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GLOBAL ATTRACTIVITY OF SOLUTIONS OF FIRST-ORDER DELAY DIFFERENTIAL EQUATIONS WITH APPLICATIONS IN POPULATION DYNAMICS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. In this paper, we obtain a sufficient condition for the global attractivity of solution to the differential equation $x'(t)+p(t)x(t-\tau)=0$, where the delay τ is a positive constant. Further, the results are applied to non-linear models arising in ecology.

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1. INTRODUCTION

First, we consider the linear differential equation

$$x'(t) + p(t)x(t - \tau) = 0, \quad t \geq 0, \tag{1.1}$$

where the delay τ is a positive constant and $p \in C([0, \infty), [0, \infty))$. Later we extend our results to a non-linear model in ecology.

The asymptotic behaviour of solutions of (1.1), and of more general equations, has been studied by several authors; see for example, Burton and Haddock [2], Cooke [3, 4], Gyori [9], Yorke [31] and the references cited there in. It is well known that if there exists a positive number p such that $p(t) \leq p$, for all $t \geq 0$, and $p\tau < 3/2$, then the zero solution of (1.1) is uniformly stable. We note that the upper bound $3/2$ is sharp in the sense that if $p\tau > 3/2$, then there are equations with unbounded solutions.

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Gopalsamy [6] showed that under the assumpitons

$$\int_{t-\tau}^{t} p(s)ds \to 0 \quad \text{as } t \to \infty, \tag{1.2}$$

$$\int_{0}^{\infty} p(t)dt = \infty, \tag{1.3}$$

the trivial solution of (1.1) is asymptotically stable. This result was improved by many authors. For example, Graef and Qian [8] proved that if (1.3) holds and

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)ds < 1, \tag{1.4}$$

then every solution of (1.1) tends to zero as $t \to \infty$. Liu and Ge [15] proved that if (1.3) holds and

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)ds < 3/2, \tag{1.5}$$

then every solution of (1.1) tends to zero as $t \to \infty$. Graef and Qian [8] obtained a result different from the ones above: If (1.3) holds and

$$\lim_{t\to\infty} \int_{t-\tau}^{t} p(s)ds < \pi/2, \tag{1.6}$$

then every solution of (1.1) tend to zero as $t \to \infty$. Later, Qian and Sun [21] proved that if (1.3) holds and

$$\limsup_{t\geq 0} \int_{t-\tau}^{t} p(s)ds = \lambda \leq 3/2, \tag{1.7}$$

then every solution of (1.1) tends to zero as $t \to \infty$. An example given in [19] shows that the condition $\int_{t-\tau}^{t} p(s)ds \leq 3/2$ is the best possible (the 2/3-criterion).

Our aim of this article is to establish a new sufficient condition for every solution of (1.1) to tend to zero as $t \to \infty$. Instead of integrating $p(t)$ from $t-\tau$ to t, as in (1.4)–(1.7), we integrate from $t-2\tau$ to t. Specifically, our main result reads as follows.

Theorem 1.1. *If* (1.3) *holds and*

$$\limsup_{t\to\infty} \int_{t-2\tau}^{t} p(s)ds < 2, \tag{1.8}$$

then every solution of (1.1) *tends to zero as* $t \to \infty$.

Condition (1.8) is different from the existing conditions (1.2), (1.4)–(1.7). However when $p(t)$ equals a positive constant, (1.8) is equivalent to (1.4). Although both conditions yield the same conclusion for autonomous equation, they yield different results for a non-autonomous equation corresponding to (1.1).

From the existing 3/2-criterion, it seems that (1.8) may be replaced by the assumption $\limsup_{t\to\infty}\int_{t-2\tau}^{t} p(s)ds < 3$. However, we have not been able to prove this claim.

This work is divided into three sections: Section 1 is the introduction. Section 2 shows the proof of global attractivity. Section 3 shows some examples that illustrate our results.

In this article, we assume an initial condition $x(t) = \phi(t)$ is attached to (1.1), where ϕ is a continuous function on $[-\tau, 0]$. Solutions can be obtained by the method of steps; i.e., using ϕ on $[-\tau, 0]$ define x on $[0, \tau]$; then using x on $[0, \tau]$ define x on $[\tau, 2\tau]$.

2. PROOF OF THE MAIN THEOREM

Note that from (1.8) there are real numbers $c > 1$ and $t_0 \geq 0$, such that

$$\int_{t-2\tau}^{t} p(s)ds \leq c < 2, \quad \forall t \geq t_0. \tag{2.1}$$

Lemma 2.1. *Assume that* (2.1) *holds and* $x(t)$ *is a solution of* (1.1)*, such that* $|x| \leq M$ *on an interval* $[t_n - \tau, t_n + \tau]$*. Then* $|x(t)| \leq M(c-1)$ *or* $x(t)x'(t) < 0$ *for each* t *in* $[t_n + \tau, t_n + 2\tau]$*.*

Proof. By contradiction, suppose that there exists $t^* \in [t_n + \tau, t_n + 2\tau]$ such that $x(t^*) > M(c-1)$ and $x'(t^*) \geq 0$. Then from (1.1) and $p(s) \geq 0$, we have $x(t^* - \tau) \leq 0$.

Let s^* be the minimizer of s on $[t^* - 2\tau, t^* - \tau]$. Then

$$x(s^*) = \min\{x(s) : t^* - 2\tau \leq s \leq t^* - \tau\} \leq x(t^* - \tau) \leq 0\,.$$

Let $p^* = \int_{t^*-\tau}^{t^*} p(s)ds$. Then p^* satisfies the following three conditions.

- $p^* \leq c$.
- $p^* > 1$ which is proven by integrating (1.1):
$$x(t^*) = x(t^* - \tau) - \int_{t^*-\tau}^{t^*} p(s)x(s-\tau)ds \leq x(t^* - \tau) - x(s^*)p^*\,;$$
if $p^* \leq 1$, we have the contradiction $0 < x(t^*) \leq x(t^* - \tau) - x(s^*) \leq 0$.
- From $t^* - 2\tau \leq s^* \leq t^* - \tau$ and $p(s) \geq 0$, we have
$$\int_{s^*}^{t^*-\tau} p(s)ds = \int_{s^*}^{t^*} p(s)ds - p^* \leq c - p^*. \tag{2.2}$$

Integrating (1.1),

$$\begin{aligned} x(t^*) &= x(s^*) - \int_{s^*}^{t^*-\tau} p(s)x(s-\tau)ds - \int_{t^*-\tau}^{t^*} p(s)x(s-\tau)ds \\ &\leq x(s^*)(1-p^*) - \int_{s^*}^{t^*-\tau} p(s)x(s-\tau)ds\,. \end{aligned}$$

Then using that $p^* > 1$ and (2.2), we have

$$x(t^*) \le |x(s^*)(1-p^*)| + M(c-p^*) \le M(p^*-1) + M(c-p^*) = M(c-1).$$

This contradicts the existence of t^*.

Now suppose that there exists $t^* \in [t_n+\tau, t_n+2\tau]$ such that $x(t^*) < -M(c-1)$ and $x'(t^*) \le 0$. Then from (1.1), we have $x(t^*-\tau) \ge 0$. Let s^* be the maximizer of s on $[t^*-2\tau, t^*-\tau]$. Then

$$x(s^*) = \max\{x(s) : t^*-2\tau \le s \le t^*-\tau\} \ge x(t^*-\tau) \ge 0\,.$$

Let p^* be as above. Then $p^* \le c$. Also $p^* > 1$ which is proven by integrating (1.1):

$$x(t^*) = x(t^*-\tau) - \int_{t^*-\tau}^{t^*} p(s)x(s-\tau)ds \ge x(t^*-\tau) - x(s^*)p^*\,;$$

if $p^* \le 1$, we have the contradiction $0 > x(t^*) \ge x(t^*-\tau) - x(s^*) \ge 0$. Also (2.2) holds as above.

Integrating (1.1),

$$\begin{aligned} x(t^*) &= x(s^*) - \int_{s^*}^{t^*-\tau} p(s)x(s-\tau)ds - \int_{t^*-\tau}^{t^*} p(s)x(s-\tau)ds \\ &\ge x(s^*)(1-p^*) - \int_{s^*}^{t^*-\tau} p(s)x(s-\tau)ds\,. \end{aligned}$$

Then using that $p^* > 1$ and (2.2), we have

$$x(t^*) \ge -|x(s^*)(1-p^*)| - M(c-p^*) \ge -M(p^*-1) + M(c-p^*) = -M(c-1).$$

This contradicts the existence of t^*. The proof is complete. □

For non-oscillatory solutions, we have the following result.

Lemma 2.2. *Under assumption* (1.3), *if $x(t)$ is a non-oscillatory solution of* (1.1), *then $x(t) \to 0$ as $t \to \infty$.*

Sketch of the proof. Assume that $x(t) > 0$ for all t large enough. Then by (1.1), the solution is non-increasing and bounded below; hence convergent. Assuming that $\lim_{t\to\infty} = \alpha > 0$, integration of (1.1) yields

$$x(t) = x(t_0) - \int_{t_0}^{t} p(s)x(s-\tau)ds \le x(t_0) - \alpha \int_{t_0}^{t} p(s)ds\,.$$

By (1.3), we arrive to a contradiction: The right-hand side approaches $-\infty$, as $t \to \infty$, while the left-hand side is positive. □

Lemma 2.3. *Assume* (2.1) *holds and $x(t)$ is an oscillatory solution of* (1.1). *Then $x(t) \to 0$ as $t \to \infty$.*

Proof. Let t_0 be defined by (2.1) and $M = \max\{|x(s)| : t_0-\tau \le s \le t_0+\tau\}$. Repeated applications of Lemma 2.1 at $t_0, t_0+\tau, t_0+2\tau, \dots$ yield $|x(t)| \le M$ for all $t \ge t_0-\tau$; i.e., the solution is bounded.

Using that the solution is oscillatory, we select the smallest time $t_1 \ge t_0$ such that $|x(t_1+\tau)| \le M(c-1)$. Then $|x(t)| \le M$ on $[t_1-\tau, t_1+\tau]$ and $|x(t_1+\tau)| \le M(c-1)$. By Lemma 2.1, $|x(t)| \le M(c-1)$ on $[t_1+\tau, t_1+2\tau]$.

Define $t_2 = t_1+\tau$. Then $|x(t)| \le M$ on $[t_2-\tau, t_2+\tau]$ and $|x(t_2+\tau)| \le M(c-1)$. By Lemma 2.1, $|x(t)| \le M(c-1)$ on $[t_2+\tau, t_2+2\tau]$. From these two applications of Lemma 2.1, we have $|x(t)| \le M(c-1)$ on $[t_2, t_2+2\tau]$. Repeated applications of Lemma 2.1 at $t_2+\tau, t_2+2\tau, \dots$ yield

$$|x(t)| \le M(c-1) \quad \text{for } t \ge t_2.$$

Using that the solution is oscillatory, we select the smallest time $t_3 \ge t_2+\tau$ such that $|x(t_3+\tau)| \le M(c-1)^2$. Then $|x(t)| \le M(c-1)$ on $[t_3-\tau, t_3+\tau]$ and $|x(t_3+\tau)| \le M(c-1)^2$. By Lemma 2.1, $|x(t)| \le M(c-1)^2$ on $[t_3+\tau, t_3+2\tau]$.

Define $t_4 = t_3+\tau$. Then $|x(t)| \le M(c-1)$ on $[t_4-\tau, t_4+\tau]$ and $|x(t_4+\tau)| \le M(c-1)^2$. By Lemma 2.1, $|x(t)| \le M(c-1)^2$ on $[t_4+\tau, t_4+2\tau]$. From these two applications of Lemma 2.1, we have $|x(t)| \le M(c-1)^2$ on $[t_4, t_4+2\tau]$. Repeated applications of Lemma 2.1 at $t_4+\tau, t_4+2\tau, \dots$ yield

$$|x(t)| \le M(c-1)^2 \quad \text{for } t \ge t_4.$$

By repeating the process above, we obtain

$$|x(t)| \le M(c-1)^k \quad \text{for } t \ge t_{2k}.$$

Since $1 < c < 2$, $\lim_{t\to\infty} |x(t)| = 0$. The proof is complete. □

Now the proof of Theorem 1.1 follows from Lemmas 2.2 and 2.3.

3. APPLICATIONS TO SOME MATHEMATICAL MODELS

In this section, we shall apply Theorem 1.1 to study the global attractivity of some mathematical models arising in ecological dynamics. In particular, we consider the mathematical models of Lasota-Wazewska and Hematopoiesis.

3.1. APPLICATION TO LASOTA-WAZEWSKA MODEL.

Consider the equation

$$x'(t) = -a(t)x(t) + b(t)e^{-\gamma(t)x(t-\tau)}, \quad t \ge 0, \tag{3.1}$$

where a, b and $\gamma \in C(R^+, (0,\infty))$ are T-periodic functions, $\tau > 0$ is a real number. For the given initial condition

$$x(t) = \phi(t), \quad t \in [-\tau, 0], \tag{3.2}$$

where $\phi \in C([-\tau, 0], R^+)$ and $\phi \neq 0$, one can prove, by the method of steps that the problem (3.1)–(3.2) has a unique nonnegative solution $x(t)$ on $[0, \infty)$, and that $x(t) > 0$ for $t \geq \tau$.

In the rest part of this problem, we regard the solution of (3.1) as a solution of (3.1)–(3.2).

If $a(t) \equiv a, b(t) \equiv b$ and $\gamma(t) \equiv \gamma$ are constants, then (3.1) reduces to

$$x'(t) = -ax(t) + be^{-\gamma x(t-\tau)}. \tag{3.3}$$

Equation (3.3) was considered first by Wazewska-Czyzewska and Lasota [25] as a model for the survival of red blood cells in an animal. Here $x(t)$ denotes the number of red cells at time t, a is the probability of death of a red blood cell, b and γ are positive constants related to the production of red blood cells per unit time, and τ is the time required to produce a red blood cell.

Global attractivity of (3.1) has been studied by many authors, for instance, one may refer to [6, 8, 12, 13, 17] and the references cited there in. Graef et al [8] proved that (3.1) has a positive T-periodic solution, let it be $\overline{x}(t)$. Further, they proved that the equilibrium solution $\overline{x}(t)$ is a global attractor to all other positive solutions of (3.1) if

$$\int_0^T b(t)e^{-\overline{x}(t)}dt < 1 \tag{3.4}$$

holds. In [13], Li and Wang proved that if

$$b(t)\gamma(t) \leq a(t), \tag{3.5}$$

then any solution of (3.1) is a global attractor. In an another attempt, Liu et al [17] showed that $\overline{x}(t)$ is a global attractor to all other solutions of (3.1) if

$$MB\overline{\gamma} \leq 1 \tag{3.6}$$

holds, where

$$M = \frac{e^{\int_0^T a(s)ds}}{e^{\int_0^T a(s)ds} - 1}, \quad B = \int_0^T b(s)ds \quad \text{and } \overline{\gamma} = \max_{0 \leq t \leq T} \gamma(t).$$

In this work, we shall obtain a different condition for the global attractivity of the equilibrium solution $\overline{x}(t)$. An example will also be given showing that our result can be applied where as (3.5) and (3.6) fails to hold, that is, the results obtained in [13, 17] cannot be applied. The condition required to prove our result is independent of the equilibrium solution $\overline{x}(t)$ and hence our condition is different from (3.4). In view of the above, our result may be treated as a different one than the results obtained in [8, 13, 17].

A simple calculation shows that every solution of (3.1) is positive and bounded. Further, from

$$x(t) = x(0)e^{-a_* t} + \int_0^t b(s)e^{-\int_s^t a(\theta)d\theta} e^{-\gamma(s)x(s-\tau)} ds,$$

it follows that

$$\limsup_{t\to\infty} x(t) \le \frac{b^*}{a_*}, \tag{3.7}$$

where $b^* = \max_{t\in[0,T]} b(t)$ and $a_* = \min_{t\in[0,T]} a(t)$.

Now, we give the following sufficient condition for the global attractivity of (3.1).

Theorem 3.1. *If*

$$\limsup_{t\to\infty} \int_{t-2\tau}^t b(s)\gamma(s)ds < 2e^{-a^*\tau}, \tag{3.8}$$

then every solution of (3.1) *tend to* $\overline{x}(t)$ *as* $t \to \infty$*.*

Proof. Let $x(t)$ be a positive solution of (3.1), set $z(t) = x(t) - \overline{x}(t)$. Then $z(t)$ is a solution of the differential equation

$$z'(t) = -a(t)z(t) + b(t)\left\{\frac{1}{e^{\gamma(t)x(t-\tau)}} - \frac{1}{e^{\gamma(t)\overline{x}(t-\tau)}}\right\}.$$

An application of the mean value theorem to the above equation yields

$$z'(t) = -a(t)z(t) + b(t)(x(t-\tau) - \overline{x}(t-\tau))\frac{(-\gamma(t))}{e^{\gamma(t)\theta(t)}},$$

$$z'(t) = -a(t)z(t) - b(t)\gamma(t)\frac{1}{e^{\gamma(t)\theta(t)}}z(t-\tau),$$

which can be written as

$$(z(t)e^{\int_0^t a(s)ds})' = -b(t)\gamma(t)\frac{1}{e^{\gamma(t)\theta(t)}}e^{\int_0^t a(s)ds}z(t-\tau),$$

where $\theta(t)$ lies between $x(t-\tau)$ and $\overline{x}(t-\tau)$. Assume that $N(t) = z(t)e^{\int_0^t a(s)ds}$. Then the above equation becomes

$$N'(t) + G(t)N(t-\tau) = 0, \tag{3.9}$$

where

$$G(t) = \frac{b(t)\gamma(t)e^{\int_{t-\tau}^t a(s)ds}}{e^{\gamma(t)\theta(t)}}.$$

To complete the proof of the theorem, it is enough to prove that $N(t) \to 0$ as $t \to \infty$. Since every solution of (3.1) is bounded, then both $x(t-\tau)$ and $\overline{x}(t-\tau)$ are also bounded. This in turn implies that there exists a real $K > 0$ such that $e^{\gamma(t)\theta(t)} < K$. Consequently,

$$\int_0^\infty G(t)dt = \int_0^\infty \frac{b(t)\gamma(t)e^{\int_{t-\tau}^t a(s)ds}}{e^{\gamma(t)\theta(t)}}dt$$

$$> \frac{b_* \gamma_* e^{a_* \tau}}{K} \int_0^\infty dt = \infty.$$

Further,

$$\begin{aligned}\limsup_{t\to\infty} \int_{t-2\tau}^{t} G(s)ds &= \limsup_{t\to\infty} \int_{t-2\tau}^{t} \frac{b(s)\gamma(s) e^{\int_{s-\tau}^{s} a(u)du}}{e^{\gamma(s)\theta(s)}} ds \\ &< \limsup_{t\to\infty} \int_{t-2\tau}^{t} b(s)\gamma(s) e^{a^*\tau} ds \\ &< 2\end{aligned}$$

holds. Applying Theorem (1.1) to (3.9), we see that $N(t) \to 0$ as $t \to \infty$. Hence the theorem is proved. □

Remark 3.2. Applying Theorem 3.1 to (3.3), we have that, if $\tau b\gamma e^{a\tau} < 1$, then $\overline{x}$ is globally attractive to all other solutions of (3.3), where $\overline{x}$ satisfies the property $a\overline{x} = be^{-\gamma\overline{x}}$. Consider the equation

$$x'(t) = -ax(t) + be^{-\gamma x(t-\tau)}, \quad t \geq 0. \tag{3.10}$$

where $a = 1/2$, $b = 2$, $\gamma = 1$ and $\tau = \log 2$. Since $\tau b\gamma e^{a\tau} = 2\log 2\, e^{\frac{1}{2}\log 2} = 2\log 2\sqrt{2} = 0.85144 < 1$, then $\overline{x}$ is a global attractor to all other solutions of (3.10), where $\overline{x}$ is a solution of $\overline{x} = 4e^{-\overline{x}}$. On the other hand, $b\gamma = 2 > 1/2 = a$ implies that Theorem 3.1 due to Li and Wang [13] can not be applied to (3.10).

In the following, we give an example in which our Theorem 3.1 can be applied where as the results in [17] can not be applied.

Example 3.3. Consider the equation

$$x'(t) = -\frac{(2+\sin t)}{4} x(t) + \frac{(2+\cos t)}{2} e^{-(\frac{2+\sin t}{18}) x(t-\frac{\pi}{2})}, \quad t \geq 0. \tag{3.11}$$

Here $a(t) = \frac{2+\sin t}{4}$, $b(t) = \frac{2+\cos t}{2}$, $\gamma(t) = \frac{2+\sin t}{18}$, $\tau = \frac{\pi}{2}$ and $T = 2\pi$. Clearly

$$\begin{aligned}\limsup_{t\to\infty} \int_{t-2\tau}^{t} b(s)\gamma(s)ds &= \limsup_{t\to\infty} \int_{t-\pi}^{t} \frac{(2+\cos s)}{2} \frac{(2+\sin s)}{18} ds \\ &\leq \frac{1}{36}(4\pi + 4 + 4) = 0.571288072\end{aligned}$$

and

$$2e^{-a^*\tau} = 2e^{-\frac{3}{4}\cdot\frac{\pi}{2}} = 0.6158$$

imply that our Theorem 3.1 can be applied to this example. On the other hand, $MB\overline{\gamma} = 1.094496 > 1$ implies that (3.6) fails to hold and hence Theorem 3.1 due to Liu et al. [17] cannot be applied to Eq. (3.11).

The following example shows that our Theorem 3.1 can be applied where as the results in [13] and [17] cannot be applied.

Example 3.4. Consider

$$x'(t) = -\frac{(2+\sin t)}{40}x(t) + \frac{(2+\cos t)}{2}e^{-(\frac{2+\sin t}{18})x(t-\frac{\pi}{2})}, \quad t \geq 0. \tag{3.12}$$

Here $a(t) = \frac{2+\sin t}{40}$, $b(t) = \frac{2+\cos t}{2}$, $\gamma(t) = \frac{2+\sin t}{18}$, $\tau = \frac{\pi}{2}$ and $T = 2\pi$. It is easy to verify that

$$\limsup_{t\to\infty} \int_{t-2\tau}^{t} b(s)\gamma(s)ds = 0.5713 < 1.778 \simeq 2e^{-a^*\tau}.$$

Hence our Theorem 3.1 can be applied to this example. On the other hand, a simple calculation shows that $MB\overline{\gamma} = 3.7093 \times 2\pi \times \frac{1}{6} = 3.8843 > 1$ and $b(t)\gamma(t) = \frac{(2+\cos t)(2+\sin t)}{36} > \frac{(2+\sin t)}{40} = a(t)$ hold, that is, (3.5) and (3.6) fail to hold. This in turn implies that Theorem 3.1 in [13] and [17] cannot be applied to this example.

3.2. APPLICATION TO HEMATOPOIESIS MODEL-1.

Consider the blood cell production model

$$x'(t) = -a(t)x(t) + \frac{b(t)}{1+x^n(t-\tau)}, \quad n > 0, \tag{3.13}$$

where a and $b \in C(R^+, (0,\infty))$ are T-periodic functions with the initial condition as defined in (3.2). Equation (3.13) was proposed by Mackey and Glass [20] in order to describe some psychological control system on blood cell production model. It is proved in [26, 29, 32] that if $a(t)$ and $b(t)$ are periodic with period $T > 0$, then (3.13) has a positive T-periodic solution. Let it be $\overline{x}(t)$. Wang and Li [27] proved that if $n > 1, \alpha n < 1$ and $Mb \leq \sqrt[n]{\frac{1}{(n-1)}}$, then $\overline{x}(t)$ is a global attractor to all other positive solutions of (3.13) (See Corollary 3.4 in [27]), where $b = \int_0^T b(t)dt$, $\alpha = \frac{b^*}{a_*}$, $M = \frac{\int_0^T a(s)ds}{\int_0^T a(s)ds - 1}$, $b^* = \max_{0\leq t\leq T} b(t)$ and $a_* = \min_{0\leq t\leq T} a(t)$. In the following, we obtain sufficient conditions for the global attractivity of solutions of (3.13) on both the cases $0 < n \leq 1$ and $n \geq 1$. We shall give an example to which our result applies where as Corollary 3.4 in [27] cannot be applied. The technique used in [27] were applied in [16] to find the global attractivity of solution of

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{m} \frac{b_i(t)}{1+x^n(t-\tau_i)}, \quad n > 0, \tag{3.14}$$

where $n > 0$ is a constant. They proved that, if one of the following condition is satisfied:

$$n \leq 1 \quad \text{and} \quad \frac{nX_1^{(n-1)}}{1+X_1^n}Mb \leq 1 \tag{3.15}$$

or

$$n > 1 \quad \text{and} \quad (n-1)^{(n-1)}(Mb)^n \leq 1, \tag{3.16}$$

then (3.14) has a unique positive T-periodic solution $\overline{x}(t)$. Further, every solution $x(t)$ to(3.14) satisfies

$$\lim_{t\to\infty}[x(t)-\overline{x}(t)]=0,$$

where $X_1 = x_1 e^{-\int_\tau^0 a(s)ds}$ and x_1 is the unique solution of $f(x) = -a^*x + \frac{b_*}{1+x^n}$, $x \in [0,\infty)$ such that

$$f(x)=\begin{cases} >0, & \text{if } 0<x<x_1\\ <0, & \text{if } x>x_1. \end{cases}$$

We note that the conditions (3.15) and (3.16) are independent of m. Thus the result can be applied to (3.13).

Let $a(t) \equiv a$ and $b(t) \equiv b$ be positive constants. Then (3.13) reduces to the autonomous equation

$$x'(t) = -ax(t) + \frac{b}{1+x^n(t-\tau)}, \quad n>0. \tag{3.17}$$

Applying the conditions (3.15) and (3.16) to the Eq. (3.17), they proved that if either

$$n \le 1 \text{ and } n\frac{b}{a} < \overline{x}e^{-a\tau} + (\overline{x}e^{-a\tau})^{(1-n)} \tag{3.18}$$

or

$$n > 1 \text{ and } (n-1)^{(n-1)}(\frac{b}{a})^n < 1, \tag{3.19}$$

then every positive solution $x(t)$ of (3.17) satisfies

$$\lim_{t\to\infty} x(t) = \overline{x},$$

where $\overline{x}$ is the unique equilibrium to (3.17), that is, $\overline{x}$ is the unique solution of

$$\frac{b}{1+\overline{x}^n} = a\overline{x}.$$

One may refer [18] for some global attractivity results on the solutions of (3.13) and (3.17). It is proved in [27] that every solution of (3.13) with the initial condition (3.1) satisfies

$$\beta \le \liminf_{t\to\infty} x(t) \le \limsup_{t\to\infty} x(t) \le \alpha \tag{3.20}$$

and $x(t)>0$ for $t \ge 0$, where $\alpha = \frac{b^*}{a_*}, \beta = \frac{\gamma}{1+\alpha^n}$ and $\gamma = \frac{b_*}{a^*}$.

Theorem 3.5. *Assume that either*

$$n \ge 1 \text{ and } n\tau {b^*}^n e^{a^*\tau} < a_*^{n-1} \tag{3.21}$$

or

$$n \le 1 \text{ and } nb^*\tau e^{a^*\tau}(a_*^n + {b^*}^n)^{(1-n)}{a^*}^{1-n} < a_*^{n(1-n)} b_*^{1-n} \tag{3.22}$$

hold. Then $\overline{x}(t)$ is a global attractor to all other positive solutions of (3.13).

Proof. Let $x(t)$ be any solution of (3.13) and set $z(t) = x(t) - \overline{x}(t)$. Then $z(t)$ is a solution of

$$z'(t) = -a(t)z(t) + b(t)\left\{\frac{1}{1+x^n(t-\tau)} - \frac{1}{1+\overline{x}^n(t-\tau)}\right\}.$$

Applying mean value theorem to the above equation, we have

$$z'(t) = -a(t)z(t) - nb(t)z(t-\tau)\frac{\theta^{n-1}}{(1+\theta^n)^2},$$

where θ lies in between $x(t-\tau)$ and $\overline{x}(t-\tau)$ and θ satisfies the persistent property (3.20). Again set $N(t) = z(t)e^{\int_0^t a(s)ds}$. Then $N(t)$ is a solution of the first order delay differential equation

$$N'(t) + G(t)N(t-\tau) = 0, \tag{3.23}$$

where

$$G(t) = nb(t)\frac{\theta^{n-1}}{(1+\theta^n)^2}e^{\int_{t-\tau}^t a(s)ds}.$$

To complete the proof of the theorem, it is enough to show that $N(t) \to 0$ as $t \to \infty$. For this, we shall show that $G(t)$ satisfies the conditions assumed in Theorem 1.1. Using (3.20), we have

$$\int_0^\infty G(t)dt > \int_0^\infty \frac{b_* n\beta^{n-1}}{(1+\alpha^n)^2}e^{a_*\tau}dt = \infty.$$

Now, let $n \geq 1$. Then

$$\begin{aligned}
\limsup_{t\to\infty}\int_{t-2\tau}^t G(s)ds &= \limsup_{t\to\infty}\int_{t-2\tau}^t \frac{b(s)n\theta^{n-1}}{(1+\theta^n)^2}e^{\int_{s-\tau}^s a(u)du}ds \\
&< \limsup_{t\to\infty} ne^{a^*\tau}\int_{t-2\tau}^t b(s)\theta^{n-1}ds \\
&< ne^{a^*\tau}b^*2\tau(\frac{b^*}{a_*})^{n-1} \\
&< 2.
\end{aligned}$$

Next, suppose that $n \leq 1$. Then

$$\begin{aligned}
\limsup_{t\to\infty}\int_{t-2\tau}^t G(s)ds &= \limsup_{t\to\infty}\int_{t-2\tau}^t \frac{b(s)n\theta^{n-1}}{(1+\theta^n)^2}e^{\int_{s-\tau}^s a(u)\,du}ds \\
&< ne^{a^*\tau}\limsup_{t\to\infty}\int_{t-2\tau}^t b(s)\theta^{1-n}ds \\
&< ne^{a^*\tau}b^*2\tau\frac{(1+\alpha^n)^{1-n}}{\gamma^{1-n}} \\
&= \frac{2nb^*\tau e^{a^*\tau}(a_*^n + b^{*^n})^{1-n}}{a_*^{n(1-n)}}\frac{{a^*}^{1-n}}{b_*^{1-n}} \\
&< 2.
\end{aligned}$$

Thus, in either case $n \geq 1$ or $n \leq 1$, $\limsup_{t\to\infty} \int_{t-2\tau}^{t} G(s)\,ds < 2$ holds. Hence by Theorem (1.1), every solution of (3.23) tends to zero as $t \to \infty$. In particular, $N(t) \to 0$ as $t \to \infty$. Thus the theorem is proved. □

Remark 3.6. Applying Theorem (3.5) to Eq. (3.17), we have that, if either of the following

$$n \leq 1 \text{ and } ne^{a\tau}(a^n + b^n)^{1-n} a^{1-n^2} b^n \tau < 1 \tag{3.24}$$

or

$$n \geq 1 \text{ and } n\tau b^n e^{a\tau} < a^{n-1} \tag{3.25}$$

hold, then $\overline{x}$ is a global attractor to all other positive solutions of (3.17). Observe that (3.18) depends upon the equilibrium point $\overline{x}$ where as the condition (3.24) is independent of $\overline{x}$. Hence the conditions (3.18) and (3.24) are different. Next, we compare the conditions (3.19) and (3.25). The condition (3.25) holds for $n = 1$ where as (3.19) fails to hold for $n = 1$. Further, our condition (3.25) depends upon τ. Thus, it is possible to find an example where (3.25) holds and (3.19) fails when $n \geq 1$. Now, we give examples to strengthen our result.

Example 3.7. Consider

$$x'(t) = -\left(\frac{2+\sin(2t)}{13}\right) x(t) + \frac{\left(\frac{2+\cos(2t)}{4}\right)}{1 + x^2(t - \frac{\pi}{100})}, \quad t \geq 0. \tag{3.26}$$

Here $a(t) = \frac{2+\sin(2t)}{13}$, $b(t) = \frac{2+\cos(2t)}{4}$, $\tau = \frac{\pi}{100}$ and $n = 2$. Since

$$\begin{aligned} n\tau b^{*n} e^{a^*\tau} &= 2\frac{\pi}{100}\frac{9}{16} e^{\frac{3\pi}{1300}} = 0.035597 \\ &< 0.076923 = \frac{1}{13} = a_*^{n-1}, \end{aligned}$$

then our Theorem 3.5 can be applied to this example. On the other hand, $\alpha_n = \frac{b^*}{a_*} n = 19.5 > 1$ implies that Corollary 3.4 due to Wang and Li [27] cannot be applied.

Example 3.8. Consider the equation

$$x'(t) = -\left(\frac{5+\sin^2 t}{20}\right) x(t) + \frac{\left(\frac{5+\sin^2 t}{10}\right)}{1 + x^{1/5}(t - \frac{\pi}{100})}, \quad t \geq 0. \tag{3.27}$$

A simple calculation shows that (3.22) holds. In fact,

$$\begin{aligned} nb^*\tau e^{a^*\tau}(a_*^n + b^{*n})^{1-n} a^{*1-n} &= 0.0113 \\ &< 0.4595 = a_*^{n(1-n)} b_*^{1-n}. \end{aligned}$$

Hence Theorem 3.5 can be applied to Eq. (3.27).

3.3. **APPLICATION TO HEMATOPOIESIS MODEL-2.** Next, we consider the model

$$x'(t) = -a(t)x(t) + b(t)\frac{x(t-\tau)}{1+x^n(t-\tau)}, \quad t \geq 0, n > 1. \tag{3.28}$$

Here we give a sufficient condition for the global attractivity of periodic solution of (3.28) under the initial condition (3.2). It is proved in [28, 32] that if $b(t) > a(t)$ for $t \in [0.T]$, then (3.28) has at least one positive T-periodic solution. Let it be $\overline{x}(t)$. Further, they [28] proved that if $b(t) \leq a(t)$ for $t \in [0.T]$, then (3.28) has no positive T-periodic solution.

Berezansky and Braverman [1] proved that, if $n > 0, b(t) \geq 0, a(t) \geq 0$ are lebesgue measurable essentially locally bounded functions and $\limsup_{t\to\infty} \frac{b(t)}{a(t)} = \lambda < 1$, $\int_0^\infty a(t)dt = \infty$ and $\sup_{t<0} x(t) < \infty$ hold, then the solution $x(t)$ of (3.28)–(3.2) tends to zero, that is, $\lim_{t\to\infty} x(t) = 0$. Clearly, this condition implies that $b(t) < a(t)$ and hence by [28] (3.28) has no positive T-periodic solution. Saker in [23] obtained a sufficient condition for the global attractivity of a positive T-periodic solution $\overline{x}(t)$ of (3.28) (see Theorem 2.5 in [23].) This condition in Theorem 2.5 in [23] contains some lower and upper bound on $\overline{x}(t)$.

In this paper, we shall obtain a different sufficient condition for the global attractivity of a positive T-periodic solution $\overline{x}(t)$. Our condition is independent of $\overline{x}(t)$ and hence as a remark, every positive periodic solution of (3.28) is globally attractive.

Lemma 3.9. *Every solution of* (3.28)–(3.2) *is positive and bounded.*

Proof. The proof of positivity of every solutions of (3.28)–(3.2) is easy and hence we omit the proof.

Let $f(u) = \frac{u}{1+u^n}$. It is easy to verify that $f(u)$ attains its maximum at $u = (\frac{1}{n-1})^{\frac{1}{n}}$ and the maximum of $f(u)$ is $\frac{1}{n}(n-1)^{\frac{n-1}{n}}$. From (3.28), we have

$$\begin{aligned} x(t) &= \phi(0)e^{-\int_0^t a(s)ds} + \int_0^t b(s)\frac{x(s-\tau)}{1+x^n(s-\tau)}e^{-\int_s^t a(u)du}ds \\ &\leq \phi(0)e^{-a_* t} + \frac{b^*}{a_*}\frac{1}{n}(n-1)^{\frac{n-1}{n}}[1-e^{-a_* t}]. \end{aligned}$$

Consequently,

$$\limsup_{t\to\infty} x(t) \leq \frac{b^*}{a_*}\frac{1}{n}(n-1)^{\frac{n-1}{n}} = \lambda, \quad \text{(say)}. \tag{3.29}$$

This completes the proof of the lemma. ☐

Remark 3.10. It follows from the proof of Lemma 3.9 that, if $x(t)$ is a solution of (3.28) then

$$\limsup_{t\to\infty} x(t) \leq \lambda,$$

where λ is given in (3.29).

Theorem 3.11. *If*

$$\limsup_{t\to\infty}\int_{t-2\tau}^{t} b(s)ds < 8ne^{-a^*\tau} \tag{3.30}$$

holds, then $\overline{x}(t)$ is a global attractor to all other positive solutions of (3.28)*.*

Proof. Let $x(t)$ be a solution of (3.28). Set $z(t) = x(t) - \bar{x}(t)$. Then $z(t)$ is a solution of

$$z'(t) = -a(t)z(t) + b(t)\left[\frac{x(t-\tau)}{1+x^n(t-\tau)} - \frac{\bar{x}(t-\tau)}{1+\bar{x}^n(t-\tau)}\right].$$

Applying mean value theorem to the above equation we have

$$z'(t) = -a(t)z(t) + b(t)z(t-\tau)f'(\theta), \tag{3.31}$$

where $f(\theta) = \frac{\theta}{1+\theta^n}$ and θ lies in between $x(t-\tau)$ and $\bar{x}(t-\tau)$. Finding $f'(\theta)$, (3.31) can be rewritten as

$$z'(t) = -a(t)z(t) - b(t)\frac{(n-1)\theta^n - 1}{(1+\theta^n)^2}\,z(t-\tau).$$

The above equation is equivalent to

$$[z(t)e^{\int_0^t a(s)ds}]' = -b(t)\frac{(n-1)\theta^n - 1}{(1+\theta^n)^2}e^{\int_0^t a(s)ds}\,z(t-\tau).$$

Let $N(t) = z(t)e^{\int_0^t a(s)ds}$. Then the above equation becomes

$$N'(t) + G(t)z(t-\tau) = 0, \tag{3.32}$$

where $G(t) = b(t)\frac{(n-1)\theta^n-1}{(1+\theta^n)^2}e^{\int_{t-\tau}^t a(s)ds}$. Since $\frac{(n-1)\theta^n-1}{(1+\theta^n)^2} > 0$ for $\theta > (\frac{1}{n-1})^{\frac{1}{n}}$, then there exists a real $t_1 > 0$ such that $G(t) < 0$ for $0 < t \le t_1$, $t_1 > (\frac{1}{n-1})^{\frac{1}{n}}$ and $G(t) > 0$ for $t > t_1$. Now,

$$\begin{aligned}\int_0^\infty G(t)dt &= \int_0^{t_1} G(t)dt + \int_{t_1}^\infty G(t)dt\\ &= \int_0^{t_1} G(t)dt + \int_{t_1}^\infty \frac{(n-1)\theta^n - 1}{(1+\theta^n)^2}b(t)e^{\int_{t-\tau}^t a(s)ds}dt.\end{aligned}$$

A simple calculation shows that $\frac{(n-1)\theta^n-1}{(1+\theta^n)^2}$ is nonincreasing for $\theta > (\frac{n+1}{n-1})^{\frac{1}{n}} > (\frac{1}{n-1})^{\frac{1}{n}} > t_1$. Hence

$$\int_0^\infty G(t)dt > \int_0^{t_1} G(t)dt + \int_{t_1}^\infty \frac{(n-1)\lambda^n - 1}{(1+\lambda^n)^2}b(t)e^{a_*\tau}dt = \infty.$$

Next,

$$\limsup_{t\to\infty}\int_{t-2\tau}^t G(s)ds < e^{a^*\tau}\limsup_{t\to\infty}\int_{t-2\tau}^t b(s)\frac{(n-1)\theta^n - 1}{(1+\theta^n)^2}ds.$$

One may verify that $\frac{(n-1)\theta^n-1}{(1+\theta^n)^2}$ attains its maximum $\frac{1}{4n}$ at $\theta = (\frac{n+1}{n-1})^{\frac{1}{n}}$ and hence

$$\limsup_{t\to\infty}\int_{t-2\tau}^t G(s)ds < e^{a^*\tau}\frac{1}{4n}\limsup_{t\to\infty}\int_{t-2\tau}^t b(s)ds < 2.$$

Then, by Theorem 1.1, every solution of (3.32) tends to zero as $t \to \infty$. This in turn implies that $\lim_{t\to\infty}(x(t) - \overline{x}(t)) = 0$. Thus, the theorem is proved. □

Example 3.12. Consider the equation

$$x'(t) = -(0.2 + \cos^2 t)x(t) + (1.5 + \cos^2 t)\frac{x(t - \frac{\pi}{4})}{1 + x^{\frac{3}{2}}(t - \frac{\pi}{4})}, \quad t \geq 0. \tag{3.33}$$

Here $a(t) = 0.2 + \cos^2 t$, $b(t) = 1.5 + \cos^2 t$, $\tau = \frac{\pi}{4}$, $n = \frac{3}{2}$, and $T = \pi$. Clearly $b_* > a^*$. It is easily verified that

$$\begin{aligned}\limsup_{t\to\infty} \int_{t-2\tau}^{t} b(s)ds &= \limsup_{t\to\infty} \int_{t-\frac{\pi}{4}}^{t} (1.5 + \cos^2 s)ds \\ &= \pi + \frac{1}{2} \limsup_{t\to\infty} \sin 2t \\ &= 3.64 < 4.6759 = 8ne^{-a^*\tau}.\end{aligned}$$

Hence from Theorem 3.11, Eq. (3.33) has a positive π-periodic solution, which is global attractor to all other positive solutions of (3.33).

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