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GENERALIZED EXTENSION OF THE QUASILINEARIZATION METHOD FOR RIEMANN-LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATIONS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. Existence and comparison results of the linear and nonlinear Riemann-Liouville fractional differential equations of order q, $0 < q < 1$, are recalled and modified where necessary. Using upper and lower solutions, an extension of the generalized quasilinearization method is developed for decomposed nonlinear fractional differential equations of order q containing generalized convex, concave, and nondifferentiable partitions. Quadratic convergence, and generalizations thereof, to the unique solution is proved via weighted sequences.

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1. INTRODUCTION

Fractional differential equations have various applications in widespread fields of science, such as in engineering [10], chemistry [11, 19, 20], physics [3, 4, 12], and others [13, 14]. In the majority of the literature existence results for Riemann-Liouville fractional differential equations are proven by a fixed point method. Initially we will recall existence by lower and upper solution method, which will be useful to developing our main results. Despite there being a number of existence theorems for nonlinear fractional differential equations, much as in the integer order case, this does not necessarily imply that calculating a solution explicitly will be routine, or even possible. Therefore, it may be necessary to employ an iterative technique to numerically approximate a needed solution. In this paper we construct such a method.

The iterative technique we construct is an extension of the generalized quasilinearization method for the nonlinear Riemann-Liouville fractional differential equation of order q, where $0 < q < 1$. The quasilinearization method was first developed in [1, 2, 18], but the method we construct is more closely related to those found in [17], that is a generalized quasilinearization method via lower and upper solutions. This

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method is very similar to the monotone method in that we construct monotone sequences from linear equations based on upper and lower solutions, which converge uniformly to the unique solution of the nonlinear equation. The difference is that the quasilinearization method employs a stronger hypothesis, but this results in a stronger result than the monotone method. In particular, for the quasilinearization method we require the nonlinear forcing function to be convex (or concave) as opposed to merely one-sided Lipschitzian. In the process, we are guaranteed that the constructed sequences converge quadradically to the unique solution. Note, uniqueness is not implied generally with the monotone method.

There are notable complications that arise when developing the quasilinearization method for Riemann-Liouville fractional differential equations. First of all, the iterates of the constructed sequences are solutions to the linear fractional differential equation with variable coefficients. The solution of this equation is quite unwieldy, therefore we will recall existence, comparison, and inequality results for this case, including a generalized Gronwall type inequality, which will be paramount to our main result. Another complication that stems from using the Riemann-Liouville derivative is that, in general, the sequences we construct, $\{\alpha_n\}, \{\beta_n\}$ do not converge uniformly to the unique solution, but the weighted sequences $\{t^{1-q}\alpha_n\}$, $\{t^{1-q}\beta_n\}$ converge uniformly and quadratically to $t^{1-q}x$, where x is the unique solution of the original equation.

We note that basic quasilinearization techniques have been established for the standard nonlinear Riemann-Liouville fractional differential equation in [5], that is, the cases where the nonlinear function f is convex, concave, and a final case where f is neither convex nor concave but where there exists a function ϕ such that $f + \phi$ is convex. The further case where the nonlinear function is not necessarily convex nor concave, but can be split into two functions, say $f + g$, such that f is convex and g is concave was considered in [9]. In this paper we take this generalization one step further and split the nonlinear function into three functions $f + g + h$, where f and g can be made convex and concave respectively. Much like in [5] this means there exist functions ϕ and ψ such that $f + \phi$ and $g + \psi$ are convex and concave respectively.

What makes our method an extension of the generalized quasilinearization method seen in [9] is that the function h is neither concave, convex, nor even differentiable. In our main results we assume that h is only Lipschitz, which fundamentally changes the typical quasilinearization method described above. In our first method the iterates are not even constructed by linear solutions, which is contrary to both methods described above, but ensures that convergence to the unique solution is quadratic. In our second method we are able to construct linear iterates, but at the cost of quadratic convergence, in this case convergence is only semi-quadratic. In our final case, we relax the hypothesis on h and assume it is only nonincreasing. In doing so we introduce intertwined sequences and convergence is only weakly quadratic. Due to these complexities, the methods we construct herein are not strictly quasilinearizations, but can instead be seen as bridging generalizations of both the quasilinearization and monotone methods.

2. PRELIMINARY RESULTS

In this section we consider results regarding the Riemann-Liouville (R-L) differential equations of order q, $0 < q < 1$. Specifically, we recall existence and comparison results which will be used in our main result. In the next section, we will apply these preliminary results to develop extensions of the generalized quasilinearization method for R-L fractional differential equations of order q. Note, for simplicity we only consider results on the interval $J = (0, T]$, where $T > 0$. Further, we will let $J_0 = [0, T]$, that is $J_0 = \bar{J}$.

Definition 2.1. Let $p = 1 - q$, a function $\phi(t) \in C(J, \mathbb{R})$ is a C_p function if $t^p\phi(t) \in C(J_0, \mathbb{R})$. The set of C_p functions is denoted $C_p(J, \mathbb{R})$. Further, given a function $\phi(t) \in C_p(J, \mathbb{R})$ we call the function $t^p\phi(t)$ the continuous extension of $\phi(t)$.

Now we define the R-L integral and derivative of order q on the interval J.

Definition 2.2. Let $\phi \in C_p(J, \mathbb{R})$, then $D_t^q\phi(t)$ is the q-th R-L derivative of ϕ with respect to $t \in J$ defined as

$$D_t^q\phi(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t (t-s)^{-q}\phi(s)ds,$$

and $I_t^q\phi(t)$ is the q-th R-L integral of ϕ with respect to $t \in J$ defined as

$$I_t^q\phi(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\phi(s)ds.$$

Note that in cases where the initial value may be different, or ambiguous, we will write out the definition explicitly. The next definition is related to the solution of linear R-L fractional differential equations and is also of great importance in the study of the R-L derivative.

Definition 2.3. The Mittag-Leffler function with parameters $\alpha, \beta \in \mathbb{R}$, denoted $E_{\alpha,\beta}$, is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

which is entire for $\alpha, \beta > 0$.

The next result gives us that the q-th R-L integral of a C_p continuous function is also a C_p continuous function. This result will give us that the solutions of R-L differential equations are also C_p continuous.

Lemma 2.4. *Let $f \in C_p(J,\mathbb{R})$, then $I_t^q f(t) \in C_p(J,\mathbb{R})$, i.e. the q-th integral of a C_p continuous function is C_p continuous.*

Note the proof of this theorem for $q \in \mathbb{R}^+$ can be found in [8]. Now we consider results for the nonhomogeneous linear R-L differential equation,

$$D_t^q x(t) = y(t)x(t) + z(t), \tag{2.1}$$

with initial condition

$$t^p x(t)\big|_{t=0} = x^0/\Gamma(q),$$

where x^0 is a constant, $y \in C(J_0,\mathbb{R})$, and $z \in C_p(J,\mathbb{R})$.

Theorem 2.5. *If $y \in C(J_0,\mathbb{R})$ and $z \in C_p(J,\mathbb{R})$ then equation (2.1) has a unique solution $x \in C_p(J,\mathbb{R})$, given explicitly by*

$$x(t) = \sum_{k=0}^{\infty} \frac{x^0}{\Gamma(q)} T_y^k\big[t^{q-1}\big] + T_y^k\big[I_t^q z(t)\big],$$

which converges uniformly on J and where T_y is the operator defined by

$$T_y\phi(t) = I_t^q y(t)\phi(t).$$

The proof of this theorem can be found in [6, 7], with the current refinements found in [5]. Note that if $z(t) = 0$ for all $t \in J$ then we get that

$$x(t) = \frac{x^0}{\Gamma(q)} \sum_{k=0}^{\infty} T_y^k\big[t^{q-1}\big].$$

In many cases we may have an explicit form of y that may prove too unwieldy to place in a subscript. In this case we will use the following notation

$$\mathcal{E}(y,f) = \sum_{k=0}^{\infty} T_y^k\big[f\big],$$

and since the case where $f = t^{q-1}$ occurs so often we will define $\mathcal{E}$ with a single parameter to be this case. That is $\mathcal{E}(y) = \mathcal{E}(y,t^{q-1})$. Therefore the solution of (2.1) can be written as

$$x(t) = \frac{x^0}{\Gamma(q)}\mathcal{E}(y) + \mathcal{E}(y, I_t^q z). \tag{2.2}$$

Further, if y is identically a constant, say λ, it can be shown that (2.2) can be expressed as

$$x(t) = x^0 t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) z(s)\, ds.$$

This is the result discussed in [16]; hence Theorem 2.5 generalizes the constant coefficient case, as expected.

Next, we recall a result we will utilize extensively in our proceeding comparison and existence results, and likewise in the construction of the quasilinearization method. We note that this result is similar to the well known comparison result found in literature, as in [16], but we do not require the function to be Hölder continuous of order $\lambda > q$.

Lemma 2.6. *Let $m \in C_p(J, R)$ be such that for some $t_1 \in J$ we have $m(t_1) = 0$ and $m(t) \leq 0$ for $t \in (0, t_1]$. Then*

$$D_t^q m(t)\big|_{t=t_1} \geq 0.$$

The proof of this lemma can be found in [8], along with further discussion as to why and how we weaken the Hölder continuous requirement of this known comparison result. We use this lemma in the proof of the later main comparison result, which will be critical in the construction of the quasilinearization method. First, we recall the nonlinear R-L fractional differential equation.

$$\begin{aligned} D_t^q x &= f(t, x), \\ t^p x(t)\big|_{t=0} &= x^0/\Gamma(q), \end{aligned} \tag{2.3}$$

where $f \in C(J_0 \times \mathbb{R}, \mathbb{R})$ and x^0 is a constant. Note that a solution $x \in C_p(J, \mathbb{R})$ of (2.3) also satisfies the equivalent R-L integral equation

$$x(t) = \frac{x^0}{\Gamma(q)} t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds. \tag{2.4}$$

Thus, if $f \in C(J_0 \times \mathbb{R}, \mathbb{R})$ then (2.3) is equivalent to (2.4). See [13, 16] for details. Now we will recall a Peano type existence theorem for equation (2.3).

Theorem 2.7. *Suppose $f \in C(R_0, \mathbb{R})$ and $|f(t, x)| \leq M$ on R_0, where*

$$R_0 = \{(t, x) \ : \ |t^p x(t) - x^0| \leq \eta, t \in J_0\}$$

Then the solution of (2.3) exists on J.

This result is presented in [16], and in [8] it was proven that the solution can be extended to all of J, and the set R_0 was modified for our succeeding results regarding existence by method of upper and lower solutions. In the direction of this result we will consider the following comparison result, which will in turn yield a general Gronwall type inequality.

Theorem 2.8. *Let $f \in C(J_0 \times \mathbb{R}, \mathbb{R})$ and let $v, w \in C_p(J, \mathbb{R})$ be lower and upper solutions of (2.3), i.e.*

$$\begin{aligned} D_t^q v &\leq f(t, v), \\ t^p v(t)\big|_{t=0} &= v^0/\Gamma(q) \leq x^0/\Gamma(q), \end{aligned}$$

and

$$D_t^q w \geq f(t, w),$$

$$t^p w(t)\Big|_{t=0} = w^0/\Gamma(q) \geq x^0/\Gamma(q).$$

If f satisfies the following Lipschitz condition

$$f(t,x) - f(t,y) \leq L(x-y), \quad \text{when } x \geq y,$$

where $L > 0$, then $v(t) \leq w(t)$ on J.

The proof follows as in [16] with appropriate modifications, specifically we use Lemma 2.6 and do not require local Hölder continuity of order $\lambda > q$. Next, we present a Gronwall type inequality for R-L fractional differential equations. A similar result in terms of fractional integral equations can be found in [7].

Theorem 2.9. *Let $v, z \in C_p(J,\mathbb{R})$ and $y \in C(J_0,\mathbb{R}^+)$, and suppose that*

$$D_t^q v \leq y(t)v(t) + z(t).$$

Then

$$v(t) \leq \frac{v^0}{\Gamma(q)}\mathcal{E}(y) + \mathcal{E}(y, I_t^q z).$$

The proof follows directly from Theorem 2.5 and Theorem 2.8. If y is identically a constant $\lambda \geq 0$, then we get the following corollary.

Corollary 2.10. *Let $v, z \in C_p(J,\mathbb{R})$ and let $\lambda \geq 0$ be a constant, and suppose that*

$$D_t^q v \leq \lambda v(t) + z(t),$$

then

$$v(t) \leq v^0 t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) z(s)\, ds.$$

Now, we will recall a result that gives us existence of a solution to (2.3) via lower and upper solutions.

Theorem 2.11. *Let $v, w \in C_p(J,\mathbb{R})$ be lower and upper solutions of (2.3) such that $v(t) \leq w(t)$ on J and let $f \in C(\Omega,\mathbb{R})$, where Ω is defined as*

$$\Omega = \{(t,y) : v(t) \leq y \leq w(t), t \in J_0\}.$$

Then there exists a solution $x \in C_p(J,\mathbb{R})$ of (2.3) such that $v(t) \leq x(t) \leq w(t)$ on J.

The proof of this Theorem can be found in [8]. We also note a uniqueness result here which is comparable to the analogous result for ordinary differential equations. As one might expect, if f satisfies the Lipschitz condition found in Theorem 2.8, then the solution x of (2.3) is unique. We mention this result here since it will be necesessary in the construction of the quasilinearization method.

3. EXTENSION OF GENERALIZED QUASILINEARIZATION

In this section we develop iterative techniques that extend the generalized quasilinearization method. Our methods will construct iterates that will converge uniformly to the solution of the following nonlinear IVP,

$$
\begin{aligned}
D_t^q x &= f(t,x) + g(t,x) + h(t,x),\\
t^p x(t)\big|_{t=0} &= x^0/\Gamma(q).
\end{aligned}
\tag{3.1}
$$

For convenience we will denote $N(t,x) = f(t,x) + g(t,x) + h(t,x)$. For our purposes we will assume that f, g are twice differentiable in x and can be made convex and concave respectively. Here we further extend the method by supposing that h is not twice differentiable in x, but is merely Lipschitz.

For our first iterative method we consider the case where h only attains a right-sided Lipschitz condition. In this case we construct sequences from solutions of nonlinear fractional IVPs. From here we inductively show monotonicity, then that convergence of the t^p-weighted sequences is uniform and quadratic. Though it is contrary to the typical procedure to use nonlinear iterates for the quasilinearization method, in this case it ensures the weighted sequences converge quadratically, which is an integral precept of quasilinearization. With these nonlinear sequences comes a more involved procedure as we must use Theorem 2.11 at almost every step to prove existence as we proceed, which will not be necessary in our later methods as we will construct linear iterates there, but we will also lose quadradic convergence.

Theorem 3.1. *Assume that*

(A_1) $\alpha_0, \beta_0 \in C_p(J,\mathbb{R})$ *are lower and upper solutions of* (3.1) *with* $\alpha_0 \le \beta_0$ *on* J.

(A_2) $N \in C(J_0 \times \Omega, \mathbb{R})$, f_x, f_{xx}, g_x, g_{xx}, *exist, are continuous for* $(t,x) \in J_0 \times \Omega$, *where* $\Omega = \{x \in C_p(J,\mathbb{R}) \,|\, \alpha_0 \le x \le \beta_0\}$.

Further suppose h satisfies the Lipschitz condition

$$h(t,x) - h(t,y) \le k(x-y),$$

whenever $x \ge y$.

Then there exist monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$, *such that* $\{t^p\alpha_n\}$ *and* $\{t^p\beta_n\}$ *converge uniformly and quadradically to* $t^p x$, *where* x *is the unique solution of* (3.1).

Proof. A main construct of this method is that we need two functions that when added to f and g yield convex and concave functions respectively. So let ϕ, ψ be in $C^{0,2}(J_0 \times \Omega, \mathbb{R})$ such that $\phi_{xx} \ge 0, \psi_{xx} \le 0$ and satisfy

$$f_{xx}(t,x) + \phi_{xx}(t,x) \ge 0, \quad g_{xx}(t,x) + \psi_{xx}(t,x) \le 0. \tag{3.2}$$

We note, that due to (A_2) it is always possible to find such functions, and in Remark 3.2 below we will describe how one can construct such functions.

For convenience let $F(t,x) = f(t,x) + \phi(t,x)$ and $G(t,x) = g(t,x) + \psi(t,x)$. In light of (3.2) we have that

$$
\begin{aligned}
& f(t,x) \geq f(t,y) + F_x(t,y)(x-y) + \phi(t,y) - \phi(t,x), \\
& g(t,x) \leq g(t,y) + G_x(t,y)(x-y) + \psi(t,y) - \psi(t,x),
\end{aligned}
\tag{3.3}
$$

for $x \geq y$ and $x, y \in \Omega$. It also follows that

$$N(t,x) - N(t,y) \leq L(x-y),$$

for $(t,x) \in J_0 \times \Omega$. Further, since N is Lipschitz (3.1) has a unique solution $x \in \Omega$.

In the direction of constructing the monotone sequences consider the fractional differential equation

$$
\begin{aligned}
D_t^q u &= f(t,\alpha_0) + g(t,\alpha_0) + h(t,u) \\
&\quad + [F_x(t,\alpha_0) + G_x(t,\beta_0) - \phi_x(t,\beta_0) - \psi_x(t,\alpha_0)](u-\alpha_0) \\
&= U(t,u;\alpha_0,\beta_0), \\
t^p u(t)\big|_{t=0} &= x^0/\Gamma(q).
\end{aligned}
\tag{3.4}
$$

For simplicity we introduce the following notation, for any $\alpha, \beta \in \Omega$ let

$$\Lambda(\alpha,\beta) = [F_x(t,\alpha) + G_x(t,\beta) - \phi_x(t,\beta) - \psi_x(t,\alpha)].$$

We wish to show that a solution to (3.4) exists on Ω, so consider

$$D_t^q \alpha_0 \leq N(t,x) = U(t,\alpha_0;\alpha_0,\beta_0).$$

Thus, α_0 is a lower solution of (3.4). Considering a similar argument for β_0 and utilizing (3.3), the Mean Value Theorem, and the monotonicity of ϕ_x, ψ yields,

$$
\begin{aligned}
D_t^q \beta_0 &\geq f(t,\alpha_0) + g(t,\alpha_0) + h(t,\beta_0) + [F_x(t,\alpha_0) - G_x(t,\beta_0)](\beta_0 - \alpha_0) \\
&\quad + \phi(t,\alpha_0) - \phi(\beta_0) - \psi(t,\beta_0) + \psi(t,\alpha_0) \\
&= f(t,\alpha_0) + g(t,\alpha_0) + h(t,\beta_0) \\
&\quad + [F_x(t,\alpha_0) - G_x(t,\beta_0) - \phi(t,\xi) - \psi(t,\eta)](\beta_0 - \alpha_0) \\
&\geq U(t,\beta_0;\alpha_0,\beta_0),
\end{aligned}
$$

where $\alpha_0 \leq \xi, \eta \leq \beta_0$. Thus, β_0 is an upper solution of (3.4), and therefore, by Theorem 2.11, (3.4) has a solution α_1 existing on J with $\alpha_0 \leq \alpha_1 \leq \beta_0$, which is unique since $U(t,u;\alpha_0,\beta_0)$ is Lipschitz in u.

Next, we consider the following fractional differential equation, which will also aid in the construction of our sequences.

$$
\begin{aligned}
D_t^q v &= f(t,\beta_0) + g(t,\beta_0) + h(t,v) + \Lambda(\alpha_0,\beta_0)(v - \beta_0) \\
&= V(t,v;\alpha_0,\beta_0), \\
t^p v(t)\big|_{t=0} &= x^0/\Gamma(q).
\end{aligned}
\tag{3.5}
$$

We note that one can show that (3.5) has a unique solution β_1, with $\alpha_0 \le \beta_1 \le \beta_0$ on J in the same manner as the previous case.

Now we will show that $\alpha_1 \le x \le \beta_1$ on J. Once again utilizing (3.3), the Mean Value Theorem, and the monotonicity of ϕ_x, ψ, and G_x we obtain

$$\begin{aligned}
D_t^q \alpha_1 \le\ & N(t,\alpha_1) + [G_x(t,\beta_0) - G_x(t,\alpha_1) - \phi_x(\beta_0) - \psi(t,\alpha_0)](\alpha_1 - \alpha_0) \\
& + \phi(t,\alpha_1) - \phi(t,\alpha_0) + \psi(t,\alpha_1) - \psi(t,\alpha_0) \\
\le\ & N(t,\alpha_1) + [\phi_x(t,\xi) - \phi_x(\beta_0) + \psi(t,\eta) - \psi(t,\alpha_0)](\alpha_1 - \alpha_0) \\
\le\ & N(t,\alpha_1),
\end{aligned}$$

where $\alpha_0 \le \xi, \eta \le \alpha_1$. Implying that α_1 is a lower solution of (3.1). Further note by a similar argument we can show that β_1 is an upper solution of (3.1); thus, by Theorem 2.8, $\alpha_0 \le \alpha_1 \le x \le \beta_1 \le \beta_0$ on J. For the construction of the sequences in our iterative technique we will define each iterate to be the solution of the fractional differential equations

$$D_t^q \alpha_{n+1} = U(t, \alpha_{n+1}; \alpha_n, \beta_n), \tag{3.6}$$

$$D_t^q \beta_{n+1} = V(t, \beta_{n+1}; \alpha_n, \beta_n). \tag{3.7}$$

$$t^p \alpha_{n+1}\big|_{t=0} = t^p \beta_{n+1}\big|_{t=0} = x^0/\Gamma(q).$$

Letting the previous work be our basis step, suppose that up to some $k \ge 1$, that α_k and β_k exist, are unique, and that $\alpha_{k-1} \le \alpha_k \le x \le \beta_k \le \beta_{k-1}$ on J. Now we will show that α_{k+1} and β_{k+1} exist on J. Note that uniqueness follows from the Lipschitzian nature of M and K. To do so note that

$$\begin{aligned}
D_t^q \alpha_k &\le N(t,\alpha_k) + [\phi_x(t,\xi) - \phi_x(t,\beta_{k-1}) + \psi_x(t,\eta) - \psi_x(t,\alpha_{k-1})](\alpha_k - \alpha_{k-1}) \\
&\le N(t,\alpha_k) = U(t,\alpha_k;\alpha_k,\beta_k),
\end{aligned}$$

where $\alpha_{k-1} \le \xi, \eta \le \alpha_k$. Implying α_k is a lower solution of (3.6). Next, we can show that x is an upper solution of (3.6). To do so we use similar arguments to the ones used above,

$$\begin{aligned}
D_t^q x \ge\ & f(t,\alpha_k) + g(t,\alpha_k) + h(t,x) \\
& + [F_x(t,\alpha_k) + G_x(t,x) - \phi_x(\xi) - \psi_x(\eta)](x - \alpha_k) \\
\ge\ & f(t,\alpha_k) + g(t,\alpha_k) + h(t,x) + \Lambda(\alpha_k,\beta_k)(x - \alpha_k) \\
=\ & U(t,x;\alpha_k,\beta_k),
\end{aligned}$$

where $\alpha_k \le \xi, \eta \le x$. Thus, by Theorem 2.11 the solution of (3.6) for $k+1$ exists, is unique, and is such that $\alpha_k \le \alpha_{k+1} \le x$ on J. By a similar argument we can show that $x \le \beta_{k+1} \le \beta_k$ on J; thus by induction we have that $\alpha_{n-1} \le \alpha_n \le x \le \beta_n \le \beta_{n-1}$ on J for all $n \ge 1$.

Now we will show that the weighted sequences of continuous extensions $\{t^p\alpha_n\}$, $\{t^p\beta_n\}$ converge uniformly to t^px by an application of the Arzelá-Ascoli Theorem. We have that these sequences are uniformly bounded since

$$|t^p\alpha_n| \leq |t^p(\alpha_n - \alpha_0)| + |t^p\alpha_0| \leq |t^p(\beta_0 - \alpha_0)| + |t^p\alpha_0|,$$

for all $n \geq 1$, which is also true for $\{t^p\beta_n\}$. Then we get that the sequences are equicontinuous by following the same process as found in [21]. Therefore $\{t^p\alpha_n\}$ and $\{t^p\beta_n\}$ converge monotonically and uniformly on J_0. We claim that both sequences converge to t^px. To show this suppose $t^p\alpha_n \to t^p\alpha$ on J_0, then we have that $\alpha_n \to \alpha$ pointwise on J. Now, if we consider the integral form of α_{n+1} we have that

$$\begin{aligned} t^p\alpha_{n+1} = \frac{x^0}{\Gamma(q)} &+ \frac{t^p}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[f(s,\alpha_n) + g(s,\alpha_n) + h(s,\alpha_{n+1})\big]ds \\ &+ \frac{t^p}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Lambda(\alpha_n,\beta_n)(\alpha_{n+1}-\alpha_n)ds, \end{aligned}$$

which will converge uniformly to

$$t^p\alpha = \frac{x^0}{\Gamma(q)} + \frac{t^p}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[f(s,\alpha) + g(s,\alpha) + h(s,\alpha)\big]ds$$

on J_0. Implying that $\alpha = x$, and similarly we can show that $t^p\beta_n \to t^px$.

Finally, we will show that $\{t^p\alpha_n\}, \{t^p\beta_n\}$ converge quadradically on J_0. To do so, first let $A_n = x - \alpha_n$, and $B_n = \beta_n - x$. Then by the continuity of F, G, ϕ and ψ on $C(J_0 \times \Omega, \mathbb{R})$, there exist continuous functions $\mathcal{F}, \mathcal{G}, \Phi$, and Ψ such that $\mathcal{F}(t, t^px) = F(t,x)$, $\mathcal{G}(t,t^px) = G(t,x)$, $\Phi(t,t^px) = \phi(t,x)$, and $\Psi(t,t^px) = \psi(t,x)$ which gives us that

$$F_{xx}(t,x) = t^{2p}\mathcal{F}_{xx}(t,t^px),$$

and the same result for the remaining three functions. Using (3.3), the monotonicity of F, G, ϕ, ψ, and the Mean Value theorem we obtain

$$\begin{aligned} D_t^qA_{n+1} &\leq h(t,x) - h(t,\alpha_{n+1}) + [G_x(t,\alpha_n) + F_x(t,x)]A_n - \phi(t,x) + \phi(t,\alpha_n) \\ &\quad + \psi(t,\alpha_n) - \phi(t,x) - \Lambda(\alpha_n,\beta_n)(\alpha_{n+1}-\alpha_n) \\ &\leq (K + \Lambda(\alpha_n,\beta_n))A_{n+1} + [F_x(t,x) - F_x(\alpha_n) + G_x(t,\alpha_n) - G_x(t,\beta_n)]A_n \\ &\quad + [\phi_x(t,\beta_n) - \phi_x(t,\xi_1) + \psi(t,\alpha_n) - \psi(t,\eta_1)]A_n \\ &\leq (k+m)A_{n+1} + F_{xx}(t,\xi_2)A_n^2 - G_{xx}(t,\eta_2)A_n(A_n+B_n) \\ &\quad + [\phi_x(t,\beta_n) - \phi_x(t,\alpha_n) + \psi(t,\alpha_n) - \psi(t,x)]A_n \\ &= (k+m)A_{n+1} + [\mathcal{F}_{xx}(t,t^p\xi_2) - \Psi_{xx}(t,t^p\eta_3)]t^{2p}A_n^2 \\ &\quad + [\Phi_{xx}(t,t^p\xi_3) - \mathcal{G}_{xx}(t,t^p\eta_2)]t^{2p}A_n(A_n+B_n) \\ &\leq (k+m)A_{n+1} + Rt^{2p}A_n^2 + (S/2)t^{2p}(3A_n^2 + B_n^2), \end{aligned}$$

where $\alpha_n \leq \xi_1, \xi_2, \eta_1, \eta_3 \leq x$ and $\alpha_n \leq \xi_3, \eta_2 \leq \beta_n$. Further, m, R, and S are bounds of Λ, $\mathcal{F}_{xx} - \Psi_{xx}$ and $\Phi_{xx} - \mathcal{G}_{xx}$ on J respectively. Thus, by Theorem 2.10 we have that,

$$\begin{aligned} t^p A_{n+1} &\leq \frac{t^p}{2} \int_0^t (t-s)^{q-1} E_{q,q}\big([k+m](t-s)^q\big) s^{2p} \big[(2R+3S)A_n^2 + SB_n^2\big] ds \\ &\leq \frac{t^p}{2} \big[(2R+3S)\|t^p A_n\|^2 + S\|t^p B_n\|^2\big] \int_0^t \sum_{i=0}^{\infty} \frac{(k+m)^i (t-s)^{qi+q-1}}{\Gamma(qi+q)} ds \\ &\leq \frac{t^p E_{q,1}([k+m]t^q)}{2(k+m)} \big[(2R+3S)\|t^p A_n\|^2 + S\|t^p B_n\|^2\big]. \end{aligned}$$

Here $\|\cdot\|$ is the uniform norm on $C(J_0, \mathbb{R})$. Thus we have that $\{t^p \alpha_n\}$ converges quadradically in the following way,

$$\|t^p(x - \alpha_{n+1})\| \leq \frac{T^p \widetilde{E}}{2(k+m)} \big[(2R+3S)\|t^p(x-\alpha_n)\|^2 + S\|t^p(\beta_n - x)\|^2\big],$$

where

$$\widetilde{E} = E_{q,1}([k+m]T^q).$$

Similarly, we can show that

$$\|t^p(\beta_{n+1} - x)\| \leq \frac{T^p \widetilde{E}}{2(k+m)} \big[(2S+3R)\|t^p(\beta_n - x)\|^2 + R\|t^p(x-\alpha_n)\|^2\big].$$

Thus finishing the proof. $\square$

Remark 3.2. We note that it is always possible to find a ϕ and ψ to satisfy (3.2). For example if $f(t,x)$ is not convex, then as we did previously, we note there exists a continuous function such that $\tilde{f}(t, t^p x) = f(t,x)$. Now let $A > 0$ be such that

$$\max_{J_0 \times \Omega} \{\tilde{f}_{xx}(t, t^p x)\} = -A < 0.$$

Then we need only choose

$$\phi(t,x) = At^{2p}x^2,$$

in order to meet the requirements of (3.2).

In our next method we construct iterates from solutions of linear fractional IVPs; in doing so, we develop a method that more closely resembles the quasilinearization method, but we also lose quadratic convergence for semi-quadratic convergence. In this case we must also strengthen the condition of h and assume it attains a two-sided Lipschitz condition. Though convergence will not be as fast in this case, in practice it should be far easier to implement as the iterates will be linear, and thus can be computed explicitly, which may not be the case in Theorem 3.1. This method also acts as a generalization of both the quasilinearization and monotone methods. We will discuss this in more detail following the proof.

Theorem 3.3. *Suppose hypotheses* (A_1) *and* (A_2) *from Theorem* 3.1 *are satisfied. If* h *satisfies the two-sided Lipscitz condition*

$$-k(x-y) \le h(t,x) - h(t,y) \le k(x-y),$$

for $x \ge y$*, then there exist monotone sequences* $\{\alpha_n\}$*,* $\{\beta_n\}$*, such that* $\{t^p\alpha_n\}$ *and* $\{t^p\beta_n\}$ *converge uniformly and semi-quadradically to* $t^p x$*, where* x *is the unique solution of* (3.1)*.*

Proof. The proof of this theorem follows in much the same way as Theorem 3.1. The sequences constructed in this case are solutions of the linear fractional differential equations

$$\begin{aligned} D_t^q \alpha_{n+1} &= N(t,\alpha_n) + [\Lambda(\alpha_n,\beta_n) - k](\alpha_{n+1} - \alpha_n), \\ D_t^q \beta_{n+1} &= N(t,\beta_n) + [\Lambda(\alpha_n,\beta_n) - k](\beta_{n+1} - \beta_n), \\ t^p\alpha_{n+1}\big|_{t=0} &= t^p\beta_{n+1}\big|_{t=0} = x^0/\Gamma(q), \end{aligned}$$

where Λ is defined as previously. Monotonicity and uniform convergence are proved in much the same way as the previous theorem, but we will show an example of where the two-sided Lipschitz condition is required. We shall consider the proof showing that $\alpha_1 \le x$. To do so note that (3.3) is still true in this case, using this with monotonicity, the Lipschitian nature of h, and the Mean Value Theorem we have that

$$\begin{aligned} D_t^q x &\ge f(t,\alpha_0) + g(t,\alpha_0) + h(t,x) \\ &\quad + [F_x(t,\alpha_0) + G_x(t,x) - \phi(t,\xi) - \psi(t,\eta)](x-\alpha_0) \\ &\ge N(t,\alpha_0) + h(t,x) - h(t,\alpha_0) + \Lambda(\alpha_0,\beta_0)(x-\alpha_0) \\ &\ge N(t,\alpha_0) + [\Lambda(\alpha_0,\beta_0) - k](x-\alpha_0), \end{aligned}$$

where $A_0 \le \xi,\eta \le x$. This implies by Theorem 2.8 that $\alpha_1 \le x$.

Now we will show that convergence is semi-quadratic. To do so, let A_n, B_n, $\mathcal{F}$, $\mathcal{G}$, Φ, Ψ be defined as previously. Then using (3.3), monotonicity, the Lipschitzian nature of h, and the Mean Value theorem we obtain

$$\begin{aligned} D_t^q A_{n+1} &\le [F_x(t,x) + G_x(t,\alpha_n) - \phi_x(t,\xi_1) - \phi_x(t,\eta_1) + k]A_n \\ &\quad - [\Lambda(\alpha_n,\beta_n) - k](A_{n+1} - A_n) \\ &\le F_{xx}(t,\xi_2)A_n^2 - G_{xx}(t,\eta_2)A_n(A_n+B_n) + [\Lambda(\alpha_n,\beta_n) - k]A_{n+1} \\ &\quad + [\phi(t,\beta_n) - \phi(t,\alpha_n) + \phi(t,\alpha_n) - \phi(t,x) + 2k]A_n \\ &\le [\mathcal{F}_{xx}(t,\xi_2) - \Psi_{xx}(t,\eta_3)]t^{2p}A_n^2 + [\Phi(t,\xi_3) - \mathcal{G}_{xx}(t,\eta_2)]t^{2p}A_n(A_n+B_n) \\ &\quad + [m-k]A_{n+1} + 2kA_n \\ &\le Rt^{2p}A_n^2 + (S/2)t^{2p}(3A_n^2 + B_n^2) + [m-k]A_{n+1} + 2kA_n. \end{aligned}$$

Where $\alpha_n \le \xi_1, \xi_2, \eta_1, \eta_3 \le x$ and $\alpha_n \le \xi_3, \eta_2 \le \beta_n$, and where m, R, and S are bounds of Λ, $\mathcal{F}_{xx} - \Psi_{xx}$ and $\Phi_{xx} - \mathcal{G}_{xx}$ on J respectively. By Theorem 2.10 we have

$$\begin{aligned}
t^p A_{n+1} &\le t^p \int_0^t (t-s)^{q-1} E_{q,q}\big([m-k](t-s)^q\big)\big[s^{2p}[(R+\tfrac{3}{2}S)A_n^2 + \tfrac{1}{2}SB_n^2] + 2kA_n\big]ds \\
&\le \frac{t^p E_{q,1}([m-k]t^q)}{2(m-k)}\big[(2R+3S)\|t^p A_n\|^2 + S\|t^p B_n\|^2\big] \\
&\quad + 2kt^p\|t^p A_n\| \sum_{\ell=0}^{\infty} \frac{(m-k)^\ell}{\Gamma(\ell q+q)} \int_0^t (t-s)^{\ell q+q-1} s^{q-1} ds \\
&\le \frac{T^p \bar{E}}{2(m-k)}\big[(2R+3S)\|t^p A_n\|^2 + S\|t^p B_n\|^2\big] + \frac{2kt^p\|t^p A_n\|}{m-k} \sum_{\ell=1}^{\infty} \frac{(m-k)^\ell \Gamma(q)}{\Gamma(\ell q+q)} t^{\ell q+q-1} \\
&\le \frac{T^p \bar{E}}{2(m-k)}\big[(2R+3S)\|t^p A_n\|^2 + S\|t^p B_n\|^2\big] + \frac{2k\|t^p A_n\|}{m-k} E_{q,q}([m-k]t^q),
\end{aligned}$$

where

$$\bar{E} = E_{q,1}([m-k]T^q).$$

Thus, $\{t^p \alpha_n\}$ converges semi-quadratically in the following sense

$$\|t^p(x-\alpha_{n+1})\| \le E_1\big[(2R+3S)\|t^p(x-\alpha_n)\|^2 + S\|t^p(\beta_n - x)\|^2\big] + 2kE_2\|t^p(x-\alpha_n)\|,$$

where

$$E_1 = \frac{T^p \bar{E}}{2(m-k)}, \quad \text{and} \quad E_2 = \frac{E_{q,q}([m-k]T^q)}{m-k}.$$

Similarly, we can show that $\{t^p \beta_n\}$ converges semi-quadratically in the following sense,

$$\|t^p(\beta_{n+1}-x)\| \le E_1\big[(2S+3R)\|t^p(\beta_n - x)\|^2 + R\|t^p(x-\alpha_n)\|^2\big] + 2kE_2\|t^p(\beta_n - x)\|.$$

This finishes the proof. $\square$

We note that if $h = 0$ in the previous method, then convergence will be quadratic and we will have the quasilinearization method as seen in [9]. Likewise, if $f + g = 0$, we will have a special case of the monotone method where convergence to the unique solution is linear. Therefore, Theorem 3.3 can be seen as a generalization of both the quasilinearization and monotone methods.

In the next technique we extend this idea a little further. Here, we assume that h is only nonincreasing in x. The iterates are still constructed from the solution of linear fractional IVPs, but the sequences will be intertwined. That is,

$$\alpha_{2n} \le \beta_{2n+1} \le x \le \alpha_{2n+1} \le \beta_{2n}$$

on J for all $n \ge 0$. For this method to work we must add the assumption that $\alpha_0 \le \beta_1$ and $\alpha_1 \le \beta_0$. Further, while convergence of the weighted sequences will still be uniform, it will only be weakly quadratic. This method will act as a further generalization of both the quasilinearization and monotone methods.

Theorem 3.4. *Suppose hypotheses* (A_1) *and* (A_2) *from Theorem* 3.1 *are satisfied. If* h *is nonincreasing in* x*, then there exist intertwined sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$*, such that*

$$\alpha_0 \le \beta_1 \le \alpha_2 \le ... \le \beta_{2n-1} \le \alpha_{2n} \le x \le \beta_{2n} \le \alpha_{2n-1} \le \cdots \le \beta_2 \le \alpha_1 \le \beta_0,$$

provided $\alpha_0 \le \beta_1$ *and* $\alpha_1 \le \beta_0$*. The weighted sequences* $\{t^p\alpha_{2n}, t^p\beta_{2n+1}\}$, $\{t^p\beta_{2n}, t^p\alpha_{2n+1}\}$ *converge uniformly and monotonically to* t^px*, where* x *is the unique solution of* (3.1)*. Further, convergence of these* t^p*-weighted sequences is weakly quadratic.*

Proof. The sequences we construct in this case are unique solutions of the linear fractional differential equations,

$$\begin{aligned}
D_t^q\alpha_{2n} &= N(t,\alpha_{2n-1}) + \Lambda(\alpha_{2n-1},\beta_{2n-1})(\alpha_{2n}-\alpha_{2n-1})\\
D_t^q\alpha_{2n+1} &= N(t,\alpha_{2n}) + \Lambda(\beta_{2n},\alpha_{2n})(\alpha_{2n+1}-\alpha_{2n})\\
D_t^q\beta_{2n} &= N(t,\beta_{2n-1}) + \Lambda(\alpha_{2n-1},\beta_{2n-1})(\beta_{2n}-\beta_{2n-1})\\
D_t^q\beta_{2n+1} &= N(t,\beta_{2n}) + \Lambda(\beta_{2n},\alpha_{2n})(\beta_{2n+1}-\beta_{2n}),\\
t^p\alpha_n\big|_{t=0} &= t^p\beta_n\big|_{t=0} = x^0/\Gamma(q),
\end{aligned}$$

where α_0, β_0 are given in the hypothesis. First, note that N is still Lipschitz in this case, implying that (3.1) has a unique solution, x, on J. Further, (3.3) is true in this case as well. We also note that Λ has a mixed monotonicity property, that is, if $\xi \le \eta$ then by the monotonicity properties of F_x, G_x, ϕ_x, and ψ_x we get

$$\Lambda(\xi,y) \le \Lambda(\eta,y), \quad \text{and} \quad \Lambda(y,\xi) \ge \Lambda(y,\eta). \tag{3.8}$$

This property will simplify some of our following arguments.

We will begin by showing that $x \le \alpha_1$, to do so using similar arguments employed previously and the fact that h is nonincreasing in x we can show that

$$D_t^q x \le N(t,\alpha_0) + \Lambda(\beta_0,\alpha_0)(x-\alpha_0),$$

which by Theorem 2.8 implies that $x \le \alpha_1$ on J. Similarly we can show that $\beta_1 \le x$ on J, giving us that $\alpha_0 \le \beta_1 \le x \le \alpha_1 \le \beta_0$ on J. Using this as a basis step suppose $\alpha_{2k} \le \beta_{2k+1} \le x \le \alpha_{2k+1} \le \beta_{2k}$ on J is true up to some $k \ge 0$. Using previous arguments, we can show that $\alpha_{2k+2} \le x \le \beta_{2k+2}$ on J. To show $\beta_{2k+1} \le \alpha_{2k+2}$, we use previous arguments along with (3.8) to obtain

$$\begin{aligned}
D^q\beta_{2k+1} &\le N(t,\alpha_{2k+1}) + \Lambda(\beta_{2k+1},\alpha_{2k+1})(\beta_{2k+1}-\beta_{2k})\\
&\qquad + [F_x(t,\beta_{2k}) + G_x(t,\alpha_{2k+1}) - \phi_x(t,\xi) - \psi_x(t,\eta)](\beta_{2k}-\alpha_{2k+1})\\
&\le N(t,\alpha_{2k+1}) + \Lambda(\beta_{2k+1},\alpha_{2k+1})(\beta_{2k+1}-\alpha_{2k+1}),
\end{aligned}$$

where $\alpha_{2k+1} \le \xi, \eta \le \beta_{2k}$, which by Theorem 2.8 implies that $\beta_{2k+1} \le \alpha_{2k+2}$ on J. Similarly, we can show that $\beta_{2k+2} \le \alpha_{2k+1}$ on J. Further, we can show that

$\beta_{2k+3} \leq x \leq \alpha_{2k+3}$ on J, and using similar arguments as before we have that

$$\begin{aligned} D_t^q \beta_{2k+3} &\geq N(t, \alpha_{2k+1}) + \Lambda(\alpha_{2k+1}, \beta_{2k+1})(\beta_{2k+3} - \beta_{2k+2}) \\ &\quad + [F_x(t, \alpha_{2k+1}) + G_x(\beta_{2k+1}) - \phi_x(t, \xi) - \psi_x(t, \eta)](\beta_{2k+2} - \alpha_{2k+1}) \\ &\geq N(t, \alpha_{2k+1}) + \Lambda(\alpha_{2k+1}, \beta_{2k+1})(\beta_{2k+3} - \alpha_{2K+1}), \end{aligned}$$

where $\beta_{2k+2} \leq \xi, \eta \leq \alpha_{2k+1}$. So by Theorem 2.8, we have that $\alpha_{2k+2} \leq \beta_{2k+3}$ on J. Similarly, we can show that $\alpha_{2k+3} \leq \beta_{2k+2}$ on J, thus finally giving us that

$$\alpha_{2k} \leq \beta_{2k+1} \leq \alpha_{2k+2} \leq \beta_{2k+3} \leq x \leq \alpha_{2k+3} \leq \beta_{2k+2} \leq \alpha_{2k+1} \leq \beta_{2k}$$

on J, which inductively implies that $\alpha_{2n} \leq \beta_{2n+1} \leq x \leq \alpha_{2n+1} \leq \beta_{2n}$ on J for all $n \geq 0$.

That each weighted subsequence $\{t^p\alpha_{2n}\}, \{t^p\alpha_{2n+1}\}, \{t^p\beta_{2n}\}$, and $\{t^p\beta_{2n+1}\}$ converges uniformly to $t^p x$ where x is the unique solution to (3.1) is proved in a similar manner as in Theorem 3.1 using the Arzelá-Ascoli Theorem. Thus, we will instead consider the weakly quadratic convergence. Let $\mathcal{F}, \mathcal{G}, \Phi$, and Ψ be defined as previously. Let

$$A_n = (-1)^n(x - \alpha_n), \quad \text{and} \quad B_n = (-1)^n(\beta_n - x),$$

then using previous arguments we have

$$\begin{aligned} D_t^q A_{2n} &\leq \Lambda(\alpha_{2n-1}, \beta_{2n-1})A_{2n} + [\Lambda(\alpha_{2n-1}, \beta_{2n-1}) - \Lambda(x, \alpha_{2n-1})]A_{2n-1} + 2k \\ &\leq mA_{2n} + [\mathcal{F}_{xx}(t, \xi_1) - \Phi_{xx}(t, \eta_2)]t^{2p}A_{2n-1}^2 \\ &\quad + [\Psi_{xx}(t, \xi_2) - \mathcal{G}_{xx}(t, \eta_1)]t^{2p}(A_{2n-1} + B_{2n-1})A_{2n-1} + 2k \\ &\leq mA_{2n} + Rt^{2p}A_{2n-1}^2 + (S/2)t^{2p}(3A_{2n-1}^2 + B_{2n-1}^2) + 2k, \end{aligned}$$

where $x \leq \xi_1, \eta_2 \leq \alpha_{2n-1}$ and $\beta_{2n-1} \leq \xi_2, \eta_1 \leq \alpha_{2n-1}$; further m, R, S, and k are bounds of Λ, $\mathcal{F}_{xx} - \Psi_{xx}$ and $\Phi_{xx} - \mathcal{G}_{xx}$, and h on J respectively. Following the same lines as before we can show that

$$t^p A_{2n} \leq \frac{t^p}{2m} E_{q,1}(mt^q)\big[(2R + 3S)\|t^p A_{2n-1}\|^2 + S\|t^p B_{2n-1}\|^2 + 4k\big].$$

Therefore, the uniform convergence of $\{t^p\alpha_{2n}\}$ is weakly quadratic in the following sense

$$\|t^p(x - \alpha_{2n})\| \leq \frac{T^p\widehat{E}}{2m}\big[(2R + 3S)\|t^p(\alpha_{2n-1} - x)\|^2 + S\|t^p(x - \beta_{2n-1})\|^2 + 4k\big],$$

where $\widehat{E} = E_{q,1}(mT^q)$. We can also show that the convergence of the three remaining weighted subsequences is weakly quadratic as

$$\begin{aligned} \|t^p(\alpha_{2n+1} - x)\| &\leq \tfrac{T^p\widehat{E}}{2m}\big[(2S + 3R)\|t^p(x - \alpha_{2n})\|^2 + R\|t^p(\beta_{2n} - x)\|^2 + 4k\big] \\ \|t^p(x - \beta_{2n+1})\| &\leq \tfrac{T^p\widehat{E}}{2m}\big[(2R + 3S)\|t^p(\beta_{2n} - x)\|^2 + S\|t^p(x - \alpha_{2n})\|^2 + 4k\big] \\ \|t^p(\beta_{2n} - x)\| &\leq \tfrac{T^p\widehat{E}}{2m}\big[(2S + 3R)\|t^p(x - \beta_{2n-1})\|^2 + R\|t^p(\alpha_{2n-1} - x)\|^2 + 4k\big], \end{aligned}$$

which completes the proof. □

We remark that a similar result can be constructed from intertwined iterates of just alphas as solutions to the linear IVPs

$$D_t^q \alpha_{2n} = N(\alpha_{2n-1}) + \Lambda(\alpha_{2n-1}, \alpha_0)(\alpha_{2n} - \alpha_{2n-1}) \tag{3.9}$$
$$D_t^q \alpha_{2n+1} = N(\alpha_{2n}) + \Lambda(\beta_0, \alpha_{2n})(\alpha_{2n+1} - \alpha_{2n}),$$

with only having to assume $\alpha_0 \leq \alpha_2$. This will generate intertwined sequences of the type

$$\alpha_0 \leq \alpha_{2n} \leq x \leq \alpha_{2n+1} \leq \beta_0.$$

A similar result can also be constructed with just betas as well, assuming only $\beta_2 \leq \beta_0$. We note this because in practice it may prove difficult to construct sequences with the added assumptions found in Theorem 3.4, and in certain circumstances may be more achievable to do so using (3.9).

Finally, we note that if $h = 0$ in Theorem 3.4 then, as previously, convergence will be quadratic, and we will have the generalized quasilinearization method. Likewise if $f + g = 0$, then Theorem 3.4 will collapse into a generalized monotone method as found in [15]. Therefore, as claimed, the methods we constructed in this paper act as generalizations of both the quasilinearization and monotone methods.

REFERENCES

[1] R. Bellman. *Methods of Nonlinear Analysis*, volume II. Academic Press, New York, 1973.

[2] R. Bellman and R. Kalaba. *Quasilinearization and Nonlinear Boundary Value Problems.* American Elsevier, New York, 1965.

[3] M. Caputo. Linear models of dissipation whose Q is almost independent, II. *Geophy. J. Roy. Astronom*, 13:529–539, 1967.

[4] A. Chowdhury and C. I. Christov. Memory effects for the heat conductivity of random suspensions of spheres. *Proc. R. Soc. A*, 466:3253–3273, 2010.

[5] Z. Denton, P. W. Ng, and A. S. Vatsala. Quasilinearization method via lower and upper solutions for Riemann-Liouville fractional differential equations. *Nonlinear Dynamics and Systems Theory*, 11(3):239–251, 2011.

[6] Z. Denton and A. S. Vatsala. Fractional differential equations and numerical approximations. In G. S. Ladde, N. G. Medhin, C. Peng, and M. Sambandham, editors, *Proceedings of Neural, Parallel, and Scientific Computations*, volume 4, pages 119–123, Atlanta, GA, 2010. Dynamic Publishers.

[7] Z. Denton and A. S. Vatsala. Fractional integral inequalities and applications. *Computers and Mathematics with Applications*, 59:1087–1094, 2010.

[8] Z. Denton and A. S. Vatsala. Monotone iterative technique for finite systems of nonlinear Riemann-Liouville fractional differential equations. *Opuscula Mathematica*, 31(3):327–339, 2011.

[9] Z. Denton and A. S. Vatsala. Generalized quasilinearization method for RL fractional differential equations. *Nonlinear Studies*, 19(4):637–652, 2012.

[10] K. Diethelm and A. D. Freed. On the solution of nonlinear fractional differential equations used in the modeling of viscoplasticity. In F. Keil, W. Mackens, H. Vob, and J. Werther, editors, *Scientific Computing in Chemical Engineering II: Computational Fluid Dynamics, Reaction Engineering, and Molecular Properties*, pages 217–224. Heidelberg: Springer, 1999.

[11] W. G. Glöckle and T. F. Nonnenmacher. A fractional calculus approach to self similar protein dynamics. *Biophy. J.*, 68:46–53, 1995.

[12] R. Hilfer, editor. *Applications of Fractional Calculus in Physics*. World Scientific Publishing, Germany, 2000.

[13] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations*. Elsevier, North Holland, 2006.

[14] V. Kiryakova. *Generalized fractional calculus and applications*, volume 301 of *Pitman Res. Notes Math. Ser.* Longman-Wiley, New York, 1994.

[15] G. S. Ladde, V. Lakshmikantham, and A. S. Vatsala. *Monotone Iterative Techniques for Nonlinear Differential Equations*. Pitman Publishing Inc, 1985.

[16] V. Lakshmikantham, S. Leela, and D. J. Vasundhara. *Theory of Fractional Dynamic Systems*. Cambridge Scientific Publishers, 2009.

[17] V. Lakshmikantham and A. S. Vatsala. *Generalized Quasilinearization for Nonlinear Problems*. Kluwer Academic Publishers, Boston, 1998.

[18] E. S. Lee. *Quasilinearization and Invariant Imbedding*. Academic Press, New York, 1968.

[19] R. Metzler, W. Schick, H. G. Kilian, and T. F. Nonnenmacher. Relaxation in filled polymers: A fractional calculus approach. *J. Chem. Phy.*, 103:7180–7186, 1995.

[20] B. Oldham and J. Spanier. *The Fractional Calculus*. Academic Press, New York - London, 2002.

[21] J. D. Ramírez and A. S. Vatsala. Monotone iterative technique for fractional differential equations with periodic boundary conditions. *Opuscula Mathematica*, 29(3):289–304, 2009.