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OSCILLATION RESULTS FOR SECOND ORDER NEUTRAL DYNAMIC EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. In this paper, we establish some new criteria for the oscillation of the second-order nonlinear neutral dynamic equations with distributed deviating arguments of the form

$$\left(r(t)\left|y^{\Delta}(t)\right|^{\gamma-1}y^{\Delta}(t)\right)^{\Delta}+\int_a^b f(t,x(\theta(t,\xi)))\Delta\xi=0$$

on a time scale $\mathbb{T}$, where $y(t):=x(t)+p(t)x(\tau(t))$, $\gamma\geq 1$ is a constant, $r(t)$, $p(t)$ are rd-continuous functions on $\mathbb{T}$, and $f:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is continuous. The results obtained are illustrated with a number of examples.

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1. INTRODUCTION

In this paper, we are concerned with the oscillatory behavior of solutions of the second order nonlinear neutral dynamic equation

$$\left(r(t)\left|y^{\Delta}(t)\right|^{\gamma-1}y^{\Delta}(t)\right)^{\Delta}+\int_a^b f(t,x(\theta(t,\xi)))\Delta\xi=0 \tag{1.1}$$

on a time scale $\mathbb{T}$, where $\gamma\geq 1$ is a constant, and

$$y(t):=x(t)+p(t)x(\tau(t)),\quad t\in\mathbb{T}. \tag{1.2}$$

Throughout this paper, we will assume, without further mention, that the following conditions hold:

(C1) $r:\mathbb{T}\to(0,\infty)$ is a real valued rd-continuous function with $r^{\Delta}(t)\geq 0$, and $p:\mathbb{T}\to[0,1)$ is a increasing real valued rd-continuous function;

(C2) $\tau:\mathbb{T}\to\mathbb{T}$ is a real valued rd-continuous function such that $\tau(t)\leq t$, and $\lim_{t\to\infty}\tau(t)=\infty$;

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(C3) $\theta(t,\xi) : \mathbb{T} \times [a,b]_{\mathbb{T}} \to \mathbb{T}$ is a real valued rd-continuous function such that $\theta(t,\xi)$ is decreasing with respect to ξ, and $\theta(t,\xi) \to \infty$ as $t \to \infty$, where $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$

(C4) $f(t,u) : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $uf(t,u) > 0$ for all $u \neq 0$ and there exists a positive rd-continuous function $q(t)$ defined on $\mathbb{T}$ such that $|f(t,u)| \geq q(t)\left|u^{\beta}\right|$, where $0 < \beta < 1$ is the quotient of odd positive integers.

We shall also consider the two cases

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{1/\gamma}(t)} = \infty \tag{1.3}$$

and

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{1/\gamma}(t)} < \infty. \tag{1.4}$$

Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty) \cap \mathbb{T}$. By a solution of (1.1), we mean a nontrivial real-valued function $x(t)$ which has the properties $y \in C^1_{rd}([t_x,\infty)_{\mathbb{T}},\mathbb{R})$ and $r\left|y^{\Delta}\right|^{\gamma-1} y^{\Delta} \in C^1_{rd}([t_x,\infty)_{\mathbb{T}},\mathbb{R})$ for $t_x \geq t_0$ and satisfies equation (1.1) on $[t_x,\infty)_{\mathbb{T}}$. Our attention is restricted to those solutions of (1.1) which exist on the half-line $[t_x,\infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)| : t \in [t_1,\infty)_{\mathbb{T}}\} > 0$ for any $t_1 \in [t_x,\infty)_{\mathbb{T}}$. A solution $x(t)$ of (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative, and it is called *nonoscillatory* otherwise. Eq. (1.1) is said to be oscillatory if all its solutions are oscillatory. The basic concepts and notation from the time scale calculus will be used (see Bohner and Peterson [9]).

In recent years there has been an increasing interest in studying the oscillation of solutions of various dynamic equations on time scales, and we refer the reader to Saker [3], Bohner et al. [8], Hassan [11], Han et al. [14], Sahiner [15], Erbe et al. [17], Chen [19], Agarwal et al. [20], Sun et al. [22] and the references contained therein.

Regarding neutral dynamic equations, Saker [1] considered the second-order quasi-linear neutral functional dynamic equation

$$\left(p(t)\left([y(t)+r(t)y(\tau(t))]^{\Delta}\right)^{\gamma}\right)^{\Delta} + f(t,y(\delta(t))) = 0 \tag{1.5}$$

and established several sufficient conditions for oscillation of (1.5) for the case γ is a ratio of odd positive integers and when (1.3) or (1.4) holds.

Motivated by work of Saker [1], Candan [4] studied the oscillatory behavior of the equation

$$\left(r(t)\left([y(t)+p(t)y(\tau(t))]^{\Delta}\right)^{\gamma}\right)^{\Delta} + \int_c^d f(t,y(\theta(t,\xi)))\Delta\xi = 0 \tag{1.6}$$

for the case γ is a ratio of odd positive integers and when (1.3) is satisfied.

Very recently, Thandapani and Piramanantham [5] considered the second order nonlinear neutral delay dynamic equation

$$\left(r(t)\left([y(t)+p(t)y(t-\tau)]^{\Delta}\right)^{\gamma}\right)^{\Delta}+q(t)y^{\beta}(t-\delta)=0 \tag{1.7}$$

and presented some criteria for the oscillation of solutions of equation (1.7) under the assumptions (1.3) and (1.4).

Additional oscillation results for second and higher order neutral dynamic equations can be found in Saker and O'Regan [2], Thandapani and Piramanantham [6], Li and Thandapani [7], Grace et al. [12], Zhang and Wang [13], Erbe et al. [16], Chen [18], Yang and Xu [21], and the references therein.

Motivated by these papers, here we wish to determine oscillation criteria for the second-order nonlinear dynamic equation (1.1). Our results when (1.3) holds are sufficient for oscillation of all solutions of (1.1) and when (1.4) holds our results ensure that all solutions either oscillate or converge to zero. Some examples are considered to illustrate the main results.

2. SOME LEMMAS

In this section, we give two lemmas and an theorem, which will play an important role in the proof of our main results. Throughout this paper, we let

$$\varphi_{+}(t):=\max\{\varphi(t),0\},\quad \varphi_{-}(t):=\max\{-\varphi(t),0\},$$

and

$$\delta(t)=\theta(t,b),\quad Q(t)=(b-a)q(t)(1-p(\theta(t,a)))^{\beta}.$$

Theorem 2.1 (Mean Value Theorem on time scale, see [10], [23]). *If f is a continuous function on $[a,b]$ and is Δ-differentiable on $[a,b)$, then there exist $\xi,\eta\in[a,b)$ such that*

$$f^{\Delta}(\eta)(b-a)\le f(b)-f(a)\le f^{\Delta}(\xi)(b-a).$$

Lemma 2.2. *Assume that conditions (C1)–(C4) and (1.3) are satisfied, and let $x(t)$ be an eventually positive solution of (1.1). Then there exists a $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that*

(2.1)
$$y(t)>0, y^{\Delta}(t)>0, y^{\Delta\Delta}(t)<0, \left(r(t)\left|y^{\Delta}(t)\right|^{\gamma-1}y^{\Delta}(t)\right)^{\Delta}<0,\quad \textit{for } t\in[t_1,\infty)_{\mathbb{T}}.$$

Proof. Since $x(t)$ is an eventually positive solution of (1.1), then there exists a $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$, $x(\tau(t))>0$ and $x(\theta(t,\xi))>0$ for all $t\in[t_1,\infty)_{\mathbb{T}}$ and $\xi\in[a,b]_{\mathbb{T}}$. Thus, by (1.2), we have $y(t)\ge x(t)>0$ for all $t\ge t_1$. In view of (1.1) and (C4), we find

$$\left(r(t)\left|y^{\Delta}(t)\right|^{\gamma-1}y^{\Delta}(t)\right)^{\Delta}+\int_a^b q(t)x^{\beta}(\theta(t,\xi))\Delta\xi\le 0,\quad \text{for } t\ge t_1, \tag{2.2}$$

which implies that $r(t)\left|y^{\Delta}(t)\right|^{\gamma-1}y^{\Delta}(t)$ is decreasing on $[t_1,\infty)_T$ and is eventually of one sign. Hence, $y^{\Delta}(t)$ is eventually of one sign, i.e., $y^{\Delta}(t)$ is either eventually positive or eventually negative. We claim that

$$y^{\Delta}(t)>0 \quad \text{for } t\in[t_1,\infty)_{\mathbb{T}}. \tag{2.3}$$

Assume not, there exists a $t_2\in[t_1,\infty)_{\mathbb{T}}$ such that $y^{\Delta}(t_2)\le 0$. From this and the decreasing nature of $r(t)\left|y^{\Delta}(t)\right|^{\gamma-1}y^{\Delta}(t)$, there is a $t_3\ge t_2$ such that

$$r(t)\left|y^{\Delta}(t)\right|^{\gamma-1}y^{\Delta}(t)\le r(t_3)\left|y^{\Delta}(t_3)\right|^{\gamma-1}y^{\Delta}(t_3):=c<0 \quad \text{for } t\ge t_3,$$

and so

$$y^{\Delta}(t)\le-(-c)^{1/\gamma}\frac{1}{r^{1/\gamma}(t)} \qquad \text{for } t\ge t_3.$$

This implies by (1.3) that

$$y(t)\le y(t_3)-(-c)^{1/\gamma}\int_{t_3}^{t}\frac{\Delta s}{r^{1/\gamma}(s)}\to-\infty \quad \text{as } t\to\infty,$$

which contradicts the fact that $y(t)>0$ for all $t\ge t_1$. Hence, (2.3) holds. Now, (2.2) and (2.3) imply

$$\left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+\int_a^b q(t)x^{\beta}(\theta(t,\xi))\Delta\xi\le 0 \quad \text{for } t\ge t_1. \tag{2.4}$$

Thus, $r(t)(y^{\Delta})^{\gamma}(t)$ is decreasing on $[t_1,\infty)_{\mathbb{T}}$. Now, we want to show that

$$y^{\Delta\Delta}(t)<0 \quad \text{for } t\ge t_1. \tag{2.5}$$

Assume the contrary, that is,

$$y^{\Delta\Delta}(t)\ge 0 \quad \text{for } t\ge t_1. \tag{2.6}$$

Then, $y^{\Delta}(t)$ is nondecreasing and

$$y^{\Delta}(t)\le y^{\Delta}(\sigma(t)) \quad \text{for } t\ge t_1 \tag{2.7}$$

which implies by (C1) and (2.3) that

$$r(\sigma(t))(y^{\Delta}(t))^{\gamma}\le r(\sigma(t))(y^{\Delta}(\sigma(t)))^{\gamma}. \tag{2.8}$$

Since $r^{\Delta}(t)\ge 0$, we have $r(t)\le r(\sigma(t))$. This and (2.8) give

$$r(t)(y^{\Delta}(t))^{\gamma}\le r(\sigma(t))(y^{\Delta}(t))^{\gamma}\le r(\sigma(t))(y^{\Delta}(\sigma(t)))^{\gamma}$$

or

$$r(t)(y^{\Delta}(t))^{\gamma}\le r(\sigma(t))(y^{\Delta}(\sigma(t)))^{\gamma},$$

which contradicts the fact that $r(t)(y^{\Delta}(t))^{\gamma}$ is decreasing on $[t_1,\infty)_{\mathbb{T}}$, hence (2.5) holds. The proof is complete. □

Lemma 2.3. *Suppose that the following conditions are satisfied:*

(i) $u \in C^2_{rd}(I, \mathbb{R})$ *where* $I = [T, \infty)_{\mathbb{T}} \subset \mathbb{T}$ *for some* $T > 0$*;*

(ii) $u(t) > 0, u^{\Delta}(t) > 0, u^{\Delta\Delta}(t) \leq 0$ *for* $t \geq T$*.*

Then, for each $0 < k < 1$*, there is a* $T_k \geq T$ *such that*

$$u(\delta(t)) \geq k u(t) \frac{\delta_*(t)}{t}, \quad \text{for } t \geq T_k \tag{2.9}$$

where $\delta_*(t) = \min\{t, \delta(t)\}$*.*

Proof. We consider the two following case: (i) $\delta(t) \leq t$; (ii) $\delta(t) \geq t$.

Case (i). Let $\delta(t) \leq t$. When $\delta(t) = t$, (2.9) holds. Then, it suffices to consider only those t for which $\delta(t) < t$. Let $T \leq \delta(t) < t$. Then, for each $t > T$, there exists a $\xi_1 \in [\delta(t), t)$ such that

$$u(t) - u(\delta(t)) \leq u^{\Delta}(\xi_1)(t - \delta(t)) \leq u^{\Delta}(\delta(t))(t - \delta(t))$$

by the Mean Value Theorem on time scales and the monotone properties of u^{Δ}. Since $u(t) > 0$, we have

$$\frac{u(t)}{u(\delta(t))} \leq 1 + \frac{u^{\Delta}(\delta(t))}{u(\delta(t))}(t - \delta(t)) \quad \text{for } t > \delta(t) \geq T. \tag{2.10}$$

Similarly, we have, for some $\xi_2 \in [T, \delta(t))$,

$$u(\delta(t)) - u(T) \geq u^{\Delta}(\xi_2)(\delta(t) - T) \geq u^{\Delta}(\delta(t))(\delta(t) - T)$$

which gives

$$\frac{u(\delta(t))}{u^{\Delta}(\delta(t))} \geq \delta(t) - T. \tag{2.11}$$

Let $k \in (0, 1)$. Then for $t \geq T/(1-k) = T_k \geq T$ we have $t - T \geq kt$ and $\delta(t) - T \geq k\delta(t)$. Now, (2.11) implies

$$\frac{u(\delta(t))}{u^{\Delta}(\delta(t))} \geq k\delta(t) \quad \text{for } t \geq T_k. \tag{2.12}$$

From (2.10) and (2.12), we obtain

$$\begin{aligned} \frac{u(t)}{u(\delta(t))} &\leq 1 + \frac{u^{\Delta}(\delta(t))}{u(\delta(t))}(t - \delta(t)) \\ &\leq 1 + \frac{t - \delta(t)}{k\delta(t)} \\ &= \frac{t + (k-1)\delta(t)}{k\delta(t)} \\ &\leq \frac{t}{k\delta(t)} = \frac{t}{k\delta_*(t)} \qquad \text{for } t > \delta(t) \geq T_k \geq T, \end{aligned}$$

which is (2.9).

Case (ii). Let $\delta(t) \geq t$. Since $u^{\Delta}(t) > 0$, we have

$$u(\delta(t)) \geq u(t) \geq ku(t) = ku(t)\frac{\delta_*(t)}{t}, \quad \text{for } t \geq T_k$$

which is (2.9). This completes the proof of Lemma 2.3. □

3. MAIN RESULTS

To prove our main results we will make use of the following form of the chain rule on time scales. It is a simple consequence of the well-known Keller's chain rule (see Bohner and Peterson [9, Theorem 1.90]):

$$(x^{\gamma}(t))^{\Delta} = \gamma x^{\Delta}(t) \int_0^1 [(1-h)x(t) + hx^{\sigma}(t)]^{\gamma-1}\, dh. \tag{3.1}$$

Theorem 3.1. *Let* (1.3) *holds. Suppose also that there exist positive rd-continuous Δ-differentiable functions $\alpha(t)$ and $\phi(t)$ such that for all constants $L > 0$ and a positive number M*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\alpha(s)\phi(s)Q(s)\left[k\frac{\delta_*(s)}{s}\right]^{\beta} - \frac{K(s)C^2(s)}{4\beta\phi(s)\alpha(s)M^{(\gamma-1)/\gamma}}\right)\Delta s = \infty, \tag{3.2}$$

where $\delta_*(s)$ is as in Lemma 2.3*,*

$$C(s) = \left(\phi^{\Delta}(s)\right)_+ + \phi(s)\frac{\left(\alpha^{\Delta}(s)\right)_+}{\alpha^{\sigma}(s)}, \quad K(s) = (L\sigma(s))^{1-\beta}\, r^{1/\gamma}(s)\alpha^2(\sigma(s)).$$

Then, equation (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$*.*

Proof. Suppose to the contrary that x is a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that x is an eventually positive solution of equation (1.1). Then there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0, x(\tau(t)) > 0$ and $x(\theta(t,\xi)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$ and $\xi \in [a,b]_{\mathbb{T}}$. Following the same lines as in the proof of Lemma 2.2, we conclude that (2.4) is satisfied. On the other hand, by (1.2) and (2.1), we obtain

$$x(t) = y(t) - p(t)x(\tau(t)) \geq y(t) - p(t)y(\tau(t)) \geq (1-p(t))y(t),$$

which implies that

$$x^{\beta}(\theta(t,\xi)) \geq (1-p(\theta(t,\xi)))^{\beta} y^{\beta}(\theta(t,\xi)), \quad \text{for } t \geq t_2 \geq t_1, \quad \xi \in [a,b]_{\mathbb{T}}. \tag{3.3}$$

Multiplying both sides of (3.3) by $q(t)$ and integrating from a to b, we find

$$\int_a^b q(t)x^{\beta}(\theta(t,\xi))\Delta\xi \geq \int_a^b q(t)(1-p(\theta(t,\xi)))^{\beta} y^{\beta}(\theta(t,\xi))\Delta\xi. \tag{3.4}$$

Substituting (3.4) into (2.4) gives

$$\left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \int_a^b q(t)(1-p(\theta(t,\xi)))^{\beta} y^{\beta}(\theta(t,\xi))\Delta\xi \leq 0 \quad \text{for } t \geq t_2. \tag{3.5}$$

Further, by (3.1) we have

$$\begin{aligned}\left(y^{\beta}(t)\right)^{\Delta} &= \beta y^{\Delta}(t)\int_0^1 \left[(1-h)y(t)+hy^{\sigma}(t)\right]^{\beta-1} dh\\ &\geq \beta\left(y^{\sigma}(t)\right)^{\beta-1} y^{\Delta}(t) > 0. \end{aligned} \tag{3.6}$$

Using (C1), (C3) and (3.6) in (3.5), we obtain

$$\left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + (b-a)q(t)(1-p(\theta(t,a)))^{\beta} y^{\beta}(\theta(t,b)) \leq 0, \quad \text{for } t \geq t_2,$$

or

$$\left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + Q(t)y^{\beta}(\delta(t)) \leq 0, \quad \text{for } t \geq t_2. \tag{3.7}$$

Now define the function

$$w(t) = \alpha(t)\frac{r(t)\left(y^{\Delta}(t)\right)^{\gamma}}{y^{\beta}(t)} \qquad \text{for } t \geq t_2. \tag{3.8}$$

Then $w(t) > 0$ and

$$\begin{aligned} w^{\Delta}(t) &= \left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}\frac{\alpha(t)}{y^{\beta}(t)} + \left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\left[\frac{\alpha(t)}{y^{\beta}(t)}\right]^{\Delta}\\ &\leq -\frac{\alpha(t)Q(t)y^{\beta}(\delta(t))}{y^{\beta}(t)} + \left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\left[\frac{\alpha^{\Delta}(t)}{y^{\beta}(\sigma(t))} - \frac{\alpha(t)\left(y^{\beta}(t)\right)^{\Delta}}{y^{\beta}(t)y^{\beta}(\sigma(t))}\right]\\ &\leq -\frac{\alpha(t)Q(t)y^{\beta}(\delta(t))}{y^{\beta}(t)} + \frac{\left(\alpha^{\Delta}(t)\right)_+}{\alpha^{\sigma}(t)}w^{\sigma}(t) - \frac{\alpha(t)\left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\left(y^{\beta}(t)\right)^{\Delta}}{y^{\beta}(t)y^{\beta}(\sigma(t))}. \end{aligned} \tag{3.9}$$

Using (3.6) in (3.9) , we get

$$\begin{aligned} w^{\Delta}(t) &\leq -\frac{\alpha(t)Q(t)y^{\beta}(\delta(t))}{y^{\beta}(t)} + \frac{\left(\alpha^{\Delta}(t)\right)_+}{\alpha^{\sigma}(t)}w^{\sigma}(t)\\ &\quad - \frac{\beta\alpha(t)\left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\left(y^{\sigma}(t)\right)^{\beta-1}y^{\Delta}(t)}{y^{\beta}(t)y^{\beta}(\sigma(t))}\\ &= -\frac{\alpha(t)Q(t)y^{\beta}(\delta(t))}{y^{\beta}(t)} + \frac{\left(\alpha^{\Delta}(t)\right)_+}{\alpha^{\sigma}(t)}w^{\sigma}(t)\\ &\quad - \frac{\beta\alpha(t)\left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{1/\gamma}\left(y^{\sigma}(t)\right)^{\beta-1}}{r^{1/\gamma}(t)y^{\beta}(t)y^{\beta}(\sigma(t))}. \end{aligned}$$

By using the fact that $r(t)\left(y^{\Delta}(t)\right)^{\gamma}$ is decreasing and $y^{\Delta}(t) > 0$, the latter inequality yields

$$\begin{aligned} w^{\Delta}(t) &\leq -\frac{\alpha(t)Q(t)y^{\beta}(\delta(t))}{y^{\beta}(t)} + \frac{\left(\alpha^{\Delta}(t)\right)_+}{\alpha^{\sigma}(t)}w^{\sigma}(t)\\ &\quad - \frac{\beta\alpha(t)\left(r^{(1+\gamma)/\gamma}(t)\left(y^{\Delta}(t)\right)^{\gamma+1}\right)^{\sigma}\left(y^{\sigma}(t)\right)^{\beta-1}}{r^{1/\gamma}(t)y^{2\beta}(\sigma(t))}\\ &= -\frac{\alpha(t)Q(t)y^{\beta}(\delta(t))}{y^{\beta}(t)} + \frac{\left(\alpha^{\Delta}(t)\right)_+}{\alpha^{\sigma}(t)}w^{\sigma}(t) \end{aligned}$$

$$-\frac{\beta\alpha(t)r^{(1+\gamma)/\gamma}(\sigma(t))\left(y^{\Delta}(\sigma(t))\right)^{2\gamma}\left(y^{\sigma}(t)\right)^{\beta-1}}{r^{1/\gamma}(t)y^{2\beta}(\sigma(t))\left(y^{\Delta}(\sigma(t))\right)^{\gamma-1}}. \tag{3.10}$$

Thus, from (3.8), (3.10) and Lemma 2.3, we obtain

$$w^{\Delta}(t) \le -\alpha(t)Q(t)\left[k\frac{\delta_*(t)}{t}\right]^{\beta} + \frac{\left(\alpha^{\Delta}(t)\right)_+}{\alpha^{\sigma}(t)}w^{\sigma}(t)$$
$$-\frac{\beta\alpha(t)r^{(1-\gamma)/\gamma}(\sigma(t))\left(y^{\sigma}(t)\right)^{\beta-1}}{r^{1/\gamma}(t)\alpha^2(\sigma(t))\left(y^{\Delta}(\sigma(t))\right)^{\gamma-1}}w^2(\sigma(t)). \tag{3.11}$$

Since $r(t)\left(y^{\Delta}(t)\right)^{\gamma}$ is positive and decreasing on $[t_1,\infty)_{\mathbb{T}}$, there exists a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $r(t)\left(y^{\Delta}(t)\right)^{\gamma} \le \frac{1}{M}$ for some positive constant M and for $t \in [t_2,\infty)$. Hence, we have

$$\frac{1}{\left(y^{\Delta}(\sigma(t))\right)^{\gamma-1}} \ge \left(Mr(\sigma(t))\right)^{(\gamma-1)/\gamma}. \tag{3.12}$$

From (2.1), one has

$$y(t) = y(t_1) + \int_{t_1}^{t} y^{\Delta}(s)\Delta s \le y(t_1) + y^{\Delta}(t_1)(t-t_1) \le c + dt,$$

where $c = y(t_1) - t_1 y^{\Delta}(t_1)$ and $d = y^{\Delta}(t_1)$. By putting $L = |c| + d$ and $t_2 \ge \max\{t_1, 1\}$, we find that

$$y(t) \le Lt \quad \text{for all } t \ge t_2,$$

which gives

$$\left(y^{\sigma}(t)\right)^{\beta-1} \ge \left(L\sigma(t)\right)^{\beta-1}. \tag{3.13}$$

Using (3.12) and (3.13) in (3.11), we obtain

$$w^{\Delta}(t) \le -\alpha(t)Q(t)\left[k\frac{\delta_*(t)}{t}\right]^{\beta} + \frac{\left(\alpha^{\Delta}(t)\right)_+}{\alpha^{\sigma}(t)}w^{\sigma}(t)$$
$$-\frac{\beta\alpha(t)M^{(\gamma-1)/\gamma}\left(L\sigma(t)\right)^{\beta-1}}{r^{1/\gamma}(t)\alpha^2(\sigma(t))}w^2(\sigma(t)). \tag{3.14}$$

Multiplying (3.14) by $\phi(s)$ and integration from t_2 to t, we conclude that

$$\int_{t_2}^{t}\alpha(s)\phi(s)Q(s)\left[k\frac{\delta_*(s)}{s}\right]^{\beta}\Delta s \le -\int_{t_2}^{t}\phi(s)w^{\Delta}(s)\Delta s + \int_{t_2}^{t}\phi(s)\frac{\left(\alpha^{\Delta}(s)\right)_+}{\alpha^{\sigma}(s)}w^{\sigma}(s)\Delta s$$
$$-\int_{t_2}^{t}\phi(s)\frac{\beta\alpha(s)M^{(\gamma-1)/\gamma}}{K(s)}w^2(\sigma(s))\Delta s. \tag{3.15}$$

Using integration by parts, we obtain

$$-\int_{t_2}^{t}\phi(s)w^{\Delta}(s)\Delta s = -\phi(s)w(s)\mid_{t_2}^{t} + \int_{t_2}^{t}\phi^{\Delta}(s)w^{\sigma}(s)\Delta s$$
$$\le \phi(t_2)w(t_2) + \int_{t_2}^{t}\left(\phi^{\Delta}(s)\right)_+ w^{\sigma}(s)\Delta s.$$

This and (3.15) lead to

$$\begin{aligned}
&\int_{t_2}^{t} \alpha(s)\phi(s)Q(s)\left[k\frac{\delta_*(s)}{s}\right]^{\beta}\Delta s \\
&\le \phi(t_2)w(t_2) + \int_{t_2}^{t}\left[\left(\phi^{\Delta}(s)\right)_+ + \phi(s)\frac{\left(\alpha^{\Delta}(s)\right)_+}{\alpha^{\sigma}(s)}\right]w^{\sigma}(s)\Delta s \\
&\quad - \int_{t_2}^{t}\phi(s)\frac{\beta\alpha(s)M^{(\gamma-1)/\gamma}}{K(s)}w^2(\sigma(s))\Delta s \\
&= \phi(t_2)w(t_2) + \int_{t_2}^{t}\frac{K(s)C^2(s)}{4\phi(s)\beta\alpha(s)M^{(\gamma-1)/\gamma}}\Delta s \\
&\quad - \int_{t_2}^{t}\left(\sqrt{\phi(s)\frac{\beta\alpha(s)M^{(\gamma-1)/\gamma}}{K(s)}}w^{\sigma}(s) - \sqrt{\frac{K(s)}{4\phi(s)\beta\alpha(s)M^{(\gamma-1)/\gamma}}}C(s)\right)^2\Delta s.
\end{aligned}$$

Hence,

$$\int_{t_2}^{t}\left(\alpha(s)\phi(s)Q(s)\left[k\frac{\delta_*(s)}{s}\right]^{\beta} - \frac{K(s)C^2(s)}{4\phi(s)\beta\alpha(s)M^{(\gamma-1)/\gamma}}\right)\Delta s \le \phi(t_2)w(t_2) < +\infty,$$

which contradicts assumption (3.2). Therefore, Equation (1.1) is oscillatory. □

Let $D_0 = \{(t,s) \in \mathbb{T}\times\mathbb{T} : t > s \ge t_0\}$ and $D = \{(t,s) \in \mathbb{T}\times\mathbb{T} : t \ge s \ge t_0\}$. We say that a function $H \in C_{rd}(D,\mathbb{R})$ belongs to a class $\mathcal{P}$ if it satisfies the following conditions:

(i) $H(t,t) = 0,\ t \ge t_0, \quad H(t,s) > 0$ on D_0;

(ii) H has a nonpositive continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ on D_0 with respect to the second variable.

Theorem 3.2. *Assume that* (1.3) *holds. Suppose further that there exist functions* $\alpha(t) \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ *and* $H, h \in C_{rd}(D,\mathbb{R})$ *such that* H *belongs to the class* $\mathcal{P}$*,*

$$-H^{\Delta_s}(t,s) - H(t,s)\frac{\alpha^{\Delta}(s)}{\alpha^{\sigma}(s)} = \frac{h(t,s)}{\alpha^{\sigma}(s)}H^{1/2}(t,s), \tag{3.16}$$

and, for all constants $L > 0$ *and a positive number* M*, we have*

(3.17)

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\alpha(s)Q(s)\left[k\frac{\delta_*(s)}{s}\right]^{\beta} - \frac{K(s)\left(h_-(t,s)\right)^2}{4\beta\alpha(s)M^{(\gamma-1)/\gamma}\alpha^2(\sigma(s))}\right]\Delta s = \infty,$$

where $\delta_*(s), Q(s)$ *and* $K(s)$ *are as in Theorem 3.1. Then, equation* (1.1) *is oscillatory.*

Proof. Suppose to the contrary that x is a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that x is an eventually positive solution of equation (1.1). Then there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0, x(\tau(t)) > 0$ and $x(\theta(t,\xi)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$ and $\xi \in [a,b]_{\mathbb{T}}$. Define the function $w(t)$ by

(3.8). Let $\alpha^{\Delta}(t)$ be replaced by $\left(\alpha^{\Delta}(t)\right)_+$ in (3.14). Proceeding as in the proof of Theorem 3.1, we have, for $t \geq t_2$,

(3.18)
$$\alpha(t)Q(t)\left[k\frac{\delta_*(t)}{t}\right]^{\beta} \leq -w^{\Delta}(t) + \frac{\alpha^{\Delta}(t)}{\alpha^{\sigma}(t)}w^{\sigma}(t) - \frac{\beta\alpha(t)M^{(\gamma-1)/\gamma}\left(L\sigma(t)\right)^{\beta-1}}{r^{1/\gamma}(t)\alpha^2(\sigma(t))}w^2(\sigma(t)).$$

Multiplying both sides of (3.18) by $H(t,s)$ and integrating from t_2 to t, we obtain

$$\int_{t_2}^{t} H(t,s)\alpha(s)Q(s)\left[k\frac{\delta_*(s)}{s}\right]^{\beta}\Delta s \leq -\int_{t_2}^{t} H(t,s)w^{\Delta}(s)\Delta s + \int_{t_2}^{t} H(t,s)\frac{\alpha^{\Delta}(s)}{\alpha^{\sigma}(s)}w^{\sigma}(s)\Delta s$$
$$-\int_{t_2}^{t} H(t,s)\frac{\beta\alpha(s)M^{(\gamma-1)/\gamma}}{K(s)}w^2(\sigma(s))\Delta s. \tag{3.19}$$

An integration by parts yields

$$\int_{t_2}^{t} H(t,s)w^{\Delta}(s)\Delta = H(t,s)w(s)|_{t_2}^{t} - \int_{t_2}^{t} H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s$$
$$= -H(t,t_2)w(t_2) - \int_{t_2}^{t} H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s. \tag{3.20}$$

Now, in view of (3.16), (3.19) and (3.20), we see that

$$\int_{t_2}^{t} H(t,s)\alpha(s)Q(s)\left[k\frac{\delta_*(s)}{s}\right]^{\beta}\Delta s \leq H(t,t_2)w(t_2)$$
$$+\int_{t_2}^{t}\left[H^{\Delta_s}(t,s) + H(t,s)\frac{\alpha^{\Delta}(s)}{\alpha^{\sigma}(s)}\right]w^{\sigma}(s)\Delta s - \int_{t_2}^{t} H(t,s)\frac{\beta\alpha(s)M^{(\gamma-1)/\gamma}}{K(s)}w^2(\sigma(s))\Delta s$$
$$= H(t,t_2)w(t_2) - \int_{t_2}^{t}\frac{h(t,s)}{\alpha^{\sigma}(s)}H^{1/2}(t,s)w^{\sigma}(s)\Delta s - \int_{t_2}^{t} H(t,s)\frac{\beta\alpha(s)M^{(\gamma-1)/\gamma}}{K(s)}w^2(\sigma(s))\Delta s$$
$$\leq H(t,t_2)w(t_2) + \int_{t_2}^{t}\frac{h_-(t,s)}{\alpha^{\sigma}(s)}H^{1/2}(t,s)w^{\sigma}(s)\Delta s - \int_{t_2}^{t} H(t,s)\frac{\beta\alpha(s)M^{(\gamma-1)/\gamma}}{K(s)}w^2(\sigma(s))\Delta s$$
$$= H(t,t_2)w(t_2) + \int_{t_2}^{t}\frac{K(s)\left(h_-(t,s)\right)^2}{4\beta\alpha(s)M^{(\gamma-1)/\gamma}\alpha^2(\sigma(s))}\Delta s$$
$$-\int_{t_2}^{t}\left[\sqrt{H(t,s)\frac{\beta\alpha(s)M^{(\gamma-1)/\gamma}}{K(s)}}w(\sigma(s)) - \frac{1}{\alpha^{\sigma}(s)}\sqrt{\frac{K(s)}{4\beta\alpha(s)M^{(\gamma-1)/\gamma}}}h_-(t,s)\right]^2\Delta s,$$

so

$$\int_{t_2}^{t}\left[H(t,s)\alpha(s)Q(s)\left[k\frac{\delta_*(s)}{s}\right]^{\beta} - \frac{K(s)\left(h_-(t,s)\right)^2}{4\beta\alpha(s)M^{(\gamma-1)/\gamma}\alpha^2(\sigma(s))}\right]\Delta s \leq H(t,t_2)w(t_2)$$

or

$$\frac{1}{H(t,t_2)}\int_{t_2}^{t}\left[H(t,s)\alpha(s)Q(s)\left[k\frac{\tau_*(s)}{s}\right]^{\beta} - \frac{K(s)\left(h_-(t,s)\right)^2}{4\beta\alpha(s)M^{(\gamma-1)/\gamma}\alpha^2(\sigma(s))}\right]\Delta s \leq w(t_2).$$

Taking $\limsup$ as $t \to \infty$ of both sides yields a contradiction to condition (3.17). This completes the proof of Theorem 3.2. □

Theorem 3.3. *Assume that* (1.4) *holds. Suppose also that there exist positive rd-continuous* Δ*-differentiable functions* $\alpha(t)$ *and* $\phi(t)$ *such that* (3.2) *holds, and for every constant* $d \geq t_0$

$$\int_d^\infty \frac{1}{r^{1/\gamma}(s)} \left(\int_d^s Q(u)\Delta u \right)^{1/\gamma} \Delta s = \infty, \tag{3.21}$$

where $Q(s)$ *is as in Theorem* 3.1*. Then every solution* $x(t)$ *of equation* (1.1) *is either oscillatory or tends to zero as* $t \to \infty$*.*

Proof. Suppose to the contrary that equation (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (1.1). Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0, x(\tau(t)) > 0$ and $x(\theta(t,\xi)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$ and $\xi \in [a,b]_{\mathbb{T}}$. Proceeding as in the proof of Theorem 3.1, we obtain

$$\left(r(t) \left| y^\Delta(t) \right|^{\gamma-1} y^\Delta(t) \right)^\Delta + Q(t) y^\beta(\delta(t)) \leq 0, \quad \text{for } t \geq t_2 \tag{3.22}$$

Then $r(t) \left| y^\Delta(t) \right|^{\gamma-1} y^\Delta(t)$ is decreasing on $[t_{2,}\infty)_{\mathbb{T}}$ and is eventually of one sign. Therefore, $y^\Delta(t)$ is eventually of one sign, i.e., there are the following cases for the sign of $y^\Delta(t)$:

Case (I). $y^\Delta(t)$ is eventually positive;

Case (II). $y^\Delta(t)$ is eventually negative.

The proof of Case (I) is similar to that of Theorem 3.1 and hence is omitted. We next assume that Case (II) holds. In this case, there exists $t_2 \geq t_1$ such that $y^\Delta(t) < 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$. This and the fact that $y(t) > 0$ imply

$$\lim_{t \to \infty} y(t) = \eta \geq 0.$$

We assert that $\eta = 0$. If not, then $y(t) \geq \eta > 0, y(\tau(t)) \geq \eta > 0$ and $y(\theta(t,\xi)) \geq \eta > 0$ for all $t \in [t_2, \infty)_{\mathbb{T}}$ and $\xi \in [a,b]_{\mathbb{T}}$. Now, by (3.22), we have

$$\left(r(t) \left| y^\Delta(t) \right|^{\gamma-1} y^\Delta(t) \right)^\Delta + Q(t)\eta^\beta \leq 0 \qquad \text{for } t \geq t_2. \tag{3.23}$$

Since $y^\Delta(t)$ is eventually negative, (3.23) gives, for $t \geq t_2$,

$$-\left(r(t) \left(-y^\Delta(t) \right)^\gamma \right)^\Delta \leq -Q(t)\eta^\beta. \tag{3.24}$$

If we integrate (3.24) from t_2 to t to obtain

$$y^\Delta(t) \leq -\frac{\eta^{\beta/\gamma}}{r^{1/\gamma}(t)} \left(\int_{t_2}^t Q(s)\Delta s \right)^{1/\gamma},$$

and we integrate again from t_2 to t, we have

$$y(t) \leq y(t_2) - \eta^{\beta/\gamma} \int_{t_2}^t \frac{1}{r^{1/\gamma}(s)} \left(\int_{t_2}^s Q(u)\Delta u \right)^{1/\gamma} \Delta s.$$

This implies by (3.21) that $y(t)$ is eventually negative, which contradicts the fact that $y(t) \geq x(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Hence we conclude that $\lim_{t\to\infty} y(t) = 0$. Since $y(t) = x(t) + p(t)x(\tau(t))$, then $0 < x(t) \leq y(t)$. This implies that $x(t) \to 0$ as $t \to \infty$. The proof of Theorem 3.3 is complete. □

Theorem 3.4. *Assume that* (1.4) *holds. Let* $\alpha(t)$, $h(t,s)$ *and* $H(t,s)$ *be defined as in Theorem* 3.2 *such that* (3.16) *and* (3.17) *hold. Furthermore, assume that for every constant* $d \geq t_0$ (3.21) *holds. Then every solution of equation* (1.1) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. The proof is similar to that of the proof of Theorem 3.3 and therefore is omitted. □

Example 3.5. Consider the nonlinear neutral dynamic equation
(3.25)

$$\left(\left(t^3 - 3t + 3\right)\left(\left(x(t) + \left(1 - \frac{1}{t}\right)x(t-1)\right)^{\Delta}\right)^3\right)^{\Delta} + \int_0^1 \left(\frac{t^2}{k}\right)^{\beta} x^{\beta}(t-\xi)\Delta\xi = 0,$$

for $t \in [1, \infty)_{\mathbb{T}}$, where $\gamma = 3$, $0 < \beta < 1$ is the quotient of odd positive integers, $r(t) = t^3 - 3t + 3$, $p(t) = 1 - \frac{1}{t}$, $\theta(t,\xi) = t - \xi$, $\delta(t) = t - 1$, $\delta_*(t) = t - 1$, $\alpha(t) = 1$, $\phi(t) = 1$ and $q(t) = \left(\frac{t^2}{k}\right)^{\beta}$. Now,

$$\begin{aligned}
\int_{t_0}^{\infty} \frac{\Delta t}{r^{1/\gamma}(t)} &= \int_{t_0}^{\infty} \frac{\Delta t}{\left(t^3 - 3t + 3\right)^{1/3}} > \int_{t_0}^{\infty} \frac{\Delta t}{\left(t^3 + 3\right)^{1/3}} \\
&\geq \int_{t_0}^{\infty} \frac{\Delta t}{\left(2t^3 + 2\right)^{1/3}} \geq \int_{t_0}^{\infty} \frac{\Delta t}{2^{1/3}\left(2t^3\right)^{1/3}} \\
&= \int_{t_0}^{\infty} \frac{\Delta t}{2^{2/3} t} = \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{t\to\infty} \int_{t_0}^{t} \left(\alpha(s)\phi(s)Q(s)\left[k\frac{\delta_*(s)}{s}\right]^{\beta} - \frac{K(s)C^2(s)}{4\phi(s)\beta\alpha(s)M^{(\gamma-1)/\gamma}}\right)\Delta s \\
&= \limsup_{t\to\infty} \int_{t_0}^{t} (b-a)\left(\frac{s^2}{k}\right)^{\beta} (1 - p(s-a))^{\beta} \left[k\frac{s-b}{s}\right]^{\beta} \Delta s \\
&= \limsup_{t\to\infty} \int_{t_0}^{t} \left(\frac{s^2}{k}\right)^{\beta} (1 - \frac{s-1}{s})^{\beta} \left[k\frac{s-1}{s}\right]^{\beta} \Delta s \\
&= \limsup_{t\to\infty} \int_{t_0}^{t} (s-1)^{\beta}\Delta s = \infty.
\end{aligned}$$

Therefore, by Theorem 3.1, equation (3.25) is oscillatory.

Example 3.6. Consider the dynamic equation

(3.26)

$$\left(t\left(x(t)+\frac{t}{t+1}x(t-1)\right)^{\Delta}\right)^{\Delta}+\int_1^2\sqrt[5]{3t}\left(1+\frac{5(L\sigma(t))^{4/5}}{t}\right)x^{1/5}(3t-\xi)\Delta\xi=0,$$

for $t\in[1,\infty)_{\mathbb{T}}$, where $\gamma=1$, $\beta=1/5$, $r(t)=t$, $p(t)=\frac{t}{t+1}$, $\theta(t,\xi)=3t-\xi$, $\delta(t)=3t-2$, $\delta_*(t)=t$, $\alpha(t)=t$, $\phi(t)=1$ and $q(t)=\sqrt[5]{3t}\left(1+\frac{5(L\sigma(t))^{4/5}}{t}\right)$. Let $k=\frac{1}{4^5}$. Now,

$$\int_{t_0}^{\infty}\frac{\Delta t}{r^{1/\gamma}(t)}=\int_{t_0}^{\infty}\frac{\Delta t}{t}=\infty$$

and

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^{t}\left(\alpha(s)\phi(s)Q(s)\left[k\frac{\delta_*(s)}{s}\right]^{\beta}-\frac{K(s)C^2(s)}{4\phi(s)\beta\alpha(s)M^{(\gamma-1)/\gamma}}\right)\Delta s\\&=\limsup_{t\to\infty}\int_{t_0}^{t}\left(s\sqrt[5]{3s}\left(1+\frac{5(L\sigma(s))^{4/5}}{s}\right)(1-p(3s-1))^{1/5}(\frac{1}{4^5})^{1/5}-\frac{5(L\sigma(s))^{4/5}}{4}\right)\Delta s\\&=\limsup_{t\to\infty}\int_{t_0}^{t}\left(s\frac{1}{4}\sqrt[5]{3s}\left(1+\frac{5(L\sigma(s))^{4/5}}{s}\right)(\frac{1}{3s})^{1/5}-\frac{5(L\sigma(s))^{4/5}}{4}\right)\Delta s\\&=\limsup_{t\to\infty}\int_{t_0}^{t}\left(\frac{1}{4}\left(s+5(L\sigma(s))^{4/5}\right)-\frac{5(L\sigma(s))^{4/5}}{4}\right)\Delta s=\limsup_{t\to\infty}\int_{t_0}^{t}\frac{s}{4}\Delta s=\infty.\end{aligned}$$

Therefore, by Theorem 3.1, equation (3.26) is oscillatory.

Example 3.7. Consider the equation

(3.27)

$$\left(t^2\left(x(t)+\frac{t-1}{t+2}x(t-1)\right)^{\Delta}\right)^{\Delta}+\int_1^2(t+3)^{1/3}(t+\sigma(t))\,x^{1/3}(t+\frac{1}{\xi})\Delta\xi=0,\quad t\in[1,\infty),$$

where $\gamma=1$, $\beta=1/3$, $r(t)=t^2$, $p(t)=\frac{t-1}{t+2}$, $\theta(t,\xi)=t+\frac{1}{\xi}$, $\delta(t)=t+\frac{1}{2}$, $\delta_*(t)=t$, $\alpha(t)=1$, $\phi(t)=1$ and $q(t)=(t+3)^{1/3}(t+\sigma(t))$. Let $k=1/3$. Now,

$$\int_{t_0}^{\infty}\frac{\Delta t}{r^{1/\gamma}(t)}=\int_{t_0}^{\infty}\frac{\Delta t}{t^2}<\infty$$

and

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_0}^{t}\left(\alpha(s)\phi(s)Q(s)\left[k\frac{\delta_*(s)}{s}\right]^{\beta}-\frac{K(s)C^2(s)}{4\phi(s)\beta\alpha(s)M^{(\gamma-1)/\gamma}}\right)\Delta s\\&=\limsup_{t\to\infty}\int_{t_0}^{t}(s+3)^{1/3}(s+\sigma(s))\left(\frac{3}{s+3}\right)^{1/3}\frac{1}{\sqrt[3]{3}}\Delta s\\&=\limsup_{t\to\infty}\int_{t_0}^{t}(s+\sigma(s))\,\Delta s=\limsup_{t\to\infty}\int_{t_0}^{t}\left(s^2\right)^{\Delta}\Delta s\\&=\limsup_{t\to\infty}s^2\mid_{t_0}^{t}=\limsup_{t\to\infty}(t^2-t_0^2)=\infty.\end{aligned}$$

Thus, (1.4) and (3.2) hold. Moreover, for every constant $d \geq t_0$, we can find $0 < A < 1$ and $t_A \geq d$ such that $t - d \geq At$ for $t \in [t_A, \infty)$. Now, we get

$$\begin{aligned}
&\int_d^\infty \frac{1}{r^{1/\gamma}(s)} \left(\int_d^s Q(u)\Delta u \right)^{1/\gamma} \Delta s \\
&= \int_d^\infty \frac{1}{s^2} \int_d^s (u+3)^{1/3} (u+\sigma(u)) (1 - p(u+1))^{1/3} \Delta u \Delta s \\
&= \int_d^\infty \frac{1}{s^2} \int_d^s (u+3)^{1/3} (u+\sigma(u)) (1 - \frac{u}{u+3})^{1/3} \Delta u \Delta s \\
&= \int_d^\infty \frac{1}{s^2} \int_d^s (u+3)^{1/3} (u+\sigma(u)) \frac{\sqrt[3]{3}}{(u+3)^{1/3}} \Delta u \Delta s \\
&= \int_d^\infty \frac{\sqrt[3]{3}(s^2 - d^2)}{s^2} \Delta s = \int_d^\infty \frac{\sqrt[3]{3}(s-d)(s+d)}{s^2} \Delta s \\
&\geq 2\sqrt[3]{3} dA \int_{t_A}^\infty \frac{1}{s} \Delta s = \infty
\end{aligned}$$

which implies that (3.21) holds. Hence Eq. (3.27) is oscillatory by Theorem 3.3.

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