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EXISTENCE OF SOLUTIONS TO MIXED BOUNDARY VALUE PROBLEMS VIA VARIATIONAL METHOD

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. In our previous paper, we discussed the existence of solutions to mixed boundary value problems of 2nd-order differential systems with a p-Laplacian, where we confined the p in the interval $[2,\infty)$. Now we give result for the case $p \in (1,2)$, via the mountain pass theorem.

KEYWORDS. Mixed boundary value problem; p-Laplacian; Duality; Mountain pass theorem.

AMS (MOS) Subject Classification. 34B15.

1. INTRODUCTION

The variational method is now a powerful tool in the study of boundary value problems of differential equations and systems [10–9]. In this paper, we research the existence of the solutions to a mixed boundary value problem of ordinary differential system with a p-Laplacian in the form

$$\begin{cases} (\phi_p(u'))' + \nabla F(t,u) = 0, \\ u(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

where $\phi_p(x) = |x|^{p-2}x$ for $x \in R^n$ with $|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ and $1 < p < 2$.

Throughout the paper, we assume that the following conditions hold.

(A_1) $F \in C([0,1] \times R^n)$, $F(t,\cdot)$ is strictly convex, lower semi continuous and continuously differentiable.

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(A_2) For $q = p/(p-1)$, there are K, N, N_0 and $\alpha \geq 0$ such that

$$(\nabla F(t,x), x) \geq q(F(t,x) - N),$$

$$-N_0 \leq F(t,x) \leq \alpha |x|^q + K.$$

(A_3) There is $r > 0$ such that

$$\inf_{\substack{0 \leq t \leq 1 \\ |x| = r}} F(t,x) = C > \frac{N}{q}.$$

Let

$$g(\alpha) = \frac{1}{p}\left(1 - \frac{p}{2}\right) q^{\frac{4-3p}{q-p}} \alpha^{-\frac{p}{q-p}} \tag{1.2}$$

and $m = C - \frac{N}{q}$. Suppose without loss of generality that

$$\alpha > \left(\frac{mq^{\frac{1}{q}}}{r^q}\right)^p.$$

The main result of this paper is

Theorem 1.1. *Suppose* $1 < p < 2$ *and assumptions* (A_1)*–*(A_3)*hold. Then BVP* (1.1) *has at least one solution if* $K < g(\alpha)$ *and* $N_0 < \left(\frac{1}{\alpha q}\right)^{\frac{1}{q}} [g(\alpha) - K]$.

Note, the expression of $g(\alpha)$ in (1.2) implies that K, N_0 may be arbitrary large if $\alpha \to 0$. On the other hand, if $K = N_0 = 0$, then α may be arbitrary large.

In order to prove the above theorem, we first make transform

$$u_1 = u, \quad u_2 = -\lambda \phi_p(u'), \tag{1.3}$$

where $\lambda = \left(\frac{1}{\alpha q}\right)^{\frac{1}{q}} > 0$. Denote (u_1, u_2) by w. Then BVP (1.1) becomes

$$\begin{cases} J\dot{w} + \nabla G(t,w) = 0, \\ u_1(0) = u_2(1) = 0, \end{cases} \tag{1.4}$$

where

$$w = (u_1, u_2) = (u_{11}, \ldots, u_{1n}; u_{21}, \ldots, u_{2n}),$$

$$G(t,w) = \frac{1}{q\lambda^{\frac{1}{q}}} |u_2|^q + \lambda F(t, u_1),$$

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \text{ with } I_n \text{ being the identity matrix in } R^n.$$

Hence, we have $\nabla G : [0,1] \times R^{2n} \to R^{2n}$ given by

$$\nabla G(t,w) = \left(\frac{1}{\lambda^{\frac{1}{q}}} \phi_q(u_2), \lambda \nabla F(t, u_1)\right).$$

And under the assumptions $(A_1)-(A_3)$, it holds that

$$\begin{aligned} G(t,w) &\le \lambda\alpha|u_1|^q + \frac{1}{\lambda^{q-1}q}|u_2|^q + \lambda K \\ &= \beta\left(|u_1|^q + |u_2|^q\right) + \left(\frac{1}{\alpha q}\right)^{\frac{1}{q}} K \\ &\le \beta|w|^q + c, \end{aligned} \tag{1.5}$$

with $\beta = \alpha^{\frac{1}{p}}/q^{\frac{1}{q}}$, $c = \left(\frac{1}{\alpha q}\right)^{\frac{1}{q}} K$, and

$$\begin{aligned} (\nabla G(t,w),w) &= (\lambda\nabla F(t,u_1),u_1) + (\lambda^{-\frac{1}{q}}\phi_q(u_2),u_2) \\ &\ge \lambda q[F(t,u_1)-N] + \lambda^{-\frac{1}{q}}|u_2|^q \\ &= qG(t,w) - \lambda qN. \end{aligned} \tag{1.6}$$

From the strict convexity of F respect to u, we can easily obtain the strict convexity of G respect to w. Therefore, we can define the Fenchel transform $G^*(t,\cdot)$ of $G(t,\cdot)$ by

$$G^*(t,\dot{v}) = \sup_{w\in R^{2n}}[(\dot{v},w) - G(t,w)],$$

where $\dot{v}$ denotes a vector in R^{2n}. The strict convexity of $G(t,w)$ in w implies the same property of $G^*(t,\dot{v})$ in $\dot{v}$. Therefore, the following three relations are equivalent,

$$\begin{aligned} &G(t,w) + G^*(t,\dot{v}) = (\dot{v},w), \\ &\dot{v} = \nabla G(t,w), \\ &w = \nabla G^*(t,\dot{v}). \end{aligned}$$

Furthermore, from (1.5) and (1.6), we have

$$\begin{aligned} G^*(t,\dot{v}) &\ge \sup_{u\in R^{2n}}[(\dot{v},w) - \beta|w|^q - c] \\ &= (q\beta\phi_q(w),w) - \beta|w|^q - c \\ &= \beta(q-1)|w|^q - c \\ &= \frac{1}{p}\left(\frac{1}{\beta q}\right)^{p-1}|\dot{v}|^p - c. \end{aligned} \tag{1.7}$$

Since $\dot{v} = \beta\nabla(|w|^q) = \beta q\phi_q(w)$ and then

$$w = \phi_p\left(\frac{\dot{v}}{\beta q}\right) = \left(\frac{1}{\beta q}\right)^{p-1}\phi_p(\dot{v}).$$

At the same time, the relation

$$\begin{aligned}(\nabla G^*(t,\dot v),\dot v) &= (u,\dot v)\\ &= (u,\nabla G(t,w))\\ &\geq q(G(t,w)-\lambda N)\\ &= q[(w,\dot v)-G^*(t,\dot v)]-q\lambda N\\ &= q(\nabla G^*(t,\dot v),\dot v)-q(G^*+\lambda N)\end{aligned}$$

results in

$$(\nabla G^*(t,\dot v),\dot v)\leq p(G^*+\lambda N). \tag{1.8}$$

From assumptions (A_2) and (A_3), we can deduce other relations.

At first, it is easy to prove that there is $R>r$ such that

$$\inf_{\substack{0\leq t\leq 1\\ |w|=R}} G(t,w)\geq c>\lambda N. \tag{1.9}$$

For each $(t,w)\in[0,1]\times R^{2n}$ with $|w|=R$, let $f(s)=G(t,sw)$. Then

$$\begin{aligned}f'(s) &= (\nabla G(t,sw),w)\quad (s\geq 1)\\ &= \frac{1}{s}(\nabla G(t,sw),sw)\\ &\geq \frac{q}{s}[G(t,sw)-\lambda N]\\ &= \frac{q}{s}f(s)-\frac{\lambda qN}{s}\end{aligned}$$

and it follows that

$$(s^{-q}f(s))'\geq -\lambda qNs^{-q-1}$$

and then for $s\geq 1$,

$$\begin{aligned}f(s) &\geq f(1)s^q+\lambda N(1-s^q)\\ &= (f(1)-\lambda N)s^q+\lambda N,\end{aligned}$$

i.e.,

$$G(t,sw)\geq (G(t,w)-\lambda N)s^q+\lambda N.$$

when $|w|\geq R$, we have $|w|/R\geq 1$. Let $w_0=\frac{R}{|w|}w$. Then $w=\frac{|w|}{R}w_0=sw_0$. Consequently, $s\geq 1$ and from (1.9), it holds that

$$\begin{aligned}G(t,w) &\geq (G(t,w_0)-\lambda N)\left(\frac{|w|}{R}\right)^q+\lambda N\\ &\geq (c-\lambda N)\left(\frac{1}{R}\right)^q|w|^q+\lambda N.\end{aligned}$$

Let $c_0=(c-\lambda N)/R^q$. The continuity of G implies that there is $N_0>0$ such that

$$G(t,w)\geq c_0|w|^q-N_0 \tag{1.10}$$

for all $w \in R^{2n}$.

Then we have

$$G^*(t, \dot{v}) \le \left(\frac{1}{c_0 q}\right)^{p-1} \frac{1}{p}|\dot{v}|^p + N_0. \tag{1.11}$$

Since inequality (1.2) implies

$$\alpha > \left(\frac{mq^{\frac{1}{q}}}{R^q}\right)^p,$$

we can get $\beta > c_0$. So inequalities (1.7) and (1.11) do not conflict with each other.

2. PRELIMINARIES

Let $I \subseteq R$ be an interval and k, m, n integers with $k \le mn$. Suppose $F : I \times R^{(m+1)n} \to R^n$ and $BC : (C(I))^{mn} \to R^k$ are functions. Then $F\left(t, u(t), \ldots, u^{(m)}(t)\right) = 0$ is a group of differential equations and $BC\left(u, u', \ldots, u^{(m-1)}\right) = 0$ is that of boundary conditions if for any $s \in I$. $BC\left(u, u', \ldots, u^{(m-1)}\right) \neq BC\left(u(s), u'(s), \ldots, u^{(m-1)}(s)\right)$. In this case

$$\begin{cases} F\left(t, u, u', \ldots, u^{(m)}\right) = 0, \\ BC\left(u, u', \ldots, u^{(m-1)}\right) = 0 \end{cases} \tag{2.1}$$

is a boundary value problem (BVP, for short).

Definition 2.1. If there is a $w \in (C^m(I))^n$ such that $BC\left(w, w', \ldots, w^{(m-1)}\right) = 0$ and

$$F\left(t, w(t), w'(t), \ldots, w^{(m)}(t)\right) = 0, \quad \text{for all } t \in I,$$

then w is called a classical solution to BVP (2.1) while

$$v \in \{x \in (c^{m-1}(I))^n : x^{(m)}(t) \text{ exists for a.e. } t \in I\}$$

such that $BC\left(v, v', \ldots, v^{(m-1)}\right) = 0$ and

$$F\left(t, v(t), v'(t), \ldots, v^{(m)}(t)\right) = 0, \ a.e.\ t \in I,$$

v is called a strong solution to BVP (2.1).

Once BVP (2.1) can be transformed into

$$\begin{cases} u^{(m)} = H\left(t, u, u', \ldots, u^{(m-1)}\right), \\ BC\left(u, u', \ldots, u^{(m-1)}\right) = 0, \end{cases} \tag{2.2}$$

we have

Lemma 2.1. *Suppose $H \in C(I \times R^{mn}, R^n)$ and u is a strong solution to BVP (2.2). Then u is also a classical solution to BVP (2.2).*

Proof. Since u is a strong solution to BVP (2.2), it holds that $BC\left(u, u', \ldots, u^{(m-1)}\right) = 0$ and $u^{(m)}(t)$ exists for a.e. $t \in I$.

$$u^{(m)}(t) = H\left(t, u(t), u'(t), \ldots, u^{(m-1)}(t)\right), \ a.e.\ t \in I.$$

The continuity of H means the function $\hat{H}(t) = H(t, u(t), \ldots, u^{(m-1)}(t))$ is continuous on I and then $u^{(m)} \in C(I)$. So u is a classical solution to (2.2). □

Let $X = \{w = (u_1, u_2) \in W^{1,p}([0,1], R^{2n}) : u_1(0) = u_2(1) = 0\}$ and construct a functional in the form

$$\phi(w) = \int_0^1 \left[\frac{1}{2}(J\dot{w}, w) + G(t, w)\right] dt. \tag{2.3}$$

Then we have

$$\langle \phi'(w), v\rangle = \int_0^1 (J\dot{w} + \nabla G(t, w), v) dt \tag{2.4}$$

for all $v \in X$. From (1.5) and (1.6), we can easily show that

$$\phi'(w) \in X^*.$$

We need the following lemma to prove Lemma 2.3.

Lemma 2.2 (10, p. 128). *If $u \in L^1_{loc}[0,1]$ satisfies*

$$\int_0^1 u(s)f(s)ds = 0, \quad \forall f \in C_0^\infty[0,1].$$

Then $u(t) = 0$, a.e. $t \in [0,1]$.

Now based on Lemma 2.1 and Lemma 2.2, we have

Lemma 2.3. *If there is a $w \in X$ such that*

$$\langle \phi'(w), v\rangle = 0$$

holds for all $v \in X$, then w is a classical solution to BVP (1.4).

Let $v = -Jw$ and $Y = \{x \in W^{1,p}([0,1], R^{2n}) : x_1(1) = x_2(0) = 0\}$. Then

$$\begin{aligned} \phi(w) &= -\frac{1}{2}\int_0^1 (J\dot{w}, w)dt + \int_0^1 [(J\dot{w}, w) + G(t, w)]dt \\ &= -\frac{1}{2}\int_0^1 (J\dot{w}, w)dt - \int_0^1 [(\dot{v}, w) - G(t, w)]dt \\ &= -\int_0^1 \left[\frac{1}{2}(J\dot{v}, v) + G^*(t, \dot{v})\right] dt \\ &=: -\psi(v) \end{aligned} \tag{2.5}$$

and $\psi : Y \to R$ is a differentiable real functional. Consequently,

$$\langle \psi'(v), u\rangle = \int_0^1 [(J\dot{v}, u) + (\nabla G^*(t, \dot{v}), \dot{u})]dt, \tag{2.6}$$

$\psi'(v) \in Y^*$.

The proof of our result is based on the famous mountain pass lemma.

Lemma 2.4 (10, Theorem 4.10)**.** *Let X be a Banach space and $\varphi \in C^1(X,R)$ satisfy (PS)-condition. Assume that there exist $u_0, u_1 \in X$ and a bounded neighborhood Ω of u_0 such that $u_1 \notin \bar{\Omega}$ and*

$$\inf_{v\in\partial\Omega} \varphi(v) > \max\{\varphi(u_0), \varphi(u_1)\}.$$

Then there exists a critical point u of φ, i.e., $\varphi'(u) = 0$.

Throughout we define the norm in Y by

$$\|y\| = \left[\int_0^1 |\dot{y}(t)|^p dt\right]^{\frac{1}{p}}, \quad y \in Y. \tag{2.7}$$

We have

Lemma 2.5 (11, 6.2.18)**.** *A closed subspace of a reflexive Banach space is reflexive.*

Then it follows that

Lemma 2.6. *Space Y is a reflexive Banach space.*

At the end of this section, we notice that

$$\langle \psi'(v), u\rangle = \int_0^1 (-Jv + \nabla G^*(t,\dot{v}), \dot{u})dt \tag{2.8}$$

and shall prove the following lemma.

Lemma 2.7. *Given $v \in Y$, there is $f \in L^q[0,1]$ such that*

$$\langle \psi'(v), u\rangle = \int_0^1 (f(t), \dot{u}(t))dt. \tag{2.9}$$

Proof. Let $l_v(u) = \langle \psi'(v), u\rangle$. Then $l_v \in Y^*$. Define

$$L_v(u) = \int_0^1 (-Jv + \nabla G^*(t,\dot{v}), u)dt, \quad u \in L^p[0,1].$$

Obviously, $L_v \in (L^p)^* = L^q$. According to the Riesz's representation theorem, there is $f \in L^q[0,1]$, such that

$$L_v(u) = \int_0^1 (f(t), u(t))dt.$$

The differential operator $D : Y \to L^p[0,1]$ has the inverse $D^{-1} : L^p[0,1]$ given by

$$(D^{-1}w)(t) = \left(-\int_t^1 w_1(s)ds, \int_0^t w_2(s)ds\right).$$

So

$$l_v(u) = L_v(Du) = \int_0^1 (-Jv + \nabla G^*(t,\dot{v}), Du)dt = \int_0^1 (f(t), \dot{u}(t))dt. \tag{2.10}$$

$\square$

3. PROOF OF THEOREM 1.1

Before the proof we first give some propositions.

Proposition 3.1. *For $v \in Y$, it holds that*

$$\int_0^1 (J\dot{v}, v)dt \geq -\|v\|^2. \tag{3.1}$$

Proof. It follows from

$$|v_1(t)| = \left|\int_t^1 \dot{v}_1(s)ds\right| \leq \int_0^1 |\dot{v}_1(s)|ds,$$

$$|v_2(t)| = \left|\int_0^t \dot{v}_2(s)ds\right| \leq \int_0^1 |\dot{v}_2(s)|ds$$

that

$$\|v\|_\infty = \max_{0\leq t\leq 1} |v(t)| \leq \int_0^1 |\dot{v}(s)|ds \leq \|v\|.$$

Then by Hölder inequality,

$$\int_0^1 (J\dot{v}, v)dt \geq -\int_0^1 |\dot{v}(t)||v(t)|dt \geq -\|v\|_\infty \int_0^1 |\dot{v}(t)|dt > -\|v\|^2.$$

□

Proposition 3.2. *Under the assumption (A_1)–(A_3), the functional ψ defined in* (2.5) *satisfies the (PS)-condition, i.e., every sequence (v_n) in Y such that*

$$\psi(v_n) \quad \textit{is bounded and} \quad \psi'(v_n) \to 0 \quad \textit{as} \quad n \to \infty,$$

contains a convergent subsequence.

Proof. It is clear that $\int_0^1 |v_n(t)|^2 dt \leq \|v_n\|^2$.

Applying (2.5), (2.6), (1.7) and Lemma 2.7, we have

$$\begin{aligned}
\psi(v_n) &= \int_0^1 G^*(t, \dot{v}_n)dt - \frac{1}{2}\int_0^1 (\nabla G^*(t, \dot{v}_n), \dot{v}_n)dt + \frac{1}{2}\langle \psi'(v_n), v_n\rangle \\
&\geq \left(1 - \frac{p}{2}\right)\int_0^1 G^*(t, \dot{v}_n)dt + \frac{1}{2}\int_0^1 (f_n(t), v_n(t))dt \qquad (3.2)\\
&\geq \left(1 - \frac{p}{2}\right)\frac{1}{p}\left(\frac{1}{\beta q}\right)^{p-1}\|v_n\|^p - \left(1 - \frac{p}{2}\right)c - \frac{1}{2}\|f_n\| \cdot \|v_n\|.
\end{aligned}$$

Hence, (v_n) is bounded in Y. Because Y is a reflexive Banach space (see Lemma 2.5), going if necessary to a sub-sequence, we assume $v_n \rightharpoonup v$ in Y, which implies that

$$v_n(t) \to v(t) \quad \text{uniformly on} \quad [0, 1].$$

Using (2.9), we have

$$\int_0^1 (-Jv_n + \nabla G^*(t, \dot{v}_n) - f_n(t), \dot{u}(t))dt = 0$$

for all $u \in Y$, which implies

$$\int_0^1 (-Jv_n + \nabla G^*(t, \dot{v}_n) - f_n(t), w(t))dt = 0$$

for all $w \in L^p[0,1]$. Then

$$-Jv_n(t) + \nabla G^*(t, \dot{v}_n(t)) = f_n(t), \quad a.e.\ t \in [0,1]$$

and $\|f_n\|_{L^q} \to 0$ since $\psi'(v_n) \to 0$. By duality, we have

$$\dot{v}_n(t) = \nabla G(t, Jv_n(t) + f_n(t)), \quad a.e.\ t \in [0,1].$$

Therefore,

$$\dot{v}_n \to \nabla G(\cdot, Jv(\cdot)) = \dot{v}, \quad \text{in} \quad L^p[0,1],$$

i.e., $v_n \to v$ in Y. □

Proof of Theorem 1.1.

Proof. We first prove that ψ defined in (2.5) has a critical point v in Y.

Obviously, Y is a real Banach space and $\psi \in C^1(Y, R)$.

Using (1.7) and Proposition 3.1, one has

$$\psi(v) \geq -\frac{1}{2}\|v\|^2 + \frac{1}{p}\left(\frac{1}{\beta q}\right)^{p-1}\|v\|^p - \left(\frac{1}{\alpha q}\right)^{\frac{1}{q}} K.$$

Take $v_0 = 0$ and $\Omega = \{v \in Y : \|v\| < r_0\}$, where

$$r_0 = \left(\frac{1}{\alpha q^{p-1}}\right)^{\frac{1}{q-p}},$$

then for $v \in \partial\Omega$,

$$\psi(v) \geq \left(\frac{1}{\alpha q}\right)^{\frac{1}{q}} g(\alpha) - \left(\frac{1}{\alpha q}\right)^{\frac{1}{q}} K =: d > 0$$

and $v_0 \in \text{int}\Omega$, $\psi(v_0) \leq N_0 < d$.

On the other hand, using (1.11) one gets

$$\psi(v) \leq \frac{1}{2}\int_0^1 (J\dot{v}, v)dt + \left(\frac{1}{c_0 q}\right)^{p-1}\frac{1}{p}\|v\|^p + N_0, \quad v \in Y.$$

Let $e \in R^n$ such that $|e| = 1$ and

$$v = r\bar{v} = r\left(\cos\frac{\pi}{2}t \cdot e,\ \sin\frac{\pi}{2}t \cdot e\right) \in Y \tag{3.3}$$

with $r > 0$. Then $(J\dot{v}, v) = -\frac{\pi}{2}r^2$ and $\|v_1\| = \|r\bar{v}\| = r\|\bar{v}\| = r$. It follows that

$$\psi(v) \leq -\frac{\pi}{4}r^2 + \left(\frac{1}{c_0 q}\right)^{p-1}\frac{1}{p}r^p + N_0.$$

Clearly, we can choose $r_1 > r_0$ large enough such that

$$\psi(v_1) = \psi(r_1\bar{v}) \leq -\frac{\pi}{4}r_1^2 + \left(\frac{1}{c_0 q}\right)^{p-1}\frac{1}{p}r_1^p + N_0 < 0 < d$$

with $v_1 \notin \bar{\Omega}$. Then ψ has a critical point $v = v(t)$ so that $\psi'(v) = 0$, i.e.,

$$\begin{aligned} 0 &= \int_0^1 [(J\dot{v}, y) + (\nabla G^*(t, \dot{v}), \dot{y})]dt \\ &= \int_0^1 [-(Jv, \dot{y}) + (\nabla G^*(t, \dot{v}), \dot{y})]dt \\ &= \int_0^1 (-Jv + \nabla G^*(t, \dot{v}), \dot{y})dt \end{aligned}$$

holds for all $y \in Y$, which implies

$$Jv = \nabla G^*(t, \dot{v}),$$

since all $\dot{u}$ make $L^p[0, 1]$.

Then by duality, one has

$$\dot{v} = \nabla G(t, Jv)$$

and the relation $v = -Jw$ yields

$$-J\dot{w} = \nabla G(t, w),$$

i.e., $w(t) = Jv(t)$ is a solution to BVP (1.4). Then

$$u(t) = u_1(t)$$

is a solution to (1.1) with $1 < p < 2$. $\square$

4. **Example**

Consider the following boundary value problem

$$\text{(4.1)} \quad \begin{cases} \left(\dfrac{u_1'}{\sqrt{u_1'^2 + u_2'^2}}\right)' + \dfrac{\sqrt{u_1^2 + u_2^2}}{6} u_1 - \dfrac{2u_1}{(u_1^2 + u_2^2 + 1)^2} \cos(\dfrac{1}{u_1^2 + u_2^2 + 1} + t) = 0, \\ \left(\dfrac{u_2'}{\sqrt{u_1'^2 + u_2'^2}}\right)' + \dfrac{\sqrt{u_1^2 + u_2^2}}{6} u_2 - \dfrac{2u_2}{(u_1^2 + u_2^2 + 1)^2} \cos(\dfrac{1}{u_1^2 + u_2^2 + 1} + t) = 0, \\ u_1(0) = u_2(0) = u_1'(1) = u_2'(1) = 0. \end{cases}$$

Let $u = (u_1, u_2)$ and

$$F(t, u) = \frac{1}{18}\left(\sqrt{u_1^2 + u_2^2}\right)^3 + \sin\left(\frac{1}{u_1^2 + u_2^2 + 1} + t\right).$$

Then BVP (4.1) becomes

$$\text{(4.2)} \quad \begin{cases} \left(\phi_{\frac{3}{2}}(u')\right)' + \nabla F(t, u) = 0, \\ u(0) = u'(1) = 0. \end{cases}$$

Obviously, $p = \frac{3}{2}$ and then $q = 3$. We have $F \in C([0, 1] \times R^2, R)$.

$$-1 \le F(t, u) \le \frac{1}{18}|u|^3 + 1,$$

$$(\nabla F(t,u),u) \geq 3\left(F(t,u)-\frac{2}{3}\right).$$

As $K = N_0 = 1$, $N = \frac{2}{3}$, $\alpha = \frac{1}{18}$, one has

$$g\left(\frac{1}{18}\right) = \frac{2}{3}\cdot\frac{1}{4}\cdot 3^{-\frac{1}{3}}\cdot 18 = \sqrt[3]{9} > 2,$$

and then

$$K = 1 < g\left(\frac{1}{18}\right), \quad N_0 = 1 < \sqrt[3]{6}\left(g\left(\frac{1}{18}\right)-1\right)$$

satisfy the conditions given in Theorem 1.1. Therefore, BVP (4.1) has at least a classical solution.

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