9 also hold for the positive and negative solutions of BVPs (1.1), (1.9) and (1.1), (1.10). Let $\rho_i^{[1]}(c) = y_i^{[1]}(c;c)$ with $(\rho_i^{[1]}(c), c)$ being on the continuous curve Λ_i^c and $\rho_j^{[2]}(d) = y_j^{[2]}(d;d)$ with $(\rho_j^{[2]}(d), d)$ being on the continuous curve Λ_j^d. By the continuity of the curves Λ^c and Λ^d, there exists $c^* = d^* \in (a,b)$ such that $(y^{[1]})'(c^*;c^*) = (y^{[2]})'(d^*;d^*)$. Also note that $y^{[1]}(c^*;c^*) = 0$ and $y^{[2]}(d^*;d^*) = 0$. By the uniqueness of solutions of IVPs, we have $y^{[1]}(t,c^*) \equiv y^{[2]}(t,d^*)$ for $t \in [a,b]$. We denote $y(t) = y^{[1]}(t,c^*) = y^{[2]}(t,d^*)$ on $[a,b]$. Thus we have $y \in \mathcal{S}_0^+[c^*,b] \cap \mathcal{S}_0^-[a,d^*]$. Therefore y has one zero on (a,b) and $y(t)$

is negative in a right-neighborhood of a. Thus $y \in \mathcal{S}_1^-$. It is easy to see $-y$ is also a solution of BVP (1.1), (1.2) since f is an odd function. Thus $-y$ has one zero in (a, b) and hence $-y \in \mathcal{S}_1^+$. □

To prove Theorem 2.4 on the non-existence of solutions of BVP (1.1), (1.2), let $\{\zeta_n(\alpha)\}_{n=0}^{\infty}$ denote the eigenvalues of the SLP consisting of Eq. (1.4) and the BC

$$\begin{aligned} &\cos\alpha\ y(a) - \sin\alpha\ y'(a) = 0, \quad \alpha \in [0, \pi), \\ &y(b) = 0. \end{aligned}$$

We note that for $n \in \mathbb{N}_0$, $\zeta_n(0) = \nu_n$, where ν_n is the n-th eigenvalue of SLP (1.4), (2.4). From [17, Lemma 3.32] and [19, Theorem 4.2], $\zeta_n(\alpha)$ is continuous and $\zeta_n(\alpha)$ is strictly decreasing in α on $[0, \pi)$; moreover,

$$\lim_{\alpha\to\pi^-} \zeta_n(\alpha) = -\infty \quad \text{and} \quad \lim_{\alpha\to\pi^-} \zeta_n(\alpha) = \zeta_{n-1}(0). \tag{3.26}$$

Consider the BVP consisting of Eq. (1.1) and the BC

$$\begin{aligned} &\cos\alpha\ y(a) - \sin\alpha\ y'(a) = 0, \quad \alpha \in [0, \pi), \\ &y'(b) - \sum_{i=1}^{m} k_i y'(\eta_i) = 0. \end{aligned} \tag{3.27}$$

The following result from [12, Theorem 2.2] plays a key role in the proof of Theorem 2.3.

Lemma 3.13. *(i) Assume $f(y)/y < \zeta_n(\alpha)$ for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP* (1.1), (3.27) *has no solution with i zeros on (a, b) for $i \geq n + 1$.*

(ii) Assume $f(y)/y > \zeta_n(\alpha)$ for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP (1.1), (3.27) *has no solution with i zeros on (a, b) for $i \leq n$.*

Proof of Theorem 2.3. (i) By contradiction, suppose BVP (1.1), (1.2) has a solution $y \in \mathcal{S}_i$ for some $i \geq n + 2$. Then there exists $\alpha^* \in [0, \pi)$ such that $\cos\alpha^*\ y(a) - \sin\alpha^*\ y'(a) = 0$. This means that y is also a solution of BVP (1.1), (3.27) for $\alpha = \alpha^*$. From our assumptions, along with the monotonicity of $\zeta_n(\alpha)$, we have that for any $\alpha \in [0, \pi)$

$$f(y)/y < \nu_n = \zeta_n(0) < \zeta_{n+1}(\alpha).$$

By Lemma 3.13, (i), BVP (1.1), (3.27) has no solution with i zeros on (a, b) for all $i \geq n + 2$. We have reached a contradiction to the assumption that $y \in \mathcal{S}_i$.

(ii) The proof is similar to above except that Lemma 3.13, (ii) instead of Lemma 3.13, (i), is used. □

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