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A NOTE ON THE COEXISTENCE OF HOMOCLINIC ORBIT AND SADDLE FOCUS POINT FOR SOME TYPICAL CHAOTIC SYSTEMS

LINGLI XIE

Department of Mathematics, Sun Yat-sen University
Guangzhou 510275, P.R. China

ABSTRACT. The coexistence of homoclinic orbit and saddle focus point is the basic assumption in Shil'nikov homoclinic theorem. We attempt to study the existence of homoclinic orbit to saddle focus point and give the necessary conditions for it. Firstly, the geometrical properties of homoclinic orbit to saddle focus point are exposed by some lemmas which are used to drive the main theorem. Consequently, the necessary conditions for the existence of homoclinic orbit to saddle focus point are obtained. The result is applied to Lorenz-type systems. Finally, the conclusions for some typical chaotic systems is presented.

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1. Introduction

Many typical chaotic systems, such as Lorenz system and Rössler system, Chua electronics circuits and Duffing oscillator, Chen system and so on, are successively discovered in past years. But as well known, it is not easy to give out rigorous proof for these systems to be chaotic. In order to apply Shil'nikov homoclinic theorem [1, 2, 3, 4, 5] to proving the existence of chaos, the key point is to show the existence of homoclinic orbit to a saddle focus point of the system.

For this purpose, some of the contributions adopt qualitative analysis including analytical method [6, 7, 8, 9] to look for the homoclinic orbit of the third-order continuous system. More of the contributions construct homoclinic orbits by series approximation method [10, 11, 12, 13]. All these method are based on the fact that the homoclinic solution is the joint of the stable and unstable manifolds for the considered dynamical system. Once the stable and unstable manifolds have intersections, the third-order continuous system can not be structurally stable [14]. However, the proof of the uniform convergence of series expansions of the homoclinic orbit is obtained based on the assumptions of structurally stable in an open set of parameters space. In addition, Algaba A. et al. [15] show another problem in the series approximation

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when proving the existence of homoclinic orbit for some chaotic third-order systems. The approximate series expression for homoclinic orbit is wrongly supposed to be symmetry and homoclinic orbit is necessarily non-symmetry. Usually, it spirals outward along the two-dimensional unstable manifold and approaches the equilibrium along its one-dimensional stable manifold. It should be noted that the homoclinic solution expressed by series expressions are used in many contributions, which is at question.

Even so, the coexistence of homoclinic orbit and saddle focus point has not been found for a third-order continuous system with C^2 vector field. For example, the famous Lorenz system is considered as follows

$$\begin{aligned}\dot{x} &= a(y-x)\\ \dot{y} &= cx - y - xz\\ \dot{z} &= xy - bz\end{aligned}$$

with $a = 10, b = 8/3$ and $c = 13.9265$. Li and Zhu in [13] provide the series expression of the homoclinic orbit to the origin $(0,0,0)^T$. It is easy to calculate that the origin is not a saddle focus point of the Lorenz system with above parameters. [16] shows the general existence conditions of homoclinic trajectories to saddle node, which is different from to saddle focus point.

To my knowledge, no example of a single third-order continuous time autonomous system with C^2 vector field is given out to show the existence of exact homoclinic orbit to saddle focus point. The author also notice that Chua's circuit are reported to be chaotic in Shil'nikov sense in [5] in 1993 and it's form is:

$$\begin{aligned}\dot{x} &= \alpha(y - x - k(x))\\ \dot{y} &= x - y + z\\ \dot{z} &= -\beta y\end{aligned} \tag{1.1}$$

where

$$k(x) = bx + \frac{1}{2}(a-b)(|x+1| - |x-1|).$$

In [17], the author gives a detailed homoclinic orbit bifurcation analysis including numerical simulation of a homoclinic orbit when $\alpha = 23.64051$, $\beta = 52, a = -8/7$ and $b = -5/7$. And with these parameters, the eigenvalues of the Jacobi matrix at equilibrium point of the system show that the origin point is saddle focus point. However, the vector field of the system (1.1) is piecewise-linear. Actually, it is a switch linear system, not a single system with C^2 vector field. That is, there is an piecewise-linear system having homoclinic orbits to its saddle focus points. Above all, it is important to study the coexistence of homoclinic orbit and saddle focus point.

The paper is organized as follows. In Section 2, the preliminary knowledge is introduced. In Section 3, some lemmas are given to expose the geometrical properties of homoclinic orbit to saddle focus point and it is also useful in the proof of main theorem. In Section 4, the main theorem is proved. Consequently the necessary condition of existing homoclinic orbit to saddle fucus point is obtained. The results are applied to Lorenz-type systems. Finally, the conclusion is presented in Section 5.

2. **PRELIMINARIES**

Consider a third-order autonomous system

$$\dot{x}(t) = f(x(t)),\ t \in \mathbb{R},\ x(t) \in \mathbb{R}^3 \tag{2.1}$$

where the vector field $f : \mathbb{R}^3 \to \mathbb{R}^3$, belongs to class $C^r (r \geqslant 2)$.

Let $x_e \in \mathbb{R}^3$ be the equilibrium of system (2.1) and $Df|_{x_e}$ be the Jacobi matrix of system (2.1) at x_e.

Definition 2.1. x_e is called a saddle foci point if $Df|_{x_e}$ possesses three eigenvalues with the following form

$$\nu,\ \sigma \pm i\omega$$

where ν, σ and ω all are real numbers and

$$\sigma\nu < 0,\ \omega \neq 0 \tag{2.2}$$

The Shil'nikov homoclinic criterion for chaos to be existed is described by the following lemma (referred to [5]).

Lemma 2.2 (Shil'nikov homoclinic theorem)**.** *Suppose that there is a saddle foci point x_e of system* (2.1)*, and $Df|_{x_e}$ possesses three eigenvalues as in Definition* 1 *satisfying the following Shil'nikov inequality*

$$|\nu| > |\sigma| > 0. \tag{2.3}$$

Suppose also that there exists a homoclinic orbit based at x_e. Then

(1) *The Shil'nikov map, defined in a neighborhood of the homoclinic orbit, has a countable number of Smale horseshoes in its discrete dynamics;*

(2) *For any sufficiently small C^1 perturbation g of f, the perturbed system*

$$\dot{x}(t) = g(x(t)),\ x(t) \in \mathbb{R}^3 \tag{2.4}$$

has at least a finite number of Smale horseshoes in the discrete dynamics of Shil'nikov map defined near the homoclinic orbit;

(3) *Both the original system* (2.1) *and perturbed system* (2.4) *have horseshoes type of chaos.*

No loss of generality, one can assume $\lambda < 0$ and $\rho > 0$. The system (2.1) is described by

$$\begin{aligned} \dot{x}_1(t) &= \lambda x_1 + f_1(x_1, x_2, x_3) \\ \dot{x}_2(t) &= \rho x_2 - \omega x_3 + f_2(x_1, x_2, x_3) \\ \dot{x}_3(t) &= \omega x_2 + \rho x_3 + f_3(x_1, x_2, x_3) \end{aligned} \tag{2.5}$$

with the equilibrium point $E = (0, 0, 0)^T$, where $f_i(0, 0, 0) = 0 (i = 1, 2, 3)$. Any solution $X(t) = (x_1(t), x_2(t), x_3(t))^T$ with initial value $X^0 = (x_1(t_0), x_2(t_0), x_3(t_0))^T$ of the system (2.5) can be formulated as the following

$$\begin{aligned} x_1(t) &= \exp(\lambda(t - t_0))x_1(t_0) + \int_{t_0}^{t} \exp(\lambda(t - s))f_1(x_1(s), x_2(s), x_3(s))ds \\ x_2(t) &= \exp(\rho(t - t_0))x_2(t_0) \cos\omega(t - t_0) - \exp(\rho(t - t_0))x_3(t_0) \sin\omega(t - t_0) \\ &\quad + \int_{t_0}^{t} \exp(\rho(t - s))[\cos\omega(t - s)f_2(x_1(s), x_2(s), x_3(s))]ds \\ &\quad - \int_{t_0}^{t} \exp(\rho(t - s))[\sin\omega(t - s)f_3(x_1(s), x_2(s), x_3(s))]ds \\ x_3(t) &= \exp(\rho(t - t_0))x_2(t_0) \sin\omega(t - t_0) + \exp(\rho(t - t_0))x_3(t_0) \cos\omega(t - t_0) \\ &\quad + \int_{t_0}^{t} \exp(\rho(t - s))[\sin\omega(t - s)f_2(x_1(s), x_2(s), x_3(s))]ds \\ &\quad + \int_{t_0}^{t} \exp(\rho(t - s))[\cos\omega(t - s)f_3(x_1(s), x_2(s), x_3(s))]ds \end{aligned} \tag{2.6}$$

Let $(h_1(t), h_2(t), h_3(t))^T$ be a homoclinic orbit of the system (2.5). We have

$$h_i(t) \to 0 \quad \text{as } t \to \pm\infty, \ i = 1, 2, 3 \tag{2.7}$$

and

$$|h_i(t)| \leq M \ \forall t \in (-\infty, +\infty) \quad i = 1, 2, 3 \tag{2.8}$$

for some constant M, and $x_i(t) = h_i(t)$ $(i = 1, 2, 3)$ satisfies (2.6).

Let $g_i (i = 1, 2, 3)$ be notated by

$$\begin{aligned} g_1(t) &= \int_{t_0}^{t} \exp(\lambda(t - s))f_1(x_1(s), x_2(s), x_3(s))ds \\ g_2(t) &= \int_{t_0}^{t} \exp(\rho(t - s))[\cos\omega(t - s)f_2(x_1(s), x_2(s), x_3(s))]ds \\ &\quad - \int_{t_0}^{t} \exp(\rho(t - s))[\sin\omega(t - s)f_3(x_1(s), x_2(s), x_3(s))]ds \\ g_3(t) &= \int_{t_0}^{t} \exp(\rho(t - s))[\sin\omega(t - s)f_2(x_1(s), x_2(s), x_3(s))]ds \\ &\quad + \int_{t_0}^{t} \exp(\rho(t - s))[\cos\omega(t - s)f_3(x_1(s), x_2(s), x_3(s))]ds. \end{aligned} \tag{2.9}$$

3. The geometrical properties of homoclinic orbit to saddle focus point for system (2.5)

In order to obtain some properties of the solutions of the system (2.5) in the ε-neighbourhood of E, we consider the linearized system of (2.5) at E as follows:

$$\begin{aligned} \dot{y}_1(t) &= \lambda y_1 \\ \dot{y}_2(t) &= \rho y_2 - \omega y_3 \\ \dot{y}_3(t) &= \omega y_2 + \rho y_3 \end{aligned} \tag{3.1}$$

Naturally one has

$$\begin{aligned} y_1(t) &= \exp(\lambda(t-t_0))y_1(t_0) \\ y_2(t) &= \exp(\rho(t-t_0))y_2(t_0)\cos\omega(t-t_0) - \exp(\rho(t-t_0))y_3(t_0)\sin\omega(t-t_0) \\ y_3(t) &= \exp(\rho(t-t_0))y_2(t_0)\sin\omega(t-t_0) + \exp(\rho(t-t_0))y_3(t_0)\cos\omega(t-t_0) \end{aligned} \tag{3.2}$$

by (2.6).

Therefore

$$y_1(t)y_1(t_0) > 0, \quad |y_1(t)| = \exp(\lambda(t-t_0))|y_1(t_0)|. \tag{3.3}$$

In addition it follows from (3.2) that

$$y_2(t_0)y_2(t) + y_3(t_0)y_3(t) = \exp(\rho(t-t_0))(y_2^2(t_0) + y_3^2(t_0))\cos\omega(t-t_0), \tag{3.4}$$

$$y_2(t_0)y_3(t) - y_3(t_0)y_2(t) = \exp(\rho(t-t_0))(y_2^2(t_0) + y_3^2(t_0))\sin\omega(t-t_0). \tag{3.5}$$

Consequently, we have

$$(3.4)^2 + (3.5)^2 = (y_2^2(t_0) + y_3^2(t_0))(y_2^2(t) + y_3^2(t)) = \exp(2\rho(t-t_0))(y_2^2(t_0) + y_3^2(t_0))^2$$

so that

$$y_2^2(t) + y_3^2(t) = \exp(2\rho(t-t_0))(y_2^2(t_0) + y_3^2(t_0)). \tag{3.6}$$

By the assumption that E is a saddle focus point, system (2.5) is topologically equivalent to the system (3.1) in the ε-neighbourhood of E as ε is small enough. Therefore we obtain the following lemma by (3.3) and (3.6).

Lemma 3.1. *For any solution $x(t) = (x_1(t), x_2(t), x_3(t))^T$ of the system (2.5) in the ε-neighbourhood of E where ε is small enough, the followings hold*

(1) $$x_1(t)x_1(t_0) > 0,\ |x_1(t)| = O(\exp(\lambda(t-t_0))|x_1(t_0)|)$$

and

$$\begin{aligned} |x_1(t)| &\le |x_1(t_0)| \quad \text{for } t \ge t_0 \\ |x_1(t)| &\ge |x_1(t_0)| \quad \text{for } t \le t_0; \end{aligned}$$

(2) $$|x_2^2(t)+x_3^2(t)| = O(\exp(2\rho(t-t_0))|x_2^2(t_0)+x_3^2(t_0)|)$$

and

$$x_2^2(t)+x_3^2(t) \geq x_2^2(t_0)+x_3^2(t_0) \quad \text{for } t \geq t_0$$
$$x_2^2(t)+x_3^2(t) \leq x_2^2(t_0)+x_3^2(t_0) \quad \text{for } t \leq t_0.$$

In the following discussion notated a homoclinic orbit of the system (2.5) by $h(t) = (h_1(t), h_2(t), h_3(t))^T$, and let ε be small enough and $T(\varepsilon)$ be such that

(3.7) $$|h_i(t)| \leq \varepsilon, \ \forall \ |t| \geq T(\varepsilon).$$

Lemma 3.2. *If $h(t)$ is a homoclinic orbit to E for the system* (2.5), *then*

(1) *it will cross the plane $x_1 = 0$ as $t \to -\infty$ and then stay on it;*

(2) *it will cross the x_1-axis as $t \to +\infty$ and then stay on it.*

Proof. It would be noted that for $t_0 < -T(\varepsilon)$ and $t < t_0$, $h(t)$ is in the ε-neighbourhood of E. For $h_1(t_0) \neq 0$, one can see that on the one hand $h_1(t) \to 0$ as $t \to -\infty$ by (2.7), on the other hand $|h_1(t)| \geq |h_1(t_0)|$ as $t < t_0$ based on Lemma 3.1. Hence it must be held that $h_1(t_0) = 0$. This implies that $h(t)$ cross the plane $x_1 = 0$ as $t \to -\infty$.

We assert that $h_1(t) = 0$ for $t < t_0$. Otherwise there exists $\bar{t}_0 < t_0$ such that $h_1(\bar{t}_0) \neq 0$. Using the same argument in proving $h_1(t_0) = 0$, we can also obtain $h_1(\bar{t}_0) = 0$. Consequently, the conclusion (1) in this lemma is valid.

Similarly, for $t_0 > T(\varepsilon)$ and $t > t_0$, $h(t)$ is also in the ε-neighbourhood of E. For $h_2^2(t_0) + h_3^2(t_0) \neq 0$, according to Lemma 3.1,

$$h_2^2(t) + h_3^2(t) \geq h_2^2(t_0) + h_3^2(t_0) \quad \text{as } t > t_0.$$

Meanwhile $h_2^2(t) + h_3^2(t) \to 0$ as $t \to +\infty$ by (2.7). Consequently, it must be held that $h_2(t_0) = h_3(t_0) = 0$. This implies that $h(t)$ cross the x_1-axis as $t \to +\infty$.

We can also assert that $h_2(t) = h_3(t) = 0$ for $t > t_0$. Otherwise there exists $\bar{t}_0 > t_0$ such that $h_2^2(\bar{t}_0) + h_3^2(\bar{t}_0) \neq 0$. Using the same argument in proving $h_2^2(t_0) + h_3^2(t_0) = 0$, we can obtain $h_2^2(\bar{t}_0) + h_3^2(\bar{t}_0) = 0$. So, the proof of this lemma is finished. □

Based on Lemma 3.2, it is easy to obtain the following

Corollary 3.3. *If $h(t)$ is a homoclinic orbit to E for the system* (2.5) *then it will not cross plane $x_1 = 0$ as $t \to +\infty$ and the x_1-axis as $t \to -\infty$.*

4. The main theorem and its applications

Theorem 4.1. *Assume that*

(4.1) $$f_i(x_1, x_2, x_3) = f_i^{(1)}(x_1) f_i^{(2)}(x_2, x_3), \quad i = 2.3$$

where

(4.2) $$f_i^{(1)}(0) = 0, \quad i = 2, 3$$

Then no homoclinic orbit to E exists for the system (2.5).

Proof. Suppose that the conclusion of the theorem 1 is not true, one can assume that there exists a homoclinic orbit $h(t)$ to E for the system (2.5). Noting that (2.8) holds and $f_i \in C^2$ $(i = 1, 2, 3)$ we can assume

$$|f_i(h_1(t), h_2(t), h_3(t))| \leq M_1 \quad i = 1, 2, 3. \tag{4.3}$$

where M_1 is a constant and $t \in (-\infty, +\infty)$.

According to Lemma 3.2, we can assume that $h_1(\bar{t}_0) = 0$ here $\bar{t}_0 < -T(\varepsilon)$ is fixed. Let $t_0 < \bar{t}_0$ satisfying

$$t_0 + \delta < \bar{t}_0 \tag{4.4}$$

where $\delta = \frac{(2\bar{K}+1)\pi}{\omega}$, and $\bar{K}$ is some positive integer satisfying

$$\exp\left(\rho\left(\frac{(2\bar{K}+1)\pi}{\omega}\right)\right) - \frac{8MM_1}{\rho} > 0. \tag{4.5}$$

Set $q(t) = h_2(t_0)h_2(t) + h_3(t_0)h_3(t)$. It is obviously that

$$q(t) \to 0 \quad \text{as } t \to \pm\infty \tag{4.6}$$

by (2.7). And we have $q(t) = q_1(t) + q_2(t)$ where

$$\begin{aligned} q_1(t) &= \exp(\rho(t - t_0))(h_2^2(t_0) + h_3^2(t_0)) \cos\omega(t - t_0) \\ q_2(t) &= h_2(t_0)g_2(t) + h_3(t_0)g_3(t) \end{aligned} \tag{4.7}$$

with $h_i(t) = x_i(t)$, $(i = 1, 2, 3)$.

Let k be a positive integer and t_k be

$$t_k - t_0 = 2k\pi/\omega. \tag{4.8}$$

In such case, we have

$$q_1(t_k) = \exp(\rho(2k\pi/\omega))(h_2^2(t_0) + h_3^2(t_0)). \tag{4.9}$$

Suppose that $h_2^2(t_0) + h_3^2(t_0) \neq 0$. One can see that

$$\begin{aligned} q_2(t_k) = {} & h_2(t_0)\int_{t_0}^{t_k} \exp(\rho(t_k - s))[\cos\omega(t_0 - s)f_2 + \sin\omega(t_0 - s)f_3]ds \\ & + h_3(t_0)\int_{t_0}^{t_k} \exp(\rho(t_k - s))[\cos\omega(t_0 - s)f_3 - \sin\omega(t_0 - s)f_2]ds, \end{aligned} \tag{4.10}$$

where $f_2 = f_2(h_1(s), h_2(s), h_3(s))$ and $f_3 = f_3(h_1(s), h_2(s), h_3(s))$.

Let $\bar{t}_k = t_k + \delta$. It can be deduced that

$$q_1(\bar{t}_k) = -\exp(\rho(\bar{t}_k - t_0))(h_2^2(t_0) + h_3^2(t_0)), \tag{4.11}$$

and

$$q_2(\bar{t}_k) = -h_2(t_0)\int_{t_0}^{\bar{t}_k} \exp(\rho(\bar{t}_k - s))[\cos\omega(t_0 - s)f_2 + \sin\omega(t_0 - s)f_3]ds \\ - h_3(t_0)\int_{t_0}^{\bar{t}_k} \exp(\rho(\bar{t}_k - s))[\cos\omega(t_0 - s)f_3 - \sin\omega(t_0 - s)f_2]ds. \tag{4.12}$$

Set $s = \tau + \delta$ in (4.12) one has

$$q_2(\bar{t}_k) = h_2(t_0)\int_{t_0-\delta}^{t_k} \exp(\rho(t_k - \tau))[\cos\omega(t_0 - \tau)f_2) + \sin\omega(t_0 - \tau)f_3]d\tau \\ + h_3(t_0)\int_{t_0-\delta}^{t_k} \exp(\rho(t_k - \tau))[\cos\omega(t_0 - \tau)f_3) - \sin\omega(t_0 - \tau)f_2]d\tau. \tag{4.13}$$

According to (4.6),

$$q(t_k) - q(\bar{t}_k) \to 0 \quad \text{as } k \to +\infty. \tag{4.14}$$

This implies that

$$\lim_{K\to+\infty}[q_1(t_k) - q_1(\bar{t}_k) - (q_2(\bar{t}_k) - q_2(t_k))] = 0. \tag{4.15}$$

It follows from (4.9) and (4.11) that

$$q_1(t_k) - q_1(\bar{t}_k) = \exp(\rho(t_k - t_0))(\exp(\delta) + 1)(h_2^2(t_0) + h_3^2(t_0)). \tag{4.16}$$

Based on (4.10) and (4.13) we can estimate for $q_2(\bar{t}_k) - q_2(t_k)$ that

$$|q_2(\bar{t}_k) - q_2(t_k)| \le \frac{8M_1M}{\rho}\exp(\rho(t_k - t_0)) + |G(t_k)| \tag{4.17}$$

by (2.8) and (4.3) where

$$G(t_k) = h_2(t_0)\int_{t_0-\delta}^{t_0} \exp(\rho(t_k - \tau))[\cos\omega(t_0 - \tau)\bar{f}_2 + \sin\omega(t_0 - \tau)\bar{f}_3]d\tau \\ + h_3(t_0)\int_{t_0-\delta}^{t_0} \exp(\rho(t_k - \tau))[\cos\omega(t_0 - \tau)\bar{f}_3 - \sin\omega(t_0 - \tau)\bar{f}_2]d\tau,$$

here $\bar{f}_2 = f_2(h_1(\bar{\tau}), h_2(\bar{\tau}), h_3(\bar{\tau}))$, $\bar{f}_3 = f_3(h_1(\bar{\tau}), h_2(\bar{\tau}), h_3(\bar{\tau}))$ and $\bar{\tau} = \tau + \delta$.

One can see that $\bar{\tau} \in (t_0, t_0 + \delta)$, it implies that $h_1(\bar{\tau}) = 0$ by (4.4) as Lemma 3.2 is employed. Hence $G(t_k) = 0$ due to (4.1) and (4.2).

Now we have

$$(4.16) - (4.17) \ge \exp(\rho(t_k - t_0))(h_2^2(t_0) + h_3^2(t_0))\left[\exp(\rho\delta) - \frac{8MM_1}{\rho}\right] \\ \to +\infty \quad \text{as } k \to +\infty \tag{4.18}$$

in terms of (4.5).

It is obviously that (4.18) contradicts (4.15). Consequently, the conclusion of the Theorem 4.1 is held. $\square$

From the proof of the Theorem 4.1 it is easy to obtain the following:

Corollary 4.2. *If $f_2 = f_3 = 0$ then the system* (2.5) *has no homoclinic orbit to E.*

Now consider Lorenz-type system

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= cx_1 - x_2 - x_1x_3 \\ \dot{x}_3 &= -bx_3 + x_1x_2 \end{aligned} \tag{4.19}$$

with $a = -3$, $c = 28$ and $b = 8/3$. The Jacobi matrix at equilibrium $E = (0, 0, 0)^T$ of the system (4.19) is

$$J = \begin{pmatrix} 3 & -3 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix}$$

The computation results show that the eigenvalues of J are $-2.6667,\ 1 \pm 8.9443i$. Hence E is a saddle focus point. Set $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T$.

Let $y = Wx$ where

$$W = \begin{pmatrix} 0 & 0.744 & 0 \\ 2.3292 & -0.1664 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can obtain

$$\begin{aligned} \dot{y}_1 &= y_1 + 8.9443y_2 - 0.744y_3(0.096y_1 + 0.4293y_2) \\ \dot{y}_2 &= -8.9443y_1 + y_2 + 0.1664y_3(0.096y_1 + 0.4293y_2) \\ \dot{y}_3 &= -2.6667y_3 + 1.344y_1(0.096y_1 + 0.4293y_2) \end{aligned} \tag{4.20}$$

Reset $y_3 = x_1$, $y_1 = x_2$ and $y_2 = x_3$, the system (4.20) takes into the form of the system (2.5) with $\lambda = -2.6667$, $\rho = 1$ and $\omega = 8.9443$. In particular,

$$\begin{aligned} f_2(x_1, x_2, x_3) &= -0.744x_1(0.096x_2 + 0.4293x_3), \\ f_3(x_1, x_2, x_3) &= 0.1664x_1(0.096x_2 + 0.4293x_3). \end{aligned}$$

It is easy to see that f_2 and f_3 satisfy the assumptions of the theorem 4.1. Consequently the system (4.20) has no homoclinic orbit.

From Theorem 4.1, we can give the following:

Corollary 4.3. *The necessary conditions for the system* (2.5) *to have homoclinic orbit to E are that f_i* ($i = 2, 3$) *cannot be taken into the form given by* (4.1) *and* (4.2).

5. Conclusion

In this paper, the conditions for third-order ODE systems with C^2 vector field to have homoclinic orbit to saddle focus are studied. The relative result is given in the Corollary 4.3.

Meanwhile the result is applied to Lorenz-type system with saddle focus point. It would be noted that any Lorenz-type system with saddle focus point can be transformed into the form like (4.20). So, we can assert that any Lorenz-type system satisfied 4.1 has no homoclinic orbit to saddle focus. This conclusion is also suitable to Chen-type system and so on.

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REFERENCES

[1] Shil'nikov, L. P. A case of the existence of a countable number of periodic motions, *Soviet Mathematics Docklady* **6**, 163–166, 1965 (translated by S. Puckette).

[2] Shil'nikov, L. P. A contribution of the problem of the structure of an extended neighborhood of rough equilibrium state of saddle-focus type, *Mathematics U.S.S.R.-Shornik* **10** 91–102,1970 (translated by F. A. Cezus).

[3] Shil'nikov, L. P. On a new type of bifurcation of multidimensional dynamical systems, *Sov. Math.* **10** 1368–1371, 1969.

[4] Shil'nikov, L. P. Shi'lnikov, A. L., Turaev, D. V. and Chua, L. O., Method qualitative theory in nonlinear dynamics (Part I), *Singapore World Scientific Pub.*1998.

[5] Silva, C. P., Shi'lnikov theorem-atutorial, *IEEE Transactions on Circuits and Systems-I* **40** 675–682,1993.

[6] Leonov, G. A., Bounds for attractors and existence of homoclinic orbits in the Lorenz system, em PMM J Appl Math Mecn. **65** 19–32, 2001.

[7] Sandstede, B., Center manifolds for homoclinic solution. Journal of Dynamics and Differential Equations, **12**, 449–501, 2000.

[8] Wilczak, D., The existence of Shilnikov homoclinic orbits in the Michelson system: A computer assisted proof. *Found. Compt. Math.*495–535,2006.

[9] Sandstede, B., Convergence estimates for the numerical approximation of homoclinic solution, *IMA J Numer Anal.* **17** 437–462, 1997.

[10] Vakakis, A. F. and Azeez, M. F. A., Analytic approximation of the homoclinic orbits of the Lorenz system at $\sigma = 10$, $b = \frac{8}{3}$ and $\rho = 13.926$, *Nolinear Dyn* **15** 245–257, 1998.

[11] Wai, W., Zhang, Q. C. and Tian, R. L., Shilnikov sense chaos in a simple three-dimensional system, *Chin. Phys. B* **19** 030517-1–030517-10, 2010.

[12] Zhou, T. S., Chen, G. and Yang, Q. G.,Constructing a new chaotic system based on Shilnikov criterion, *Chaos Solitons and Fractals* **19** 985–993, 2004.

[13] Li Y. H. and Zhu S. M., n-Dimensional stable and unstable manifolds of hyperbolic singular point, *Chaos, Solitons and Fractals*, **29**, 1155-1164, 2006.

[14] Chen Y. S., *Bifurcation and Chaos Theory of Nonlinear Vibration System* (High Education Press, China), 1993.

[15] Algaba A., Fernández-Sánchez, F., Merino, M. and A. J. Rodriguez-Lius, Comment on 'Shil'nikov chaos of LIu system' [Chaos 18, 013113 (2008)], *Chaos* **21**,048101, 2011.

[16] Leonov, G. A., General existence conditions of homoclinic trajectories in dissipative systems: Lorenz, Shimizu-Morioka, Lu and Chen systems, *Physics Letters* **376**, 3045-3050, 2012.

[17] Mefrano-T. R. O., Baptista M. S. and Caldas I. L., Shilnikov homoclinic orbit bifurcations in the Chua's circuit, *Chaos*, **16**, 043119, 2006.