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MULTIPLE SOLUTIONS FOR A NONLINEAR PERTURBED FRACTIONAL BOUNDARY VALUE PROBLEM

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. The existence of three distinct solutions for a nonlinear perturbed fractional boundary value problem under suitable assumptions on the nonlinear term is established. Our approach is based on recent variational methods for smooth functionals defined on reflexive Banach spaces.

Keywords: Multiple solutions; perturbed fractional boundary value problem; critical point theory; variational methods.

AMS (MOS) Subject Classification. 58E05, 34B15, 26A33

1. INTRODUCTION

Consider the following perturbed fractional boundary value problem

(1.1)
$$\frac{d}{dt}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big) + \lambda f(u(t)) + \mu g(u(t)) = 0, \quad \text{a.e. } t \in [0,T],$$
$$u(0) = u(T) = 0$$

where $\alpha \in (1/2, 1]$, ${}_0D_t^{\alpha-1}$ and ${}_tD_T^{\alpha-1}$ are the left and right Riemann-Liouville fractional integrals of order $1-\alpha$ respectively, ${}_0^cD_t^{\alpha}$ and ${}_t^cD_T^{\alpha}$ are the left and right Caputo fractional derivatives of order $0 < \alpha \le 1$ respectively, λ is a positive real parameter, μ is a non-negative real parameter and $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions.

Because of its wide application in the modeling of many phenomena in various fields of physic, chemistry, biology, engineering and economics, the theory of fractional differential equations has recently been attracting increasing interest, see for instance the monographs of Miller and Ross [30], Samko et al [34], Podlubny [31], Hilfer [23], Kilbas et al [25] and the papers [1, 4, 5, 6, 7, 26, 27] and the references therein.

Critical point theory has been very useful in determining the existence of solution for integer order differential equations with some boundary conditions, for example [20, 27, 28, 29, 32, 35]. But until now, there are few results on the solution to fractional

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boundary value problems which were established by the critical point theory, since it is often very difficult to establish a suitable space and variational functional for fractional boundary value problems. Recently, Jiao and Zhou in [24] by using the critical point theory investigated the fractional boundary-value problem

$$\frac{d}{dt}\Big(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))\Big) + \nabla F(t, u(t)) = 0, \quad \text{a.e. } t \in [0, T],$$
$$u(0) = u(T) = 0 \tag{1.2}$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \leq \beta < 1$ respectively, $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function and $\nabla F(t, x)$ is the gradient of F at x. Also, Chen and Tang in [19] studied the existence and multiplicity of solutions for the fractional boundary value problem (1.2) where $F(t, \cdot)$ are superquadratic, asymptotically quadratic, and subquadratic, respectively. In particular, Bai in [3], by using a local minimum theorem due to Bonanno ([8]), investigated the existence of at least one non-trivial solution to the problem (1.1).

In the present paper, motivated by [3], using two kinds of three critical points theorems obtained in [9, 13] which we recall in the next section (Theorems 2.10 and 2.11), we ensure the existence of at least three solutions for the problem (1.1); see Theorems 3.1 and 3.2. These theorems have been successfully employed to establish the existence of at least three solutions for perturbed boundary value problems in the papers [10, 11, 18, 21, 22].

A special case of Theorem 3.1 is the following theorem.

Theorem 1.1. *Let $\frac{1}{2} < \alpha \leq 1$ and $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function. Put $F(x) := \int_0^x f(\xi)d\xi$ for each $x \in \mathbb{R}$. Assume that*

$$\liminf_{\xi\to 0} \frac{F(\xi)}{\xi^2} = \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 0.$$

Then, there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ and for every continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying the asymptotical condition

$$\limsup_{|x|\to\infty} \frac{\int_0^x g(s)ds}{x^2} < +\infty,$$

there exists $\delta^*_{\lambda,g} > 0$ such that, for each $\mu \in [0, \delta^*_{\lambda,g}[$, the problem (1.1) admits at least three solutions.

Moreover, the following result is a consequence of Theorem 3.2.

Theorem 1.2. *Let $\frac{1}{2} < \alpha \leq 1$ and $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function such that*

$$\lim_{t\to 0^+} \frac{f(t)}{t} = 0,$$

and

$$\int_0^1 f(\xi)d\xi < \frac{1}{24}\frac{|\cos(\pi\alpha)|}{\frac{4}{\Gamma^2(\alpha)(2(\alpha-1)+1)}\frac{\Gamma^3(2-\alpha)}{\Gamma(4-2\alpha)}(2^{2\alpha-1}-1)}\int_0^{2\Gamma(2-\alpha)}\int_0^x f(\xi)d\xi dx.$$

Then, for every

$$\lambda \in \left]\frac{24\frac{\Gamma^3(2-\alpha)}{\Gamma(4-2\alpha)}2^{1-2\alpha}(2^{2\alpha-1}-1)}{\int_0^{2\Gamma(2-\alpha)}\int_0^x f(\xi)d\xi dx},\ \frac{|\cos(\pi\alpha)|}{4\frac{2^{2\alpha-1}}{\Gamma^2(\alpha)(2(\alpha-1)+1)}}\frac{1}{\int_0^1 f(\xi)d\xi}\right[$$

and for every non-negative continuous function $g:\mathbb{R}\to\mathbb{R}$, *there exists* $\delta^*_{\lambda,g}>0$ *such that, for each* $\mu\in[0,\delta^*_{\lambda,g}[$, *the problem*

$$\frac{d}{dt}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big) + \lambda f(u(t)) + \mu g(u(t)) = 0, \quad a.e.\ t\in[0,2],$$
$$u(0)=u(2)=0$$

admits at least three solutions.

The present paper is arranged as follows. In Section 2 we recall some basic definitions and preliminary results, while Section 3 is devoted to the existence of multiple solutions for the problem (1.1).

2. PRELIMINARIES

In this section, we will introduce some notations, definitions and preliminary facts which are used throughout this paper.

Definition 2.1 ([25])**.** Let f be a function defined on $[a,b]$ and $\alpha>0$. The left and right Riemann-Liouville fractional integrals of order α for the function f are defined by

$${}_aD_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds, \quad t\in[a,b],$$
$${}_tD_b^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}f(s)ds, \quad t\in[a,b],$$

provided the right-hand sides are pointwise defined on $[a,b]$, where $\Gamma(\alpha)$ is the gamma function.

Definition 2.2 ([25])**.** Let $\gamma\geq 0$ and $n\in\mathbb{N}$.

(i) If $\gamma\in(n-1,n)$ and $f\in AC^n([a,b],\mathbb{R}^N)$, then the left and right Caputo fractional derivatives of order γ for function f denoted by ${}_a^cD_t^{\gamma}f(t)$ and ${}_t^cD_b^{\gamma}f(t)$, respectively, exist almost everywhere on $[a,b]$, ${}_a^cD_t^{\gamma}f(t)$ and ${}_t^cD_b^{\gamma}f(t)$ are represented by

$${}_a^cD_t^{\gamma}f(t) = \frac{1}{\Gamma(n-\gamma)}\int_a^t (t-s)^{n-\gamma-1}f^{(n)}(s)ds, \quad t\in[a,b],$$

$$ {}_t^cD_b^{\gamma} f(t) = \frac{(-1)^n}{\Gamma(n-\gamma)} \int_t^b (s-t)^{n-\gamma-1} f^{(n)}(s)ds, \quad t \in [a,b], $$

respectively.

(ii) If $\gamma = n-1$ and $f \in AC^{n-1}([a,b],\mathbb{R}^N)$, then ${}_a^cD_t^{n-1}f(t)$ and ${}_t^cD_b^{n-1}f(t)$ are represented by

$$ {}_a^cD_t^{n-1}f(t) = f^{(n-1)}(t), \quad \text{and} \quad {}_t^cD_b^{n-1}f(t) = (-1)^{(n-1)}f^{(n-1)}(t), \quad t \in [a,b]. $$

With these definitions, we have the rule for fractional integration by parts, and the composition of the Riemann-Liouville fractional integration operator with the Caputo fractional differentiation operator, which were proved in [25, 34].

Proposition 2.3 ([25, 34]). *We have the following property of fractional integration*

$$ \int_a^b [{}_aD_t^{-\gamma} f(t)]g(t)dt = \int_a^b [{}_tD_b^{-\gamma} g(t)]f(t)dt, \quad \gamma > 0, \tag{2.1} $$

provided that $f \in L^p([a,b],\mathbb{R}^N)$, $g \in L^q([a,b],\mathbb{R}^N)$ *and* $p \geq 1$, $q \geq 1$, $1/p + 1/q \leq 1+\gamma$ *or* $p \neq 1$, $q \neq 1$, $1/p + 1/q = 1+\gamma$.

Proposition 2.4 ([25]). *Let* $n \in \mathbb{N}$ *and* $n-1 < \gamma \leq n$. *If* $f \in AC^n([a,b],\mathbb{R}^N)$ *or* $f \in C^n([a,b],\mathbb{R}^N)$, *then*

$$ {}_aD_t^{-\gamma}({}_a^cD_t^{\gamma} f(t)) = f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!}(t-a)^j \,, $$

$$ {}_tD_b^{-\gamma}({}_t^cD_b^{\gamma} f(t)) = f(t) - \sum_{j=0}^{n-1} \frac{(-1)^j f^{(j)}(b)}{j!}(b-t)^j, $$

for $t \in [a,b]$. *In particular, if* $0 < \gamma \leq 1$ *and* $f \in AC([a,b],\mathbb{R}^N)$ *or* $f \in C^1([a,b],\mathbb{R}^N)$, *then*

$$ {}_aD_t^{-\gamma}({}_a^cD_t^{\gamma} f(t)) = f(t) - f(a), \quad \text{and} \quad {}_tD_b^{-\gamma}({}_t^cD_b^{\gamma} f(t)) = f(t) - f(b). \tag{2.2} $$

Remark 2.5. In view of (2.1) and Definition 2.2, it is obvious that $u \in AC([0,T])$ is a solution of (1.1) if and only if u is a solution of the problem

$$ \begin{aligned} &\frac{d}{dt}\left({}_0D_t^{-\beta}(u'(t)) + {}_tD_T^{-\beta}(u'(t))\right) + \lambda f(u(t)) + \mu g(u(t)) = 0, \quad \text{a.e. } t \in [0,T], \\ &u(0) = u(T) = 0, \end{aligned} \tag{2.3} $$

where $\beta = 2(1-\alpha) \in [0,1)$.

To establish a variational structure for (1.1), it is necessary to construct appropriate function spaces.

Definition 2.6 ([24]). Let $0<\alpha\le 1$. The fractional derivative space E_0^α is defined by the closure of $C_0^\infty[0,T]$ with respect to the norm

$$\|u\|_\alpha=\left(\int_0^T |{}_0^cD_t^\alpha u(t)|^2dt+\int_0^T |u(t)|^2dt\right)^{1/2},\quad \forall u\in E^\alpha,$$

where $C_0^\infty[0,T]$ denotes the set of all functions $u\in C^\infty[0,T]$ with $u(0)=u(T)=0$. It is obvious that the fractional derivative space E_0^α is the space of functions $u\in L^2[0,T]$ having an α-order Caputo fractional derivative ${}_0^cD_t^\alpha u\in L^2[0,T]$ and $u(0)=u(T)=0$.

Proposition 2.7 ([24]). *Let $0<\alpha\le 1$. The fractional derivative space E_0^α is reflexive and separable Banach space.*

Proposition 2.8 ([24]). *Let $0<\alpha\le 1$. For all $u\in E_0^\alpha$, we have*

$$\|u\|_{L^2}\le\frac{T^\alpha}{\Gamma(\alpha+1)}\|{}_0^cD_t^\alpha u\|_{L^2},\tag{2.4}$$

$$\|u\|_\infty\le\frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2(\alpha-1)+1)^{1/2}}\|{}_0^cD_t^\alpha u\|_{L^2}.\tag{2.5}$$

According to (2.4), we can consider E_0^α with respect to the norm

$$\|u\|_\alpha=\left(\int_0^T |{}_0^cD_t^\alpha u(t)|^2dt\right)^{1/2}=\|{}_0^cD_t^\alpha u\|_{L^2},\quad \forall u\in E_0^\alpha\tag{2.6}$$

in the following analysis.

Proposition 2.9 ([24]). *Let $1/2<\alpha\le 1$, then for any $u\in E_0^\alpha$, we have*

$$|\cos(\pi\alpha)|\|u\|_\alpha^2\le-\int_0^T {}_0^cD_t^\alpha u(t)\cdot{}_t^cD_T^\alpha u(t)dt\le\frac{1}{|\cos(\pi\alpha)|}\|u\|_\alpha^2.\tag{2.7}$$

By Proposition 2.8, when $\alpha>1/2$, for each $u\in E_0^\alpha$ we have

$$\|u\|_\infty\le k\left(\int_0^T |{}_0^cD_t^\alpha u(t)|^2dt\right)^{1/2}=k\|u\|_\alpha,\tag{2.8}$$

where

$$k=\frac{T^{\alpha-\frac12}}{\Gamma(\alpha)\sqrt{2(\alpha-1)+1}}.\tag{2.9}$$

Our main tools are the following three critical points theorems. In the first one the coercivity of the functional $\Phi-\lambda\Psi$ is required, in the second one a suitable sign hypothesis is assumed.

Theorem 2.10 ([13, Theorem 2.6]). *Let X be a reflexive real Banach space, $\Phi: X\longrightarrow\mathbb{R}$ be a coercive continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi: X\longrightarrow\mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0)=\Psi(0)=0$.*

Assume that there exist $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$ such that

(a_1) $\frac{\sup_{\Phi(x)\leq r}\Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$,

(a_2) *for each* $\lambda \in \Lambda_r := \left]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)\leq r}\Psi(x)}\right[$ *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 2.11 ([9, Theorem 3.3])**.** *Let X be a reflexive real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

1. $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

2. for each $\lambda > 0$ and for every $u_1, u_2 \in X$ which are local minima for the functional $\Phi - \lambda\Psi$ and such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are two positive constants r_1, r_2 and $\overline{v} \in X$, with $2r_1 < \Phi(\overline{v}) < \frac{r_2}{2}$, such that

(b_1) $\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} < \frac{2}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}$;

(b_2) $\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}$.

Then, for each $\lambda \in \left]\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \ \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}, \ \frac{\frac{r_2}{2}}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\right\}\right[$, *the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.*

Corresponding to f and g we introduce the functions $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$, respectively, as follows

$$F(x) := \int_0^x f(\xi)d\xi, \quad \forall\ x \in \mathbb{R}$$

and

$$G(x) := \int_0^x g(\xi)d\xi, \quad \forall\ x \in \mathbb{R}.$$

Moreover, set $G^\theta := T\max_{|x|\leq\theta} G(x)$, for every $\theta > 0$ and $G_\eta := \inf_{[0,\eta]} G$, for every $\eta > 0$. If g is sign-changing, then $G^\theta \geq 0$ and $G_\eta \leq 0$.

Put

$$\omega_\alpha := \frac{4\Gamma^2(2-\alpha)}{\Gamma(4-2\alpha)}T^{1-2\alpha}(2^{2\alpha-1}-1).$$

3. MAIN RESULTS

Following the construction given in [11], in order to introduce our first result, fixing two positive constants θ and η such that

$$\frac{\omega_\alpha\Gamma(2-\alpha)\eta^3}{\int_0^{\Gamma(2-\alpha)\eta} F(x)dx} < \frac{|\cos(\pi\alpha)|\theta^2}{k^2\max_{|x|\leq\theta} F(x)},$$

and taking

$$\lambda \in \Lambda := \left] \frac{\omega_\alpha \eta^2}{\frac{T}{\Gamma(2-\alpha)\eta}\int_0^{\Gamma(2-\alpha)\eta} F(x)dx}, \ \frac{\frac{|\cos(\pi\alpha)|}{k^2}\theta^2}{T\max_{|x|\le\theta} F(x)} \right[,$$

(3.1)

$$\delta_{\lambda,g} := \min\left\{ \frac{|\cos(\pi\alpha)|\theta^2 - \lambda k^2 T \max_{|x|\le\theta} F(x)}{k^2 G^\theta}, \ \frac{\omega_\alpha\eta^2 - \lambda \frac{T}{\Gamma(2-\alpha)\eta}\int_0^{\Gamma(2-\alpha)\eta} F(x)dx}{TG_\eta} \right\}$$

and

$$\overline{\delta}_{\lambda,g} := \min\left\{ \delta_{\lambda,g}, \ \frac{1}{\max\left\{0, \frac{k^2}{|\cos(\pi\alpha)|}\limsup_{|x|\to\infty}\frac{G(x)}{x^2}\right\}} \right\}, \tag{3.2}$$

where we read $\rho/0 = +\infty$, so that, for instance, $\overline{\delta}_{\lambda,g} = +\infty$ when

$$\limsup_{|x|\to\infty} \frac{G(x)}{x^2} \le 0,$$

and $G_\eta = G^\theta = 0$.

Now, we formulate our main result.

Theorem 3.1. *Let $\frac{1}{2} < \alpha \le 1$. Assume that there exist two positive constants θ and η with $\sqrt{\frac{|\cos(\pi\alpha)|}{\omega_\alpha}}\frac{\theta}{k} < \eta$ such that*

(A_1) $\frac{\max_{|x|\le\theta} F(x)}{\theta^2} < \frac{|\cos(\pi\alpha)|}{\Gamma(2-\alpha)k^2\omega_\alpha}\frac{\int_0^{\Gamma(2-\alpha)\eta} F(x)dx}{\eta^3}$;

(A_2) $\limsup_{|x|\to+\infty} \frac{F(x)}{x^2} \le 0$.

Then, for each $\lambda \in \Lambda$ and for every continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\limsup_{|x|\to\infty} \frac{G(x)}{x^2} < +\infty,$$

there exists $\overline{\delta}_{\lambda,g} > 0$ given by (3.2) such that, for each $\mu \in [0, \overline{\delta}_{\lambda,g}[$, the problem (1.1) admits at least three distinct solutions in E_0^α.

Proof. In order to apply Theorem 2.10 to our problem, let X be the fractional derivative space E_0^α equipped with the norm

$$\|u\|_\alpha = \Big(\int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt\Big)^{1/2},$$

and we introduce the functionals Φ, $\Psi : X \to \mathbb{R}$ for each $u \in X$ as follows

$$\Phi(u) := -\int_0^T {}_0^cD_t^\alpha u(t) \cdot {}_t^cD_T^\alpha u(t)dt \ \text{ and } \ \Psi(u) := \int_0^T (F(u(t)) + \frac{\mu}{\lambda}G(u(t)))dt.$$

Clearly, Φ and Ψ are Gâteaux differentiable functionals whose Gâteaux derivative at the point $u \in X$ are given by

$$\Phi'(u)v = -\int_0^T ({}_0^cD_t^\alpha u(t) \cdot {}_t^cD_T^\alpha v(t) + {}_t^cD_T^\alpha u(t) \cdot {}_0^cD_t^\alpha v(t))dt$$

and

$$\Psi'(u)v = \int_0^T (f(u(t)) + \frac{\overline{\mu}}{\lambda} g(u(t)))v(t)dt$$
$$= -\int_0^T \int_0^t f(u(s))ds \cdot v'(t)dt - \frac{\mu}{\lambda}\int_0^T \int_0^t g(u(s))ds \cdot v'(t)dt$$

for every $v \in X$. By Definition 2.2 and (2.2), we have

$$\Phi'(u)v = \int_0^T ({}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))) \cdot v'(t)dt.$$

It is well known that the functionals Φ and Ψ satisfy all regularity assumptions requested in Theorem 2.10. Put $I_\lambda := \Phi - \lambda\Psi$. The solutions of the problem (1.1) are exactly the solutions of the equation $I'_\lambda(u) = 0$ (see [3]). Put $r := \frac{|\cos(\pi\alpha)|}{k^2}\theta^2$ and

$$w(t) = \begin{cases} \frac{2\Gamma(2-\alpha)\eta}{T}t, & t \in [0, T/2), \\ \frac{2\Gamma(2-\alpha)\eta}{T}(T-t), & t \in [T/2, T]. \end{cases} \tag{3.3}$$

It is easy to check that $w(0) = w(T) = 0$ and $w \in L^2[0, T]$. The direct calculation shows that

$${}_0^cD_t^\alpha w(t) = \begin{cases} \frac{2\eta}{T}t^{1-\alpha}, & t \in [0, T/2), \\ \frac{2\eta}{T}(t^{1-\alpha} - 2(t-\frac{T}{2})^{1-\alpha}), & t \in [T/2, T] \end{cases}$$

and

$$\|w\|_\alpha^2 = \int_0^T ({}_0^cD_t^\alpha w(t))^2 dt = \int_0^{\frac{T}{2}} ({}_0^cD_t^\alpha w(t))^2 dt + \int_{T/2}^T ({}_0^cD_t^\alpha w(t))^2 dt$$
$$= \frac{4\eta^2}{T^2}\left[\int_0^T t^{2(1-\alpha)}dt - 4\int_{T/2}^T t^{1-\alpha}\left(t-\frac{T}{2}\right)^{1-\alpha} dt + 4\int_{T/2}^T \left(t-\frac{T}{2}\right)^{2(1-\alpha)} dt\right]$$
$$= \frac{4(1+2^{2\alpha-1})\eta^2}{3-2\alpha}T^{1-2\alpha} - \frac{16\eta^2}{T^2}\int_{T/2}^T t^{1-\alpha}\left(t-\frac{T}{2}\right)^{1-\alpha} dt < \infty.$$

That is, ${}_0^cD_t^\alpha w \in L^2[0, T]$. Thus, $w \in X$. Moreover, the direct calculation shows

$${}_t^cD_T^\alpha w(t) = \begin{cases} \frac{2\eta}{T}((T-t)^{1-\alpha} - 2(\frac{T}{2}-t)^{1-\alpha}), & t \in [0, T/2), \\ \frac{2\eta}{T}(T-t)^{1-\alpha}, & t \in [T/2, T] \end{cases}$$

and

$$\Phi(w) = -\int_0^T {}_0^cD_t^\alpha w(t) \cdot {}_t^cD_T^\alpha w(t)dt$$
$$= -\left(\frac{2\eta}{T}\right)^2 \left[\int_0^{\frac{T}{2}} t^{1-\alpha}\left((T-t)^{1-\alpha} - 2\left(\frac{T}{2}-t\right)^{1-\alpha}\right) dt\right.$$
$$\left. + \int_{T/2}^T (T-t)^{1-\alpha} \cdot \left(t^{1-\alpha} - 2(t-\frac{T}{2})^{1-\alpha}\right) dt\right]$$

$$= -\left(\frac{2\eta}{T}\right)^2 \left[\int_0^T t^{1-\alpha}(T-t)^{1-\alpha}dt - 4\int_0^{\frac{T}{2}} t^{1-\alpha}\left(\frac{T}{2}-t\right)^{1-\alpha} dt\right]$$

$$= -\left(\frac{2\eta}{T}\right)^2 \left[\frac{\Gamma^2(2-\alpha)}{\Gamma(4-2\alpha)}T^{3-2\alpha} - 4\frac{\Gamma^2(2-\alpha)}{\Gamma(4-2\alpha)}\left(\frac{T}{2}\right)^{3-2\alpha}\right]$$

$$= \frac{4\Gamma^2(2-\alpha)}{\Gamma(4-2\alpha)}T^{1-2\alpha}(2^{2\alpha-1}-1)\eta^2 = \omega_\alpha \eta^2. \tag{3.4}$$

From the condition $\sqrt{\frac{|\cos(\pi\alpha)|}{\omega_\alpha}}\frac{\theta}{k} < \eta$ we get

$$0 < r < \Phi(w).$$

Taking (2.7) into account, for all $u \in X$ such that $u \in \Phi^{-1}(]-\infty, r])$, we have

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \le \Phi(u) \le r,$$

which implies

$$\|u\|_\alpha^2 \le \frac{1}{|\cos(\pi\alpha)|}r. \tag{3.5}$$

Thus, in view of (2.8) and (3.5) we have

$$|u(t)| < k\|u\|_\alpha \le k\sqrt{\frac{r}{|\cos(\pi\alpha)|}} = \theta, \quad \forall t \in [0,T],$$

which from the definition of Ψ follows that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) = \sup_{u\in\Phi^{-1}(]-\infty,r])} \int_0^T \left[F(u(t)) + \frac{\mu}{\lambda}G(u(t))\right] dt$$

$$\le T\left(\max_{|x|\le\theta} F(x) + \frac{\mu}{\lambda}G^\theta\right).$$

On the other hand, by using Assumption (A_1), we infer

$$\Psi(w) = \int_0^T \left(F(w(t)) + \frac{\mu}{\lambda}G(w(t))\right) dt$$

$$= \frac{T}{\Gamma(2-\alpha)\eta}\int_0^{\Gamma(2-\alpha)\eta} F(x)dx + \frac{\mu}{\lambda}\int_0^T G(w(t))dt$$

$$\ge \frac{T}{\Gamma(2-\alpha)\eta}\int_0^{\Gamma(2-\alpha)\eta} F(x)dx + T\frac{\mu}{\lambda}\inf_{[0,\eta]} G$$

$$= \frac{T}{\Gamma(2-\alpha)\eta}\int_0^{\Gamma(2-\alpha)\eta} F(x)dx + T\frac{\mu}{\lambda}G_\eta.$$

Hence

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} = \frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\int_0^T \left[F(u(t)) + \frac{\mu}{\lambda}G(u(t))\right] dt}{r}$$

$$\le \frac{T\left(\max_{|x|\le\theta} F(x) + \frac{\mu}{\lambda}G^\theta\right)}{\frac{|\cos(\pi\alpha)|}{k^2}\theta^2} \tag{3.6}$$

and

$$\frac{\Psi(w)}{\Phi(w)} \geq \frac{\frac{T}{\Gamma(2-\alpha)\eta}\int_0^{\Gamma(2-\alpha)\eta} F(x)dx + \frac{\mu}{\lambda}\int_0^T G(w(t))dt}{\omega_\alpha \eta^2}$$

$$\geq \frac{\frac{T}{\Gamma(2-\alpha)\eta}\int_0^{\Gamma(2-\alpha)\eta} F(x)dx + T\frac{\mu}{\lambda}G_\eta}{\omega_\alpha \eta^2}. \tag{3.7}$$

Since $\mu < \delta_{\lambda,g}$, one has

$$\mu < \frac{|\cos(\pi\alpha)|\theta^2 - \lambda k^2 T \max_{|x|\leq\theta} F(x)}{k^2 G^\theta},$$

this means

$$\frac{T\max_{|x|\leq\theta} F(x) + \frac{\mu}{\lambda}G^\theta}{\frac{|\cos(\pi\alpha)|}{k^2}\theta^2} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{\omega_\alpha \eta^2 - \lambda\frac{T}{\Gamma(2-\alpha)\eta}\int_0^{\Gamma(2-\alpha)\eta} F(x)dx}{TG_\eta},$$

taking into account that $G_\eta \leq 0$, this means

$$\frac{\frac{T}{\Gamma(2-\alpha)\eta}\int_0^{\Gamma(2-\alpha)\eta} F(x)dx + T\frac{\mu}{\lambda}G_\eta}{\omega_\alpha \eta^2} > \frac{1}{\lambda}.$$

Then,

$$\frac{T\max_{|x|\leq\theta} F(x) + \frac{\mu}{\lambda}G^\theta}{\frac{|\cos(\pi\alpha)|}{k^2}\theta^2} < \frac{1}{\lambda} < \frac{\frac{T}{\Gamma(2-\alpha)\eta}\int_0^{\Gamma(2-\alpha)\eta} F(x)dx + T\frac{\mu}{\lambda}G_\eta}{\omega_\alpha \eta^2}. \tag{3.8}$$

Hence from (3.6)-(3.8), the condition (a_1) of Theorem 2.10 is verified. Finally, since $\mu < \overline{\delta}_{\lambda,g}$, we can fix $l > 0$ such that

$$\limsup_{|x|\to\infty} \frac{G(x)}{x^2} < l$$

and $\mu l < \frac{|\cos(\pi\alpha)|}{Tk^2}$. Therefore, there exists a constant τ such that

$$G(x) \leq lx^2 + \tau \tag{3.9}$$

for every $x \in \mathbb{R}$. Now, fix $0 < \epsilon < \frac{|\cos(\pi\alpha)|}{Tk^2\lambda} - \frac{\mu l}{\lambda}$. From (A_2) there is a constant τ_ϵ such that

$$F(x) \leq \epsilon x^2 + \tau_\epsilon \tag{3.10}$$

for every $x \in \mathbb{R}$. Taking (2.8) into account, from (3.9) and (3.10), it follows that, for each $u \in X$,

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &= -\int_0^T {}_0^cD_t^\alpha u(t)\cdot {}_t^cD_T^\alpha u(t)dt - \lambda\int_0^T \Big(F(u(t)) + \frac{\mu}{\lambda}G(u(t))\Big)dt \\ &\geq |\cos(\pi\alpha)|\|u\|_\alpha^2 - \lambda\epsilon\int_0^T u^2(t)dt - \lambda T\tau_\epsilon - \mu l\int_0^T u^2(t)dt - \mu T\tau \\ &\geq \big(|\cos(\pi\alpha)| - \lambda Tk^2\epsilon - \mu Tk^2 l\big)\|u\|_\alpha^2 - \lambda\tau_\epsilon - \mu\tau,\end{aligned}$$

and thus

$$\lim_{\|u\|_\alpha \to +\infty} (\Phi(u) - \lambda\Psi(u)) = +\infty,$$

which means the functional $\Phi - \lambda\Psi$ is coercive, and the condition (a_2) of Theorem 2.10 is verified. Since from (3.6)–(3.8) one also has

$$\lambda \in \left] \frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(x)\leq r} \Psi(x)} \right[,$$

Theorem 2.10 (with $\overline{x} = w$) ensures the existence of three critical points for the functional I_λ, and the proof is complete. □

Now, a variant of Theorem 3.1 where no asymptotic condition on the nonlinear term g is required, is pointed out. In such a case f and g are supposed to be non-negative.

For our goal, let us fix positive constants θ_1, θ_2 and η such that

$$\frac{3}{2}\frac{\omega_\alpha \eta^2}{\frac{1}{\Gamma(2-\alpha)\eta}\int_0^{\Gamma(2-\alpha)\eta} F(x)dx} < \frac{|\cos(\pi\alpha)|}{k^2} \min\left\{ \frac{\theta_1^2}{\max_{|x|\leq\theta_1} F(x)}, \ \frac{\theta_2^2}{2\max_{|x|\leq\theta_2} F(x)} \right\},$$

and taking

$$\lambda \in \Lambda :$$

$$= \left] \frac{3}{2}\frac{\omega_\alpha \eta^2}{\frac{T}{\Gamma(2-\alpha)\eta}\int_0^{\Gamma(2-\alpha)\eta} F(x)dx}, \ \frac{|\cos(\pi\alpha)|}{Tk^2} \min\left\{ \frac{\theta_1^2}{\max_{|x|\leq\theta_1} F(x)}, \ \frac{\theta_2^2}{2\max_{|x|\leq\theta_2} F(x)} \right\} \right[.$$

With the above notations we have the following result.

Theorem 3.2. *Let $\frac{1}{2} < \alpha \leq 1$ and $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $f(x) \geq 0$ for every $x \in \mathbb{R}^+ \cup \{0\}$. Assume that there exist three positive constants θ_1, θ_2 and η with $2^{1/2}\theta_1 < \sqrt{\frac{\omega_\alpha}{|\cos(\pi\alpha)|}}\eta k < \frac{\theta_2}{2^{1/2}}$ such that*

(B_1) $\frac{\max_{|x|\leq\theta_1} F(x)}{\theta_1^2} < \frac{2}{3}\frac{|\cos(\pi\alpha)|}{\Gamma(2-\alpha)k^2\omega_\alpha}\frac{\int_0^{\Gamma(2-\alpha)\eta} F(x)dx}{\eta^3}$;

(B_2) $\frac{\max_{|x|\leq\theta_2} F(x)}{\theta_2^2} < \frac{1}{3}\frac{|\cos(\pi\alpha)|}{\Gamma(2-\alpha)k^2\omega_\alpha}\frac{\int_0^{\Gamma(2-\alpha)\eta} F(x)dx}{\eta^3}$.

Then, for each $\lambda \in \Lambda$ and for every non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\delta^*_{\lambda,g} > 0$ given by

$$\min\left\{ \frac{|\cos(\pi\alpha)|\theta_1^2 - \lambda k^2 T \max_{|x|\leq\theta_1} F(x)}{k^2 G^{\theta_1}}, \ \frac{|\cos(\pi\alpha)|\theta_2^2 - 2\lambda k^2 T \max_{|x|\leq\theta_2} F(x)}{2k^2 G^{\theta_2}} \right\}.$$

such that, for each $\mu \in [0, \delta^*_{\lambda,g}[$, the problem (1.1) *admits at least three distinct solutions u_i for $i = 1, 2, 3$, such that*

$$0 \leq u_i(t) < \theta_2, \quad \forall\, t \in [0, T], \ (i = 1, 2, 3).$$

Proof. Fix λ, g and μ as in the conclusion and take Φ and Ψ as in the proof of Theorem 3.1. We observe that the regularity assumptions of Theorem 2.11 on Φ and Ψ are satisfied. Then, our aim is to verify (b_1) and (b_2). To this end, put w as given in (3.3),

$$r_1 := \frac{|\cos(\pi\alpha)|}{k^2}\theta_1^2,$$

and

$$r_2 := \frac{|\cos(\pi\alpha)|}{k^2}\theta_2^2.$$

Taking the condition $2^{1/2}\theta_1 < \sqrt{\frac{\omega_\alpha}{|\cos(\pi\alpha)|}}\eta k < \frac{\theta_2}{2^{1/2}}$ into account, and bearing in mind (3.4), one has $2r_1 < \Phi(w) < \frac{r_2}{2}$. Since $\mu < \delta^*_{\lambda,g}$, we have

$$\mu < \frac{|\cos(\pi\alpha)|\theta_1^2 - \lambda k^2 T \max_{|x|\leq\theta_1} F(x)}{k^2 G^{\theta_1}}$$

and

$$\mu < \frac{|\cos(\pi\alpha)|\theta_2^2 - 2\lambda k^2 T \max_{|x|\leq\theta_2} F(x)}{2k^2 G^{\theta_2}},$$

namely $\frac{T\max_{|x|\leq\theta_1} F(x)+\frac{\mu}{\lambda}G^{\theta_1}}{\frac{|\cos(\pi\alpha)|}{k^2}\theta_1^2} < \frac{1}{\lambda}$ and $\frac{2T\max_{|x|\leq\theta_2} F(x)+2\frac{\mu}{\lambda}G^{\theta_2}}{\frac{|\cos(\pi\alpha)|}{k^2}\theta_2^2} < \frac{1}{\lambda}$, therefore bearing in mind that $G_\eta = 0$, one has

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1])}\Psi(u)}{r_1} &= \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1])}\int_0^T\left[F(u(t))+\frac{\mu}{\lambda}G(u(t))\right]dt}{r_1}\\ &\leq \frac{T\max_{|x|\leq\theta_1}F(x)+\frac{\mu}{\lambda}G^{\theta_1}}{\frac{|\cos(\pi\alpha)|}{k^2}\theta_1^2}\\ &< \frac{1}{\lambda} < \frac{2}{3}\frac{\frac{T}{\Gamma(2-\alpha)\eta}\int_0^{\Gamma(2-\alpha)\eta}F(x)dx + T\frac{\mu}{\lambda}G\eta}{\omega_\alpha\eta^2}\\ &\leq \frac{2}{3}\frac{\Psi(w)}{\Phi(w)},\end{aligned}$$

and

$$\begin{aligned}\frac{2\sup_{u\in\Phi^{-1}(]-\infty,r_2])}\Psi(u)}{r_2} &= \frac{2\sup_{u\in\Phi^{-1}(]-\infty,r_2])}\int_0^T\left[F(u(t))+\frac{\mu}{\lambda}G(u(t))\right]dt}{r_2}\\ &\leq \frac{2T\max_{|x|\leq\theta_2}F(x)+2\frac{\mu}{\lambda}G^{\theta_2}}{\frac{|\cos(\pi\alpha)|}{k^2}\theta_2^2}\\ &< \frac{1}{\lambda} < \frac{2}{3}\frac{\frac{T}{\Gamma(2-\alpha)\eta}\int_0^{\Gamma(2-\alpha)\eta}F(x)dx + T\frac{\mu}{\lambda}G\eta}{\omega_\alpha\eta^2}\\ &\leq \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.\end{aligned}$$

Therefore, (b_1) and (b_2) of Theorem 2.11 are verified.

Finally, we verify that $\Phi-\lambda\Psi$ satisfies the assumption *2.* of Theorem 2.11. Let u^* and $u^{\star\star}$ be two local minima for $\Phi-\lambda\Psi$. Then u^* and $u^{\star\star}$ are critical points for $\Phi-\lambda\Psi$, and so, they are solutions for the problem (1.1). Since $f(x) \geq 0$ for all $x \in \mathbb{R}^+\cup\{0\}$, from the Weak Maximum Principle (see for instance [17]) we deduce $u^\star(x) \geq 0$ and $u^{\star\star}(t) \geq 0$ for every $t \in [0, T]$. So, it follows that $su^\star + (1-s)u^{\star\star} \geq 0$ for all $s \in [0,1]$, and that $f(su^\star + (1-s)u^{\star\star}) \geq 0$, and consequently, $\Psi(su^\star + (1-s)u^{\star\star}) \geq 0$ for all $s \in [0,1]$.

By using Theorem 2.11, for every

$$\lambda \in \left]\frac{3}{2}\frac{\Phi(w)}{\Psi(w)},\ \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)},\ \frac{r_2/2}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\right\}\right[,$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which are the solutions of the problem (1.1) and the conclusion is achieved. □

Finally, we prove Theorems 1.1 and 1.2 in Introduction.

Proof of Theorem 1.1: Fix $\lambda > \lambda^* := \frac{\omega_\alpha \eta^2}{\frac{T}{\Gamma(2-\alpha)\eta}\int_0^{\Gamma(2-\alpha)\eta} F(x)dx}$ for some $\eta > 0$. Recalling that

$$\liminf_{\xi\to 0}\frac{F(\xi)}{\xi^2} = 0,$$

there is a sequence $\{\theta_n\} \subset]0,+\infty[$ such that $\lim_{n\to\infty}\theta_n = 0$ and

$$\lim_{n\to\infty}\frac{\max_{|\xi|\leq\theta_n} F(\xi)}{\theta_n^2} = 0.$$

Indeed, one has

$$\lim_{n\to\infty}\frac{\max_{|\xi|\leq\theta_n} F(\xi)}{\theta_n^2} = \lim_{n\to\infty}\frac{F(\xi_{\theta_n})}{\xi_{\theta_n}^2}\frac{\xi_{\theta_n}^2}{\theta_n^2} = 0,$$

where $F(\xi_{\theta_n}) = \max_{|\xi|\leq\theta_n} F(\xi)$.

Hence, there exists $\overline{\theta} > 0$ such that

$$\frac{\max_{|\xi|\leq\overline{\theta}} F(\xi)}{\overline{\theta}^2} < \min\left\{\frac{|\cos(\pi\alpha)|}{\Gamma(2-\alpha)k^2\omega_\alpha}\frac{\int_0^{\Gamma(2-\alpha)\eta} F(x)dx}{\eta^3};\ \frac{|\cos(\pi\alpha)|}{\lambda T k^2}\right\}$$

and $\sqrt{\frac{|\cos(\pi\alpha)|}{\omega_\alpha}}\frac{\overline{\theta}}{k} < \eta$.

The conclusion follows by using Theorem 3.1. □

Proof of Theorem 1.2: Our aim is to employ Theorem 3.2 by choosing $T = 2$, $\theta_2 = 1$ and $\eta = 2$. Therefore, we see that

$$\frac{3}{2}\frac{\omega_\alpha\eta^2}{\frac{T}{\Gamma(2-\alpha)\eta}\int_0^{\Gamma(2-\alpha)\eta} F(x)dx} = \frac{24\frac{\Gamma^3(2-\alpha)}{\Gamma(4-2\alpha)}2^{1-2\alpha}(2^{2\alpha-1}-1)}{\int_0^{2\Gamma(2-\alpha)}\int_0^x f(\xi)d\xi dx}$$

and

$$\frac{|\cos(\pi\alpha)|}{Tk^2}\frac{\theta_2^2}{2\max_{|x|\leq\theta_2} F(x)} = \frac{|\cos(\pi\alpha)|}{4\frac{2^{2\alpha-1}}{\Gamma^2(\alpha)(2(\alpha-1)+1)}}\frac{1}{\int_0^1 f(\xi)d\xi}.$$

Moreover, since $\lim_{t\to 0^+}\frac{f(t)}{t} = 0$, one has

$$\lim_{t\to 0^+}\frac{\int_0^t f(\xi)d\xi}{t^2} = 0.$$

Then, there exists a positive constant $\theta_1 < 2\sqrt{2}\sqrt{\frac{\frac{\Gamma^2(2-\alpha)}{\Gamma(4-2\alpha)}(2^{2\alpha-1}-1)}{|\cos(\pi\alpha)|}}\frac{1}{\Gamma(\alpha)\sqrt{2(\alpha-1)+1}} < \frac{1}{2}$ for some $\frac{1}{2} < \alpha \leq 1$, such that

$$\frac{\int_0^{\theta_1} f(\xi)d\xi}{\theta_1^2} < \frac{1}{12}\frac{|\cos(\pi\alpha)|}{\frac{1}{\Gamma^2(\alpha)(2(\alpha-1)+1)}\frac{4\Gamma^3(2-\alpha)}{\Gamma(4-2\alpha)}(2^{2\alpha-1}-1)}\int_0^{2\Gamma(2-\alpha)}\int_0^x f(\xi)d\xi dx$$

and

$$\frac{1}{2\int_0^1 f(\xi)d\xi} < \frac{\theta_1^2}{\int_0^{\theta_1} f(\xi)d\xi}.$$

Finally, a simple computation shows that all assumptions of Theorem 3.2 are satisfied. The conclusion follows from Theorem 3.2. □

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