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ROBUST EXPONENTIAL STABILITY OF STOCHASTIC REACTION-DIFFUSION RECURRENT NEURAL NETWORKS WITH MARKOVIAN JUMPING PARAMETERS AND MODE-DEPENDENT DELAYS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. This paper is devoted to investigating global robust exponential stability for a class of delayed stochastic reaction-diffusion recurrent neural networks. The network parameters are governed by a continuous-time discrete-state Markov process which takes values in a finite set. By employing a Lyapunov-Krasovskii functional and some inequalities, some easy-to-test criteria on global exponential stability for this kind of stochastic neural networks are established in the form of linear matrix inequalities. An example is presented to illustrate the effectiveness of the theoretical results.

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1. INTRODUCTION

Recently, recurrent neural networks (RNNs) have attracted increasing attention due to their many important applications, such as speech recognition [1, 2], GPS measurements [3], robotic sound-source localization [4], underwater robot control [5], adaptive filtering [6], combinatorial optimization [7] and others [8, 9, 10]. Such applications heavily depend on the dynamical behaviors of RNNs. Analysis of the dynamical behaviors of RNNs, such as stability [11–24], periodic oscillation [25–29] and chaotic behaviors [2, 30–32], is essential for practical design of RNNs. Besides, there are often some unavoidable uncertainties such as modeling errors, external perturbations and parameter fluctuations, which can cause instability of the networks. Thus, many researchers have probed robust stability of the RNNs with errors and perturbations [33–37]. Yang, Gao and Shi [34] provided novel robust stability criteria for stochastic Hopfield neural networks with time delays in 2009. Raja and Samidurai

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presented new delay-dependent robust asymptotic stability for uncertain stochastic recurrent neural networks with multiple time varying delays in [37].

Markovian jump systems, introduced by Krasovskii and Lidskii [38] in 1961, have been widely studied, see [39–46] and the references therein. As we all know, during the implementation of RNNs on very-large- scale integrated chips, the phenomena, such as information latching, random failure of the components, sudden disturbances and variations of the environment, and changes of the subsystem interconnections, may result in the changes of the network parameters. Therefore, it is necessary to investigate the model of Markovian jumping neural networks, see [47–52] and the references therein. Wu and Shi etc. [47] have investigated passivity analysis for discrete-time stochastic Markovian jump neural networks with mixed time delays. Zhao and Zhang etc. [49] have established robust stability criterion for discrete-time uncertain markovian jumping neural networks with defective statistics of modes. Zhang and Wang [50] have studied the stability analysis of Markovian jumping stochastic Cohen-Grossberg neural networks (CGNNs) with mixed time delays by LMI approach. Cao and Zhu [51] have considered robust exponential stability of Markovian jump impulsive stochastic CGNNs with mixed time delays using LMI approach. However, The above have not considered reaction-diffusion effects.

Diffusion effect cannot be avoided in electric circuits and the neural network model when electrons are moving in asymmetric electromagnetic field, see [53–56] and the references therein. The stability for reaction-diffusion neural networks with Dirichlet boundary conditions was addressed in [53, 54], where some novel diffusion dependent criteria were provided. For reaction-diffusion neural networks with Neumann boundary conditions, some diffusion-independent stability criteria were established [55, 56]. Wang and Zhang [55] have considered stochastic exponential stability of the delayed reaction-diffusion RNNs with Markovian jumping parameters, and their results are diffusion-independent and delay-independent, which may lead to considerable conservativeness.

The axonal signal transmission delays often occur in practice and may cause undesirable dynamic network behaviors such as oscillation and instability. Therefore, the stability for neural networks with time delays has attracted more and more researchers, who have classified the delay type under consideration as constant, time-varying and distributed [11–27, 60–62]. Recently, it is found that, in real systems, the transmission delay may occur randomly, which can be modeled as a Markov process [64–67]. To the best of the authors' knowledge, there are few literatures on the Markovian switching reaction-diffusion RNNs with mixed model-dependent delays.

Motivated by the above discussion, in this paper, we are interested in investigating global robust exponential stability for a class of stochastic reaction-diffusion RNNs with Markovian jumping parameters and model-dependent delays. The parameters are governed by a continuous-time discrete-state Markov process which takes the values in a finite set. The parameter uncertainties are assumed to be norm bounded. In section 2, the model studied is presented, and some definitions and lemmas are introduced. In section 3, delay and diffusion dependent robust stability of the model is probed and some sufficient conditions in the form of LMIs are developed. In section 4, a numerical example is presented to illustrate the effectiveness and efficiency of the proposed method, and finally, conclusions are drawn in section 5.

Notations: $L^2(\mathbb{R}\times\Omega)$ stands for the space of real Lebesgue measurable functions of $\mathbb{R}\times\Omega$. It is a Banach space with the 2-norm $\|u(t)\|_2 = (\sum_{i=1}^{n}\|u_i(t)\|^2)^{\frac{1}{2}}$, where $\|u_i(t)\| = (\int_\Omega |u_i(t,x)|^2dx)^{\frac{1}{2}}$, and $|u_i(t,x)|$ is Euclid norm. $(\Xi,\mathbb{F},\{\mathbb{F}_t\}_{t\geq 0},\mathbb{P})$ is a complete probability space with a filtration $\{\mathbb{F}_t\}_{t\geq 0}$ satisfying the usual conditions. $L^P_{\mathbb{F}_0}([-\tau_0,0]\times\Omega;R^n)$ is the family of all $\mathbb{F}_0$-measurable $C([-\tau_0,0]\times\Omega;R^n)$-valued random variables $\xi=\{\xi(\theta,x):-\tau_0\leq\theta\leq 0, x\in\Omega\}$ such that $\sup_{-\tau_0\leq\theta\leq 0}\mathbb{E}\|\xi(\theta)\|_2^2<\infty$, where $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure $\mathbb{P}$. $A=(a_{ij})_{n\times n}>0$ (<0) is a positive (negative) definite matrix. $A=(a_{ij})_{n\times n}\geq 0$ is a semi-positive definite matrix. $A\geq B$ (respectively, $A>B$) this means $A-B$ is a semi-positive definite matrix (respectively, positive definite). I is identity matrix with compatible dimension. $\lambda_{\max}(A)$ (respectively, $\lambda_{\min}(A)$) means the largest (respectively, smallest) eigenvalue of the matrix A.

2. PRELIMINARIES

Consider the delayed stochastic reaction-diffusion RNNs with Markovian jumping parameters and mode-dependent delays as follows

$$
\left\{
\begin{aligned}
& dy(t,x) = [\nabla\bullet(D^*(t,x,y)\circ\nabla y(t,x)) - (A(r(t))+\triangle A(r(t)))y(t,x) \\
&\quad +(W_0(r(t))+\triangle W_0(r(t)))G_0(y(t,x)) + (W_1(r(t)) \\
&\quad +\triangle W_1(r(t)))G_1(y(t-\tau(r(t)),x)) \\
&\quad +(W_2(r(t))+\triangle W_2(r(t)))\int_{t-\delta(r(t))}^{t} G_2(y(s,x))ds + I(r(t))]dt \\
&\quad +[(W_3(r(t))+\triangle W_3(r(t)))y(t,x) \\
&\quad +(B(r(t))+\triangle B(r(t)))y(t-\tau(r(t)),x) \\
&\quad +(C(r(t))+\triangle C(r(t)))\int_{t-\delta(r(t))}^{t} G_2(y(s,x))ds]d\omega(t), \\
& \frac{\partial y}{\partial n} := \left(\frac{\partial y}{\partial x_1},\ldots,\frac{\partial y}{\partial x_n}\right) = 0, \quad t\geq t_0\geq 0, \quad x\in\partial\Omega, \\
& y(t_0+\theta,x,r_0) = \xi(\theta,x), \quad -\tau_0\leq\theta\leq 0, \quad r_0\in\mathbb{S}, \quad x\in\Omega.
\end{aligned}
\right. \tag{2.1}
$$

where $\Omega=\{x=(x_1,\ldots,x_m)^T, |x_i|<\pi, i=1,\ldots,m\}$ is a compact set with smooth boundary $\partial\Omega$ and measure $\mu(\Omega)>0$ in $\mathbb{R}^m$, $D^*(t,x,y)=(D^*_{ik}(t,x,y))_{n\times m}\geq 0$ diffusion operator; $y(t,x)=(y_1(t,x),\ldots,y_n(t,x))^\intercal$ is the state vector; $A(r(t)) = \mathrm{diag}(a_1(r(t)),\ldots,a_n(r(t)))>0$ are amplification functions; $W_l(r(t))=(W^l_{ij}(r(t)))_{n\times n}$

$(l = 0, 1, 2, 3)$ are connection weight matrices; $G_k(u) = (g_1^k(u_1), \ldots, g_2^k(u_n))^\intercal$, $(k = 0, 1, 2)$ are the activation functions; $I(r(t)) = (I_1(r(t)), \ldots, I_n(r(t)))^\intercal$ is an external input vector; $\tau(r(t)) = \tau_i, \delta(r(t)) = \delta_i$ are bounded time delay governed by a continuous-time discrete-state Markov process which takes values in a finite set; so, let $0 \le \tau_i \le \tau$, $0 \le \delta_i \le \delta$, $\tau_0 = \max\{\tau, \delta\}$. $\omega(t) = (\omega_1(t), \ldots, \omega_n(t))^\intercal \in \mathbb{R}^n$ is a Brownian motion defined on a completely space. $\{r(t), t > 0\}$ is a right-continuous Markov process on the probability space which takes values in the finite space $\mathbb{S} = \{1, 2, \ldots, N\}$ with generator $\pi = \{\pi_{ij}\}$ $(i, j \in \mathbb{S})$ (also called jumping transfer matrix) given by

$$\mathbb{P}\{r(t+\triangle) = j \mid r(t) = i\} = \begin{cases} \pi_{ij}\triangle + o(\triangle) & \text{if } i \neq j \\ 1 + \pi_{ii}\triangle + o(\triangle) & \text{if } i = j \end{cases} \tag{2.2}$$

$\triangle > 0$ and $\lim_{\triangle \to 0} \frac{o(\triangle)}{\triangle} = 0$. Here, $\pi_{ij} \ge 0$ is the transition rate from i to j if $i \neq j$ and $\pi_{ii} = -\sum_{j \neq i} \pi_{ij}$. As usual, we assume that the Brownian motion $\{\omega(t), t > 0\}$ is independent from the Markov process $\{r(t), t > 0\}$. For a fixed network mode $r(t) = i$, $A(r(t)), B(r(t)), C(r(t))$ and $W_l(r(t))(l = 0, 1, 2, 3)$ are know constant matrices with appropriate dimensions. The matrices $\triangle B(r(t))$, $\triangle C(r(t))$, $\triangle A(r(t))$ and $\triangle W_l(r(t))$ $(l = 0, 1, 2, 3)$ denote the parameter uncertainties. Recall that the Markov process $\{r(t), t > 0\}$ takes values in the finite space $\mathbb{S} = \{1, 2, \ldots, N\}$. For the sake of simplicity, we write $\forall i \in \mathbb{S}$,

$$\begin{gathered} A(i) = A_i, \quad W_l(i) = W_{li}, \quad \triangle W_l(i) = W_{li}, \\ \triangle B(i) = \triangle B_i, \quad \triangle C(i) = \triangle C_i, \quad (l = 0, 1, 2, 3) \end{gathered}$$

And they satisfy with the following structure

$$\begin{aligned} &[\triangle A_i, \triangle W_{0i}, \triangle W_{1i}, \triangle W_{2i}, \triangle W_{3i}, \triangle B_i, \triangle C_i] \\ &= M_i E_i [N_{1i}, N_{2i}, N_{3i}, N_{4i}, N_{5i}, N_{6i}, N_{7i}], \quad \forall i \in \mathbb{S} \end{aligned} \tag{2.3}$$

where M_i, N_{ki} $(k = 1, \ldots, 7)$ are known real constant matrices with appropriate dimensions. The uncertain matrix E_i satisfies $E_i^\intercal E_i \le I$.

Assumption 1. The neuron activation functions in (2.1), $G_k(u)$, satisfy the following Lipschitz condition

$$0 \le \frac{G_k(u_1) - G_k(u_2)}{u_1 - u_2} \le S_k, \quad \forall u_1, u_2 \in \mathbb{R}^n, \quad k = 0, 1, 2.$$

where $S_k \in \mathbb{R}^{n \times n}$, $(k = 0, 1, 2)$ are known constant matrices.

For the purpose of simplicity, system (2.1) can be rewritten as

$$(2.4)\quad \begin{cases} dy(t,x) = [\nabla \bullet (D^*(t,x,y) \circ \nabla y(t,x)) - (A_i + \triangle A_i)y(t,x) \\ \quad +(W_{0i} + \triangle W_{0i})G_0(y(t,x)) + (W_{1i} + \triangle W_{1i})G_1(y(t-\tau_i,x)) \\ \quad +(W_{2i} + \triangle W_{2i}) \int_{t-\delta_i}^{t} G_2(y(s,x))ds + I_i]dt + [(W_{3i} + \triangle W_{3i})y(t,x) \\ \quad +(B_i + \triangle B_i)y(t-\tau_i,x) + (C_i + \triangle C_i) \int_{t-\delta_i}^{t} G_2(y(s,x))ds]d\omega(t), \\ \frac{\partial y}{\partial n} := \left(\frac{\partial y}{\partial x_1}, \dots, \frac{\partial y}{\partial x_n}\right) = 0, \quad t \geq t_0 \geq 0, \quad x \in \partial\Omega, \\ y(t_0+\theta,x,\gamma_0) = \xi(\theta,x), \quad -\tau_0 \leq \theta \leq 0, \quad x \in \Omega, \quad \gamma_0 \in \mathbb{S}. \end{cases}$$

Let y^* be the equilibrium point of (2.4), we can shift the intended equilibrium y^* to the origin by letting $u = y - y^*$, and then the system (2.4) can be transformed into:

$$(2.5)\quad \begin{cases} du(t,x) = \Big[\nabla \bullet (D(t,x,u) \circ \nabla u(t,x)) - (A_i + \triangle A_i)u(t,x) \\ \quad +(W_{0i} + \triangle W_{0i})F_0(u(t,x)) + (W_{1i} + \triangle W_{1i})F_1(u(t-\tau_i,x)) \\ \quad +(W_{2i} + \triangle W_{2i}) \int_{t-\delta_i}^{t} F_2(u(s,x))ds\Big]dt + \Big[(W_{3i} + \triangle W_{3i})u(t,x) \\ \quad +(B_i + \triangle B_i)u(t-\tau_i,x) + (C_i + \triangle C_i) \int_{t-\delta_i}^{t} F_2(u(s,x))ds\Big]d\omega(t), \\ \frac{\partial u}{\partial n} = 0, \quad t \geq t_0 \geq 0, \quad x \in \partial\Omega, \\ u(t_0+\theta,x,\gamma_0) = \varphi(\theta,x), \quad -\tau_0 \leq \theta \leq 0, \quad x \in \Omega, \quad \gamma_0 \in \mathbb{S}. \end{cases}$$

where $u \in R^n$, $\varphi(\theta,x) = \xi(\theta,x) - y^*$, $D(t,x,u) = D^*(t,x,u+y^*)$, $F_k(u) = G_k(u + y^*) - G_k(y^*)$, $k = 0,1,2$. Then, it follows from Assumption 1, we can easily have

$$(2.6)\quad 0 \leq \frac{F_k(u)}{u} \leq S_k, \quad \forall u \in \mathbb{R}^n, \quad k = 0,1,2.$$

Definition 2.1. The equilibrium point u^* of system (2.5) is called robust exponentially stable on norm $\|\cdot\|_2$ in the mean square, if for every $\varphi \in L^P_{\mathbb{F}_0}([-\tau_0,0] \times \Omega; \mathbb{R}^n)$, there exist two positive scalar $\lambda > 0$ and $M > 0$ such that the following equality holds:

$$E\|u(t,\varphi)\|_2^2 \leq Me^{-\lambda t} \sup_{-\tau_0 \leq \theta \leq 0} E\|\varphi(\theta,x)\|_2^2.$$

Lemma 2.2.

$$(2.7)\quad V_1(u_t,t,\gamma(t)) = C(\gamma(t)) \int_\Omega \int_{t-\delta(\gamma(t))}^{t} u(s,x)ds\,dx$$

$$(2.8)\quad V_2(u_t,t,\gamma(t)) = \int_\Omega \int_{t-\delta(\gamma(t))}^{t} u^{\mathsf{T}}(s,x)Q(\gamma(t))u(s,x)ds\,dx$$

$$(2.9)\quad V_3(u_t,t,\gamma(t)) = \int_\Omega \int_{-\delta(\gamma(t))}^{0} \int_{t+\theta}^{t} u^{\mathsf{T}}(s,x)Ru(s,x)ds\,d\theta\,dx$$

Then, when $\gamma(t) = i$, we have

(2.10)
$$\mathcal{L}V_1(u_t, t, i) = \int_\Omega \left[C_i u(t,x) - (1-h_i) C_i u(t-\delta_i, x) + \sum_{j=1}^N \pi_{ij} C_j \int_{t-\delta_i}^t u(s,x)ds \right] dx$$

(2.11)
$$\mathcal{L}V_2(u_t, t, i) = \int_\Omega u^\intercal(t,x) Q_i u(t,x) dx + \sum_{j=1}^N \pi_{ij} \int_\Omega \int_{t-\delta_i}^t u^\intercal(s,x) Q_j u(s,x) ds\, dx - \int_\Omega (1-h_i) u^\intercal(t-\delta_i, x) Q_i u(t-\delta_i, x) dx$$

(2.12)
$$\mathcal{L}V_3(u_t, t, i) = \delta_i \int_\Omega u^\intercal(t,x) R u(t,x) dx - (1-h_i) \int_\Omega \int_{t-\delta_i}^t u^\intercal(s,x) R u(s,x) ds\, dx.$$

where $h_i = \sum_{j=1}^N \pi_{ij}\delta_j, u_t = \int_\Omega u(t,x)dx$.

Proof. According to the definition of infinitesimal operator, we have

(2.13)
$$\mathcal{L}V_1(u_t, t, i) = \lim_{\Delta \to 0^+} \frac{1}{\Delta} \left\{ \mathbb{E}\left[V_1(u_{t+\Delta}, \gamma(t+\Delta), t+\Delta) \middle| u_t, \gamma(t) = i \right] - V_1(u_t, t, i) \right\}$$

From additivity of integration on intervals, we derive

(2.14)
$$\begin{aligned} &\mathbb{E}\left[V_1(u_{t+\Delta}, \gamma(t+\Delta), t+\Delta) \middle| u_t, \gamma(t) = i\right] \\ &= \mathbb{E}\left[C(\gamma(t+\Delta)) \int_\Omega \int_{t-h_{\gamma(t+\Delta)}+\Delta}^{t+\Delta} u(s,x) ds\, dx \middle| u_t, \gamma(t) = i \right] \\ &= \mathbb{E}\left[C(\gamma(t+\Delta)) \int_\Omega \left(\int_{t-\delta_i}^t u(s,x)ds + \int_t^{t+\Delta} u(s,x)ds \right) dx \middle| u_t, \gamma(t) = i \right] \\ &\quad - \mathbb{E}\left[C(\gamma(t+\Delta)) \int_\Omega \left(\int_{t-\delta_i}^{t-\delta_i+\Delta} u(s,x)ds + \int_{t-\delta_i+\Delta}^{t-\delta_{\gamma(t+\Delta)}+\Delta} u(s,x)ds \right) dx \middle| u_t, \gamma(t) = i \right] \end{aligned}$$

By (2.2), we obtain

(2.15)
$$\begin{aligned} &\lim_{\Delta \to 0^+} \frac{1}{\Delta} \left\{ \mathbb{E}\left[C(\gamma(t+\Delta)) \int_\Omega \int_{t-\delta_i}^t u(s,x) ds\, dx \middle| u_t, \gamma(t) = i \right] \right. \\ &\qquad \left. - C_i \int_\Omega \int_{t-\delta_i}^t u(s,x) ds\, dx \right\} \\ &= \lim_{\Delta \to 0^+} \frac{1}{\Delta} \left\{ \left[\left(\sum_{j \neq i} \pi_{ij}\Delta + o(\Delta) \right) C_j + (1 + \pi_{ii}\Delta + o(\Delta)) C_i \right] \right. \\ &\quad \left. \times \int_\Omega \left[\int_{t-\delta_i}^t u(s,x)ds - C_i \int_{t-\delta_i}^t u(s,x)ds \right] dx \right\} \\ &= \sum_{j=1}^N \pi_{ij} C_j \int_\Omega \int_{t-\delta_i}^t u(s,x) ds\, dx \end{aligned}$$

Besides,

(2.16)
$$\lim_{\Delta \to 0^+} \frac{1}{\Delta} \left\{ \mathbb{E}\left[C(\gamma(t+\Delta)) \int_\Omega \left(\int_t^{t+\Delta} u(s,x)ds \right.\right.\right.$$

$$- \int_{t-\delta_i}^{t-\delta_i+\Delta} u(s,x)ds\Big)dx\big|u_t, \gamma(t) = i\Big]\Big\}$$
$$= C_i \int_\Omega u(t,x)dx - C_i \int_\Omega u(t-\delta_i, x)dx$$

Using integration mean value theorem and continuity, we get

$$\begin{aligned}
&\lim_{\Delta\to 0^+} \frac{1}{\Delta}\Big\{\mathbb{E}\Big[C(\gamma(t+\Delta)) \int_\Omega \int_{t-\delta_i+\Delta}^{t-\delta(\gamma(t+\Delta))+\Delta} u(s,x)ds\,dx\big|u_t, \gamma(t) = i\Big]\Big\} \\
&= \lim_{\Delta\to 0^+} \Big[(\textstyle\sum_{j\neq i} \pi_{ij}\Delta + o(\Delta))C_j + (1 + \pi_{ii}\Delta + o(\Delta))C_i\Big] \\
&\times \frac{1}{\Delta} \int_\Omega \int_{t-\delta_i+\Delta}^{t-\Big[(\sum_{j\neq i} \pi_{ij}\Delta + o(\Delta))\delta_j + (1+\pi_{ii}\Delta + o(\Delta))\delta_i\Big]+\Delta} u(s,x)ds\,dx \\
&= -h_i C_i \int_\Omega u(t-\delta_i, x)dx
\end{aligned} \tag{2.17}$$

Thus, $\mathcal{L}V_1(u_t, t, i)$ can be derived by computing (2.13)–(2.17). Similarly, (2.8) and (2.9) can be proofed. $\square$

Lemma 2.3 (Schur Complement)**.** *The LMI*

$$\begin{bmatrix} Q(t) & S(t) \\ S^\intercal(t) & R(t) \end{bmatrix} > 0$$

where $Q(t) = Q^\intercal(t), R(t) = R^\intercal(t)$, and $S(t)$ depend on t, is equivalent to any one of the following conditions:

$$\begin{aligned}
&(L_1) \quad R(t) > 0, \quad Q(t) - S(t)R^{-1}(t)S^\intercal(t) > 0; \\
&(L_2) \quad Q(t) > 0, \quad R(t) - S^\intercal(t)Q^{-1}(t)S(t) > 0.
\end{aligned}$$

Lemma 2.4. *Let $x \in \mathbb{R}^n, y \in \mathbb{R}^n$ and $\varepsilon > 0$. Then we have*

$$x^\intercal y + y^\intercal x \le \varepsilon x^\intercal x + \varepsilon^{-1} y^\intercal y.$$

Lemma 2.5 ([59])**.** *For any constant matrix $M > 0$, any scalars a and b with $a < b$, and a vector function $x(t) : [a, b] \to \mathbb{R}^n$ such that the integrals concerned are well defined, then the following holds*

$$\left[\int_a^b x(s)ds\right]^\intercal M \left[\int_a^b x(s)ds\right] \le (b-a) \int_a^b x^\intercal(s)Mx(s)ds.$$

3. MAIN RESULTS

Theorem 3.1. *The null solution of system (2.5) is robust stability on norm $\|\cdot\|_2$ in the mean square for any time-varying delays τ_i and δ_i satisfying $p_i \le 0$ and $h_i \le 0$,* *if there exist a sequence of positive scalars β_i ($i \in \mathbb{S}$) and positive definite matrices $Q > 0$ and $R > 0$ such that the following linear matrix inequalities*

$$\mathbf{\Gamma}_i = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ * & \Gamma_{22} & 0 & \Gamma_{24} \\ * & * & \Gamma_{33} & \Gamma_{34} \\ * & * & * & \Gamma_{44} \end{bmatrix} < 0. \tag{3.1}$$

hold, where

$$\begin{aligned}
\Gamma_{11} &= -\alpha I - \beta_i A_i - \beta_i A_i^{\mathsf{T}} + N_{1i}^{\mathsf{T}} N_{1i} + S_0^{\mathsf{T}} S_0 + S_0^{\mathsf{T}} N_{2i}^{\mathsf{T}} N_{2i} S_0 + \beta_i W_{3i}^{\mathsf{T}} W_{3i} \\
&\quad + (2\beta_i^2 + \beta_i + 1) N_{5i}^{\mathsf{T}} N_{5i} + \sum_{j=1}^{N} \pi_{ij} \beta_j I + Q + \delta_i R, \\
\Gamma_{12} &= W_{3i}^{\mathsf{T}} B_i + N_{5i}^{\mathsf{T}} N_{6i}, \\
\Gamma_{13} &= W_{3i}^{\mathsf{T}} C_i S_2 + B_i^{\mathsf{T}} C_i S_2 + N_{5i}^{\mathsf{T}} N_{7i} S_2 + N_{6i}^{\mathsf{T}} N_{7i} S_2, \\
\Gamma_{22} &= S_1^{\mathsf{T}} S_1 + S_1^{\mathsf{T}} N_{3i}^{\mathsf{T}} N_{3i} S_1 + (\beta_i + 1) B_i^{\mathsf{T}} B_i + (\beta_i + 2) N_{6i}^{\mathsf{T}} N_{6i} - (1 - p_i) Q, \\
\Gamma_{33} &= -(1 - h_i) R + S_2^{\mathsf{T}} S_2 + S_2^{\mathsf{T}} N_{4i}^{\mathsf{T}} N_{4i} S_2 + (\beta_i + 2) S_2^{\mathsf{T}} C_i^{\mathsf{T}} C_i S_2 + (\beta_i + 3) S_2^{\mathsf{T}} N_{7i}^{\mathsf{T}} N_{7i} S_2, \\
\Gamma_{14} &= [2\beta_i M_i, \beta_i W_{0i}, \beta_i W_{1i}, \beta_i W_{2i}, \beta_i B_i^{\mathsf{T}}, \beta_i W_{3i}^{\mathsf{T}}, \beta_i N_{5i}^{\mathsf{T}}, \beta_i N_{6i}^{\mathsf{T}}], \\
\Gamma_{24} &= [\beta_i B_i^{\mathsf{T}}, 0, 0, 0, 0, 0, 0, 0], \\
\Gamma_{34} &= [\beta_i S_2^{\mathsf{T}} C_i^{\mathsf{T}}, 0, 0, 0, 0, 0, 0, 0], \\
\Gamma_{44} &= diag\left[-I, -I, -I, -I, -I, -\frac{1}{3}I, -\frac{1}{2}I, -I\right].
\end{aligned}$$

Proof. Let $\mathbb{C}^{2,1}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{S}; \mathbb{R}_+)$ denote the family of all nonnegative functions $V(t, y, i)$ on $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{S}$ which are continuous twice differentiable in y and differentiable in t. Given $\varphi \in \mathcal{L}_{\mathbb{F}_0}^P([-\tau_0, 0] \times \Omega; \mathbb{R}^n)$, and fixed system mode $i \in \mathbb{S}$ arbitrarily. Write $u(t, x) = u(t, x; \varphi)$, and define a Lyapunov-Krasovskii functional candidate by

$$V(t, u(t, x), i) = V_1(t, u(t, x), i) + V_2(t, u(t, x), i) + V_3(t, u(t, x), i)$$

where

$$\begin{aligned}
V_1(t, u(t, x), i) &= \int_\Omega \beta_i u^{\mathsf{T}}(t, x) u(t, x) dx, \\
V_2(t, u(t, x), i) &= \int_\Omega \int_{t-\tau_i}^{t} u^{\mathsf{T}}(s, x) Q u(s, x) ds\, dx, \\
V_3(t, u(t, x), i) &= \int_\Omega \int_{-\delta_i}^{0} \int_{t+\theta}^{t} u^{\mathsf{T}}(s, x) R u(s, x) ds\, d\theta\, dx.
\end{aligned}$$

It is known that $\{u(t, x), r(t)\}$ ($t \geq t_0$) is a $\mathbb{C}([-\tau_0, 0] \times \Omega; \mathbb{R}^n) \times \mathbb{S}$-valued Markov process. From (6), the weak infinitesimal operator $\mathcal{L}$ of the stochastic process $\{u(t, x), r(t)\}$ ($t > 0$) can be calculated as

$$\mathcal{L}V_1(t, u(t, x), i) = 2\beta_i \int_\Omega u^{\mathsf{T}}(t, x) \Big[\nabla \cdot (D(t, x, u) \circ \nabla u(t, x)) - (A_i + \triangle A_i) u(t, x)$$

$$
\begin{aligned}
&+ (W_{0i} + \triangle W_{0i})F_0(u(t,x)) + (W_{1i} + \triangle W_{1i})F_1(y(t-\tau_i,x)) \\
&+ (W_{2i} + \triangle W_{2i}) \int_{t-\delta_i}^{t} F_2(u(s,x))ds\Big] dx \\
&+ \sum_{j=1}^{N} \pi_{ij}\beta_j \int_\Omega u^\intercal(t,x)u(t,x)dx \\
&+ \int_\Omega \Big[(W_{3i} + \triangle W_{3i})u(t,x) + (B_i + \triangle B_i)u(t-\tau_i,x) \\
&+ (C_i + \triangle C_i) \int_{t-\delta_i}^{t} F_2(u(s,x))\Big]^\intercal \beta_i \Big[(W_{3i} + \triangle W_{3i})u(t,x) \\
&+ (B_i + \triangle B_i)u(t-\tau_i,x) + (C_i + \triangle C_i) \int_{t-\delta_i}^{t} F_2(u(s,x))\Big] dx,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}V_2(t,u(t,x),i) &= \int_\Omega u^\intercal(t,x)Qu(t,x)dx - (1-p_i)\int_\Omega u^\intercal(t-\tau_i,x)Qu(t-\tau_i,x)dx \\
&+ \sum_{j=1}^{N}\pi_{ij}\int_\Omega\int_{t-\tau_i}^{t} u^\intercal(s,x)Qu(s,x)ds\,dx,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}V_3(t,u(t,x),i) &= \delta_i \int_\Omega u^\intercal(t,x)Ru(t,x)dx - (1-h_i)\int_\Omega\int_{t-\delta_i}^{t} u^\intercal(s,x)Ru(s,x)ds\,dx \\
&\le \delta_i \int_\Omega u^\intercal(t,x)Ru(t,x)dx - \frac{(1-h_i)}{\delta_i}\int_\Omega \left[\int_{t-\delta_i}^{t} u(s,x)ds\right]^\intercal \\
&\times R\left[\int_{t-\delta_i}^{t} u(s,x)ds\right] dx.
\end{aligned}
$$

where $p_i = \sum_{j=1}^{N}\pi_{ij}\tau_j$, $h_i = \sum_{j=1}^{N}\pi_{ij}\delta_j$. It follows from $\sum_{j=1}^{N}\pi_{ij} = 0$ that

$$
\sum_{j=1}^{N}\pi_{ij}\int_\Omega\int_{t-\tau_i}^{t} u^\intercal(s,x)Qu(s,x)ds\,dx = 0. \tag{3.2}
$$

Hence, from Lemma 2.5, we have

$$
2\beta_i \int_\Omega u^\intercal(t,x)\nabla\cdot(D(t,x,u)\circ\nabla u(t,x))dx \le -\alpha\int_\Omega u^\intercal(t,x)u(t,x)dx. \tag{3.3}
$$

From Lemma 2.4 and (2.3), we can easily obtain

$$
\begin{aligned}
&-2\beta_i \int_\Omega u^\intercal(t,x)(A_i + \triangle A_i)u(t,x)dx \qquad (3.4)\\
&= -2\beta_i \int_\Omega u^\intercal(t,x)A_iu(t,x)dx - 2\beta_i\int_\Omega u^\intercal(t,x)\triangle A_iu(t,x)dx \\
&= -2\beta_i \int_\Omega u^\intercal(t,x)A_iu(t,x)dx - 2\beta_i\int_\Omega u^\intercal(t,x)M_iE_iN_{1i}u(t,x)dx \\
&\le \int_\Omega u^\intercal(t,x)[-2\beta_iA_i + \beta_i^2M_iM_i^\intercal + N_{1i}^\intercal N_{1i}]u(t,x)dx.
\end{aligned}
$$

From Lemma 2.4 and (2.3), (2.6), we have

$$2\beta_i \int_\Omega u^\intercal(t,x)(W_{0i}+\triangle W_{0i})F_0(u(t,x))dx = 2\beta_i \int_\Omega u^\intercal(t,x)W_{0i}F_0(u(t,x))dx + 2\beta_i \int_\Omega u^\intercal(t,x)\triangle W_{0i}F_0(u(t,x))dx.$$

For the first part of above equation, we have

$$\begin{aligned}2\beta_i \int_\Omega u^\intercal(t,x)W_{0i}F_0(u(t,x))dx &\le \beta_i^2 \int_\Omega u^\intercal(t,x)W_{0i}W_{0i}^\intercal u(t,x)dx \\ &\quad + \int_\Omega F_0^\intercal(u(t,x))F_0^\intercal(u(t,x))dx \\ &\le \beta_i^2 \int_\Omega u^\intercal(t,x)W_{0i}W_{0i}^\intercal u(t,x)dx + \int_\Omega u^\intercal(t,x)S_0^\intercal S_0 u(t,x)dx.\end{aligned}$$

For the second part, we have

$$\begin{aligned}2\beta_i \int_\Omega u^\intercal(t,x)\triangle W_{0i}F_0(u(t,x))dx &= 2\beta_i \int_\Omega u^\intercal(t,x)M_iE_iN_{2i}F_0(u(t,x))dx \\ &\le \beta_i^2 \int_\Omega u^\intercal(t,x)M_iM_i^\intercal u(t,x)dx + \int_\Omega F_0^\intercal(u(t,x))N_{2i}^\intercal N_{2i}F_0(u(t,x))dx \\ &\le \beta_i^2 \int_\Omega u^\intercal(t,x)M_iM_i^\intercal u(t,x)dx + \int_\Omega u^\intercal(t,x)S_0^\intercal N_{2i}^\intercal N_{2i}S_0u(t,x)dx.\end{aligned}$$

So, we can obtain that

$$\begin{aligned}&2\beta_i \int_\Omega u^\intercal(t,x)(W_{0i}+\triangle W_{0i})F_0(u(t,x))dx \\ &\le \int_\Omega u^\intercal(t,x)[\beta_i^2(W_{0i}W_{0i}^\intercal + M_iM_i^\intercal)]u(t,x)dx \\ &\quad + \int_\Omega u^\intercal(t,x)[S_0^\intercal S_0 + S_0^\intercal N_{2i}^\intercal N_{2i}S_0]u(t,x)dx.\end{aligned} \tag{3.5}$$

From Lemma 2.4 and (2.3), (2.6), we have

$$2\beta_i \int_\Omega u^\intercal(t,x)(W_{1i}+\triangle W_{1i})F_1(u(t-\tau_i,x))dx = 2\beta_i \int_\Omega u^\intercal(t,x)W_{1i}F_1(u(t-\tau_i,x))dx + 2\beta_i \int_\Omega u^\intercal(t,x)\triangle W_{1i}F_1(u(t-\tau_i,x))dx.$$

For the first part, we have

$$\begin{aligned}2\beta_i \int_\Omega u^\intercal(t,x)W_{1i}F_1(u(t-\tau_i,x))dx &\le \beta_i^2 \int_\Omega u^\intercal(t,x)W_{1i}W_{1i}^\intercal u(t,x)dx \\ &\quad + \int_\Omega F_1^\intercal(u(t-\tau_i,x))F_1^\intercal(u(t-\tau_i,x))dx \\ &\le \beta_i^2 \int_\Omega u^\intercal(t,x)W_{1i}W_{1i}^\intercal u(t,x)dx + \int_\Omega u^\intercal(t-\tau_i,x)S_1^\intercal S_1u(t-\tau_i,x)dx.\end{aligned}$$

For the second part, we have

$$2\beta_i \int_\Omega u^\intercal(t,x)\triangle W_{1i}F_1(u(t-\tau_i,x))dx = 2\beta_i \int_\Omega u^\intercal(t,x)M_iE_iN_{3i}F_1(u(t-\tau_i,x))dx$$

$$
\begin{aligned}
&\le \beta_i^2 \int_\Omega u^\intercal(t,x) M_i M_i^\intercal u(t,x) dx \\
&\quad + \int_\Omega F_1^\intercal(u(t-\tau_i,x)) N_{3i}^\intercal N_{3i} F_1(u(t-\tau_i,x)) dx \\
&\le \beta_i^2 \int_\Omega u^\intercal(t,x) M_i M_i^\intercal u(t,x) dx \\
&\quad + \int_\Omega u^\intercal(t-\tau_i,x) S_1^\intercal N_{3i}^\intercal N_{3i} S_1 u(t-\tau_i,x) dx.
\end{aligned}
$$

So, we can obtain that

$$
\begin{aligned}
&2\beta_i \int_\Omega u^\intercal(t,x)(W_{1i} + \triangle W_{1i}) F_1(u(t-\tau_i,x)) dx \\
&\le \int_\Omega u^\intercal(t,x)[\beta_i^2 (W_{1i}W_{1i}^\intercal + M_i M_i^\intercal)] u(t,x) dx \\
&\quad + \int_\Omega u^\intercal(t-\tau_i,x)[S_1^\intercal S_1 + S_1^\intercal N_{3i}^\intercal N_{3i} S_1] u(t-\tau_i,x) dx.
\end{aligned} \tag{3.6}
$$

From Lemma 2.4, Lemma 2.5 and (2.3), (2.6) we have

$$
\begin{aligned}
&2\beta_i \int_\Omega u^\intercal(t,x)(W_{2i} + \triangle W_{2i}) \int_{t-\delta_i}^t F_2(u(s,x)) ds dx \\
&= 2\beta_i \int_\Omega u^\intercal(t,x) W_{2i} \int_{t-\delta_i}^t F_2(u(s,x)) ds dx \\
&\quad + 2\beta_i \int_\Omega u^\intercal(t,x) \triangle W_{2i} \int_{t-\delta_i}^t F_2(u(s,x)) ds dx.
\end{aligned}
$$

For the first part, we get

$$
\begin{aligned}
&2\beta_i \int_\Omega u^\intercal(t,x) W_{2i} \int_{t-\delta_i}^t F_2(u(s,x)) ds dx \\
&\le \beta_i^2 \int_\Omega u^\intercal(t,x) W_{2i} W_{2i}^\intercal u(t,x) dx \\
&\quad + \int_\Omega \left[\int_{t-\delta_i}^t F_2(u(s,x)) ds\right]^\intercal \left[\int_{t-\delta_i}^t F_2(u(s,x)) ds\right] dx \\
&\le \beta_i^2 \int_\Omega u^\intercal(t,x) W_{2i} W_{2i}^\intercal u(t,x) dx \\
&\quad + \int_\Omega \left[\int_{t-\delta_i}^t S_2 u(s,x) ds\right]^\intercal \left[\int_{t-\delta_i}^t S_2 u(s,x) ds\right] dx \\
&\le \beta_i^2 \int_\Omega u^\intercal(t,x) W_{2i} W_{2i}^\intercal u(t,x) dx \\
&\quad + \int_\Omega \left[\int_{t-\delta_i}^t u(s,x) ds\right]^\intercal S_2^\intercal S_2 \left[\int_{t-\delta_i}^t u(s,x) ds\right] dx.
\end{aligned}
$$

For the second part, we get

$$
2\beta_i \int_\Omega u^\intercal(t,x) \triangle W_{2i} \int_{t-\delta_i}^t F_2(u(s,x)) ds dx
$$

$$
\begin{aligned}
&= 2\beta_i \int_\Omega u^\intercal(t,x) M_i E_i N_{4i} \int_{t-\delta_i}^t F_2(u(s,x))dsdx \\
&\le \beta_i^2 \int_\Omega u^\intercal(t,x) M_i M_i^\intercal u(t,x)dx \\
&\quad + \int_\Omega \left[E_i N_{4i} \int_{t-\delta_i}^t F_2(u(s,x))ds\right]^\intercal \left[E_i N_{4i} \int_{t-\delta_i}^t F_2(u(s,x))ds\right] dx \\
&\le \beta_i^2 \int_\Omega u^\intercal(t,x) M_i M_i^\intercal u(t,x)dx \\
&\quad + \int_\Omega \left[\int_{t-\delta_i}^t F_2(u(s,x))ds\right]^\intercal N_{4i}^\intercal N_{4i} \left[\int_{t-\delta_i}^t F_2(u(s,x))ds\right] dx \\
&\le \beta_i^2 \int_\Omega u^\intercal(t,x) M_i M_i^\intercal u(t,x)dx \\
&\quad + \int_\Omega \left[\int_{t-\delta_i}^t u(s,x)ds\right] S_2^\intercal N_{4i}^\intercal N_{4i} S_2 \left[\int_{t-\delta_i}^t u(s,x)ds\right] dx.
\end{aligned}
$$

So, we can obtain that

$$
\begin{aligned}
&2\beta_i \int_\Omega u^\intercal(t,x)(W_{2i} + \triangle W_{2i}) \int_{t-\delta_i}^t F_2(u(s,x))dsdx \\
&\quad \le \beta_i^2 \int_\Omega u^\intercal(t,x)[W_{2i}W_{2i}^\intercal + M_i M_i^\intercal] u(t,x)dx \\
&\qquad + \int_\Omega \left[\int_{t-\delta_i}^t u(s,x)ds\right] [S_2^\intercal S_2 + S_2^\intercal N_{4i}^\intercal N_{4i} S_2] \left[\int_{t-\delta_i}^t u(s,x)ds\right] dx.
\end{aligned} \tag{3.7}
$$

Suppose $M_i^\intercal M_i \le I$ and from (2.3), we can easily obtain that

$$
\begin{aligned}
&\beta_i \int_\Omega [(W_{3i} + \triangle W_{3i})u(t,x) + (B_i + \triangle B_i)u(t-\tau_i,x) + (C_i + \triangle C_i) \int_{t-\delta_i}^t F_2(u(s,x))]^\intercal \\
&\cdot [(W_{3i} + \triangle W_{3i})u(t,x) + (B_i + \triangle B_i)u(t-\tau_i,x) + (C_i + \triangle C_i) \int_{t-\delta_i}^t F_2(u(s,x))]dx \\
&= \beta_i \int_\Omega u^\intercal(t,x)(W_{3i}^\intercal + \triangle W_{3i}^\intercal)(W_{3i} + \triangle W_{3i})u(t,x)dx \\
&\quad + \beta_i \int_\Omega u^\intercal(t-\tau_i,x)(B_i^\intercal + \triangle B_i^\intercal)(B_i + \triangle B_i)u(t-\tau_i,x)dx \\
&\quad + \beta_i \int_\Omega [\int_{t-\delta_i}^t F_2(u(s,x))ds]^\intercal (C_i^\intercal + \triangle C_i^\intercal)(C_i + \triangle C_i)[\int_{t-\delta_i}^t F_2(u(s,x))ds]dx \\
&\quad + 2\beta_i \int_\Omega u^\intercal(t,x)(W_{3i}^\intercal + \triangle W_{3i}^\intercal)(B_i + \triangle B_i)u(t-\tau_i,x)dx \\
&\quad + 2\beta_i \int_\Omega u^\intercal(t,x)(W_{3i}^\intercal + \triangle W_{3i}^\intercal)(C_i + \triangle C_i)[\int_{t-\delta_i}^t F_2(u(s,x))ds]dx \\
&\quad + 2\beta_i \int_\Omega u^\intercal(t-\tau_i,x)(B_i^\intercal + \triangle B_i^\intercal)(C_i + \triangle C_i)[\int_{t-\delta_i}^t F_2(u(s,x))ds]dx.
\end{aligned}
$$

For the first part, we have

$$\begin{aligned}&\beta_i \int_\Omega u^\intercal(t,x)(W_{3i}^\intercal + \triangle W_{3i}^\intercal)(W_{3i} + \triangle W_{3i})u(t,x)dx\\ &= \beta_i \int_\Omega u^\intercal(t,x)[W_{3i}^\intercal W_{3i} + W_{3i}^\intercal \triangle W_{3i} + \triangle W_{3i}^\intercal W_{3i} + \triangle W_{3i}^\intercal \triangle W_{3i}]u(t,x)dx\end{aligned}$$

Noting that

$$\begin{aligned}&\beta_i \int_\Omega u^\intercal(t,x)W_{3i}^\intercal \triangle W_{3i}u(t,x)dx = \beta_i \int_\Omega u^\intercal(t,x)W_{3i}^\intercal M_i E_i N_{5i}u(t,x)dx\\ &\le \frac{\beta_i^2}{2}\int_\Omega u^\intercal(t,x)W_{3i}^\intercal W_{3i}u(t,x)dx + \frac{1}{2}\int_\Omega u^\intercal(t,x)N_{5i}^\intercal E_i^\intercal M_i^\intercal M_i E_i N_{5i}u(t,x)dx\\ &\le \frac{\beta_i^2}{2}\int_\Omega u^\intercal(t,x)W_{3i}^\intercal W_{3i}u(t,x)dx + \frac{1}{2}\int_\Omega u^\intercal(t,x)N_{5i}^\intercal N_{5i}u(t,x)dx,\end{aligned}$$

and

$$\begin{aligned}&\beta_i \int_\Omega u^\intercal(t,x)\triangle W_{3i}^\intercal W_{3i}u(t,x)dx = \beta_i \int_\Omega u^\intercal(t,x)N_{5i}^\intercal E_i^\intercal M_i^\intercal W_{3i}u(t,x)dx\\ &\le \frac{1}{2}\int_\Omega u^\intercal(t,x)N_{5i}^\intercal E_i^\intercal M_i^\intercal M_i E_i N_{5i}u(t,x)dx + \frac{\beta_i^2}{2}\int_\Omega u^\intercal(t,x)W_{3i}^\intercal W_{3i}u(t,x)dx\\ &\le \frac{1}{2}\int_\Omega u^\intercal(t,x)N_{5i}^\intercal N_{5i}u(t,x)dx + \frac{\beta_i^2}{2}\int_\Omega u^\intercal(t,x)W_{3i}^\intercal W_{3i}u(t,x)dx,\end{aligned}$$

and

$$\begin{aligned}\beta_i \int_\Omega u^\intercal(t,x)\triangle W_{3i}^\intercal \triangle W_{3i}u(t,x)dx &= \beta_i \int_\Omega u^\intercal(t,x)N_{5i}^\intercal E_i^\intercal M_i^\intercal M_i E_i N_{5i}u(t,x)dx\\ &\le \beta_i \int_\Omega u^\intercal(t,x)N_{5i}^\intercal N_{5i}u(t,x)dx.\end{aligned}$$

So, from above inequalities, we have

$$\begin{aligned}&\beta_i \int_\Omega u^\intercal(t,x)(W_{3i}^\intercal + \triangle W_{3i}^\intercal)(W_{3i} + \triangle W_{3i})u(t,x)dx\\ &\le \int_\Omega u^\intercal(t,x)[(\beta_i^2+\beta_i)W_{3i}^\intercal W_{3i} + (\beta_i+1)N_{5i}^\intercal N_{5i}]u(t,x)dx.\end{aligned} \tag{3.8}$$

Similarly

$$\begin{aligned}&\beta_i \int_\Omega u^\intercal(t-\tau_i,x)(B_i^\intercal + \triangle B_i^\intercal)(B_i + \triangle B_i)u(t-\tau_i,x)dx\\ &\le \int_\Omega u^\intercal(t-\tau_i,x)[(\beta_i^2+\beta_i)B_i^\intercal B_i + (\beta_i+1)N_{6i}^\intercal N_{6i}]u(t-\tau_i,x)dx.\end{aligned} \tag{3.9}$$

and

(3.10)

$$\begin{aligned}&\beta_i \int_\Omega \left[\int_{t-\delta_i}^t F_2(u(s,x))ds\right]^\intercal (C_i^\intercal + \triangle C_i^\intercal)(C_i + \triangle C_i)\left[\int_{t-\delta_i}^t F_2(u(s,x))ds\right]dx\\ &\le \int_\Omega \left[\int_{t-\delta_i}^t F_2(u(s,x))ds\right]^\intercal \left[(\beta_i^2+\beta_i)C_i^\intercal C_i + (\beta_i+1)N_{7i}^\intercal N_{7i}\right]\left[\int_{t-\delta_i}^t F_2(u(s,x))ds\right]dx\end{aligned}$$

$$\leq \int_\Omega \left[\int_{t-\delta_i}^t S_2 u(s,x)ds\right]^\intercal [(\beta_i^2+\beta_i)C_i^\intercal C_i + (\beta_i+1)N_{7i}^\intercal N_{7i}]\left[\int_{t-\delta_i}^t S_2 u(s,x)ds\right]dx$$
$$\leq \int_\Omega \left[\int_{t-\delta_i}^t u(s,x)ds\right]^\intercal [(\beta_i^2+\beta_i)S_2^\intercal C_i^\intercal C_i S_2 + (\beta_i+1)S_2^\intercal N_{7i}^\intercal N_{7i} S_2]\left[\int_{t-\delta_i}^t u(s,x)ds\right]dx.$$

For the fourth part, we have

$$\begin{aligned}&2\beta_i \int_\Omega u^\intercal(t,x)(W_{3i}^\intercal + \triangle W_{3i}^\intercal)(B_i+\triangle B_i)u(t-\tau_i,x)dx\\ &\quad = 2\beta_i \int_\Omega u^\intercal(t,x)[W_{3i}^\intercal B_i + W_{3i}^\intercal \triangle B_i + \triangle W_{3i}^\intercal B_i + \triangle W_{3i}^\intercal \triangle B_i]u(t-\tau_i,x)dx.\end{aligned}$$

Noting that

$$\begin{aligned}&2\beta_i \int_\Omega u^\intercal(t,x)W_{3i}^\intercal \triangle B_i u(t,x)dx = 2\beta_i \int_\Omega u^\intercal(t,x)W_{3i}^\intercal M_i E_i N_{6i} u(t-\tau_i,x)dx\\ &\leq \beta_i^2 \int_\Omega u^\intercal(t,x)W_{3i}^\intercal W_{3i}u(t,x)dx + \int_\Omega u^\intercal(t-\tau_i,x)N_{6i}^\intercal E_i^\intercal M_i^\intercal M_i E_i N_{6i} u(t-\tau_i,x)dx\\ &\leq \beta_i^2 \int_\Omega u^\intercal(t,x)W_{3i}^\intercal W_{3i}u(t,x)dx + \int_\Omega u^\intercal(t-\tau_i,x)N_{6i}^\intercal N_{6i} u(t-\tau_i,x)dx,\end{aligned}$$

and

$$\begin{aligned}&2\beta_i \int_\Omega u^\intercal(t,x)\triangle W_{3i}^\intercal B_i u(t-\tau_i,x)dx = \beta_i \int_\Omega u^\intercal(t,x)N_{5i}^\intercal E_i^\intercal M_i^\intercal B_i u(t-\tau_i,x)dx\\ &\leq \beta_i^2 \int_\Omega u^\intercal(t,x)N_{5i}^\intercal E_i^\intercal M_i^\intercal M_i E_i N_{5i}u(t,x)dx + \int_\Omega u^\intercal(t-\tau_i,x)B_i^\intercal B_i u(t-\tau_i,x)dx\\ &\leq \beta_i^2 \int_\Omega u^\intercal(t,x)N_{5i}^\intercal N_{5i}u(t,x)dx + \int_\Omega u^\intercal(t-\tau_i,x)B_i^\intercal B_i u(t-\tau_i,x)dx,\end{aligned}$$

and

$$\begin{aligned}2\beta_i \int_\Omega u^\intercal(t,x)\triangle W_{3i}^\intercal \triangle B_i u(t-\tau_i,x)dx &= 2\beta_i \int_\Omega u^\intercal(t,x)N_{5i}^\intercal E_i^\intercal M_i^\intercal M_i E_i N_{6i}u(t-\tau_i,x)dx\\ &\leq 2\beta_i \int_\Omega u^\intercal(t,x)N_{5i}^\intercal N_{6i}u(t-\tau_i,x)dx.\end{aligned}$$

So, from above inequalities, we have

$$\begin{aligned}&2\beta_i \int_\Omega u^\intercal(t,x)(W_{3i}^\intercal + \triangle W_{3i}^\intercal)(B_i+\triangle B_i)u(t-\tau_i,x)dx\\ &\leq \beta_i^2 \int_\Omega u^\intercal(t,x)[W_{3i}^\intercal W_{3i} + N_{5i}^\intercal N_{5i}]u(t,x)dx\\ &\quad + 2\beta_i \int_\Omega u^\intercal(t,x)(W_{3i}^\intercal B_i + N_{5i}^\intercal N_{6i})u(t-\tau_i,x)dx\\ &\quad + \int_\Omega u^\intercal(t-\tau_i,x)[B_i^\intercal B_i + N_{6i}^\intercal N_{6i}]u(t-\tau_i,x)dx.\end{aligned} \tag{3.11}$$

Similarly

$$2\beta_i \int_\Omega u^\intercal(t,x)(W_{3i}^\intercal + \triangle W_{3i}^\intercal)(C_i+\triangle C_i)\int_{t-\delta_i}^t F_2(u(s,x))ds\,dx \tag{3.12}$$

$$
\begin{aligned}
&\leq \beta_i^2 \int_\Omega u^\intercal(t,x)[W_{3i}^\intercal W_{3i} + N_{5i}^\intercal N_{5i}]u(t,x)dx \\
&\quad + 2\beta_i \int_\Omega u^\intercal(t,x)[W_{3i}^\intercal C_i + N_{5i}^\intercal N_{7i}] \int_{t-\delta_i}^t F_2(u(s,x))ds\, dx \\
&\quad + \int_\Omega \left[\int_{t-\delta_i}^t F_2(u(s,x))ds\right]^\intercal [C_i^\intercal C_i + N_{7i}^\intercal N_{7i}] \left[\int_{t-\delta_i}^t F_2(u(s,x))ds\right] dx \\
&\leq \beta_i^2 \int_\Omega u^\intercal(t,x)[W_{3i}^\intercal W_{3i} + N_{5i}^\intercal N_{5i}]u(t,x)dx \\
&\quad + 2\beta_i \int_\Omega u^\intercal(t,x)[W_{3i}^\intercal C_i + N_{5i}^\intercal N_{7i}] \int_{t-\delta_i}^t S_2(u(s,x))ds\, dx \\
&\quad + \int_\Omega \left[\int_{t-\delta_i}^t S_2 u(s,x)ds\right]^\intercal [C_i^\intercal C_i + N_{7i}^\intercal N_{7i}] \left[\int_{t-\delta_i}^t S_2 u(s,x)ds\right] dx \\
&= \beta_i^2 \int_\Omega u^\intercal(t,x)[W_{3i}^\intercal W_{3i} + N_{5i}^\intercal N_{5i}]u(t,x)dx \\
&\quad + 2\beta_i \int_\Omega u^\intercal(t,x)[W_{3i}^\intercal C_i S_2 + N_{5i}^\intercal N_{7i} S_2] \int_{t-\delta_i}^t u(s,x)ds\, dx \\
&\quad + \int_\Omega \left[\int_{t-\delta_i}^t u(s,x)ds\right]^\intercal [S_2^\intercal C_i^\intercal C_i S_2 + S_2^\intercal N_{7i}^\intercal N_{7i} S_2] \left[\int_{t-\delta_i}^t u(s,x)ds\right] dx.
\end{aligned}
$$

and

$$
\begin{aligned}
(3.13)\qquad & 2\beta_i \int_\Omega u^\intercal(t,x)(B_i^\intercal + \triangle B_i^\intercal)(C_i + \triangle C_i) \int_{t-\delta_i}^t F_2(u(s,x))ds\, dx \\
&\leq \beta_i^2 \int_\Omega u^\intercal(t,x)[B_i^\intercal B_i + N_{6i}^\intercal N_{6i}]u(t,x)dx \\
&\quad + 2\beta_i \int_\Omega u^\intercal(t,x)[B_i^\intercal C_i S_2 + N_{6i}^\intercal N_{7i} S_2] \int_{t-\delta_i}^t u(s,x)ds\, dx \\
&\quad + \int_\Omega \left[\int_{t-\delta_i}^t u(s,x)ds\right]^\intercal [S_2^\intercal C_i^\intercal C_i S_2 + S_2^\intercal N_{7i}^\intercal N_{7i} S_2] \left[\int_{t-\delta_i}^t u(s,x)ds\right] dx.
\end{aligned}
$$

Hence, from (3.8)-(3.13) we have

(3.14)

$$
\begin{aligned}
&\beta_i \int_\Omega [(W_{3i} + \triangle W_{3i})u(t,x) + (B_i + \triangle B_i)u(t-\tau_i,x) + (C_i + \triangle C_i) \int_{t-\delta_i}^t F_2(u(s,x))]^\intercal \\
&\quad \times [(W_{3i} + \triangle W_{3i})u(t,x) + (B_i + \triangle B_i)u(t-\tau_i,x) + (C_i + \triangle C_i) \int_{t-\delta_i}^t F_2(u(s,x))]dx \\
&\leq \int_\Omega u^\intercal(t,x)[(3\beta_i^2 + \beta_i)W_{3i}^\intercal W_{3i} + (3\beta_i^2 + \beta_i + 1)N_{5i}^\intercal N_{5i} + \beta_i^2 N_{6i}^\intercal N_{6i} + \beta_i^2 B_i^\intercal B_i]u(t,x)dx \\
&\quad + \int_\Omega \left[\int_{t-\delta_i}^t u(s,x)ds\right] [(\beta_i^2 + \beta_i + 2)S_2^\intercal C_i^\intercal C_i S_2 + (\beta_i + 3)S_2^\intercal N_{7i}^\intercal N_{7i} S_2] \left[\int_{t-\delta_i}^t u(s,x)ds\right] dx \\
&\quad + \int_\Omega u^\intercal(t-\tau_i,x)[(\beta_i^2 + \beta_i + 1)B_i^\intercal B_i + (\beta_i + 2)N_{6i}^\intercal N_{6i}]u(t-\tau_i,x)dx
\end{aligned}
$$

$$+ 2\beta_i \int_\Omega u^\intercal(t,x)[W_{3i}^\intercal B_i + N_{5i}^\intercal N_{6i}]u(t-\tau_i,x)dx$$
$$+ 2\beta_i \int_\Omega u^\intercal(t,x)[W_{3i}^\intercal C_i S_2 + B_i^\intercal C_i S_2 + N_{5i}^\intercal N_{7i} S_2 + N_{6i}^\intercal N_{7i} S_2] \left[\int_{t-\delta_i}^t u(s,x)ds\right] dx.$$

Therefore, from (3.2)–(3.7) and (3.14), we can easily obtain

$$\begin{aligned}\mathcal{L}V(t,u(t,x),i) \le \int_\Omega u^\intercal(t,x)\Big[&-\alpha I - 2\beta_i A_i + N_{1i}^\intercal N_{1i} + S_0^\intercal S_0 + S_0^\intercal N_{2i}^\intercal N_{2i} S_0 \\ &+ 4\beta_i^2 M_i M_i^\intercal + \beta_i^2 \sum_{l=0}^{2} W_{li} W_{li}^\intercal + (3\beta_i^2+\beta_i) W_{3i}^\intercal W_{3i} + (2\beta_i^2 \\ &+ \beta_i + 1) N_{5i}^\intercal N_{5i} + \beta_i^2 N_{6i}^\intercal N_{6i} \\ &+ \beta_i^2 B_i^\intercal B_i + \sum_{j=1}^{N} \pi_{ij}\beta_j I + Q + \delta_i R\Big] u(t,x)dx \\ &+ \int_\Omega u^\intercal(t-\tau_i,x)[S_1^\intercal S_1 + S_1^\intercal N_{3i}^\intercal N_{3i} S_1 + (\beta_i^2+\beta_i+1) B_i^\intercal B_i \\ &+ (\beta_i+2) N_{6i}^\intercal N_{6i} - (1-p_i)Q] u(t-\tau_i,x)dx \\ &+ \int_\Omega \left[\int_{t-\delta_i}^t u(s,x)ds\right]^\intercal [-(1-h_i)R + S_2^\intercal S_2 + S_2^\intercal N_{4i}^\intercal N_{4i} S_2 \\ &+ (\beta_i^2+\beta_i+2) S_2^\intercal C_i^\intercal C_i S_2 + (\beta_i+3) S_2^\intercal N_{7i}^\intercal N_{7i} S_2] \left[\int_{t-\delta_i}^t u(s,x)ds\right] dx \\ &+ 2\beta_i \int_\Omega u^\intercal(t,x)[W_{3i}^\intercal B_i + N_{5i}^\intercal N_{6i}] u(t-\tau_i,x)dx \\ &+ 2\beta_i \int_\Omega u^\intercal(t,x)[W_{3i}^\intercal C_i S_2 + B_i^\intercal C_i S_2 + N_{5i}^\intercal N_{7i} S_2 + N_{6i}^\intercal N_{7i} S_2] \\ &\times \left[\int_{t-\delta_i}^t u(s,x)ds\right] dx.\end{aligned}$$

Define $\psi^\intercal(t,x) = (u^\intercal(t,x), u^\intercal(t-\tau_i),x), \left[\int_{t-\delta_i}^t u(s,x)ds\right]^\intercal$, we have

$$\mathcal{L}V(t,u(t,x),i) \le \int_\Omega \psi^\intercal(t,x)\Xi_i \psi(t,x)dx.$$

where

$$\Xi_i = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} \\ \Theta_{12}^\intercal & \Theta_{22} & 0 \\ \Theta_{13}^\intercal & 0 & \Theta_{33} \end{bmatrix} < 0.$$

hold, where

$$\begin{aligned}\Theta_{11} = &-\alpha I - 2\beta_i A_i + N_{1i}^\intercal N_{1i} + S_0^\intercal S_0 + S_0^\intercal N_{2i}^\intercal N_{2i} S_0 + 4\beta_i^2 M_i M_i^\intercal + \beta_i^2 \sum_{l=0}^{2} W_{li} W_{li}^\intercal \\ &+ \beta_i^2 B_i^\intercal B_i + (3\beta_i^2+\beta_i) W_{3i}^\intercal W_{3i} + (2\beta_i^2+\beta_i+1) N_{5i}^\intercal N_{5i} + \beta_i^2 N_{6i}^\intercal N_{6i}\end{aligned}$$

$$+\sum_{j=1}^{N}\pi_{ij}\beta_j I + Q + \delta_i R,$$

$$\Theta_{12} = W_{3i}^{\intercal}B_i + N_{5i}^{\intercal}N_{6i},$$

$$\Theta_{13} = W_{3i}^{\intercal}C_iS_2 + B_i^{\intercal}C_iS_2 + N_{5i}^{\intercal}N_{7i}S_2 + N_{6i}^{\intercal}N_{7i}S_2,$$

$$\Theta_{22} = S_1^{\intercal}S_1 + S_1^{\intercal}N_{3i}^{\intercal}N_{3i}S_1 + (\beta_i^2+\beta_i+1)B_i^{\intercal}B_i + (\beta_i+2)N_{6i}^{\intercal}N_{6i} - (1-p_i)Q,$$

$$\Theta_{33} = -(1-h_i)R + S_2^{\intercal}S_2 + S_2^{\intercal}N_{4i}^{\intercal}N_{4i}S_2 + (\beta_i^2+\beta_i+2)S_2^{\intercal}C_i^{\intercal}C_iS_2 + (\beta_i+3)S_2^{\intercal}N_{7i}^{\intercal}N_{7i}S_2.$$

when $\forall\psi^{\intercal}(t,x) = (u^{\intercal}(t,x), u^{\intercal}(t-\tau_i), x), \left[\int_{t-\delta_i}^{t} u(s,x)ds\right]^{\intercal})$, we have $\mathcal{L}V(t,u(t,x),i) \le 0$ and $\forall\psi^{\intercal}(t,x) \ne 0$, we have $\mathcal{L}V(t,u(t,x),i) < 0$.

Let $a = \min\{\lambda_{\min}(-\Theta_i)\}$, $a_1 = \max_{i\in\mathbb{S}}\{\beta_i\}$, $a_2 = \lambda_{\max}(Q)$, $a_3 = \lambda_{\max}(R)$. And, we suppose η_i is the unique solution of the following equation:

$$-a + a_1\eta_i + a_2\eta_i\tau_i e^{\eta_i\tau_i} + a_3\eta_i\delta_i^2 e^{\eta_i\delta_i} = 0.$$

By Itô formula, we have

$$\begin{aligned}\mathcal{L}[e^{\eta_i t}V(t,u(t,x),i)] &= e^{\eta_i t}[\mathcal{L}V(t,u(t,x),i) + \eta_i V(t,u(t,x),i)]\\ &\le e^{\eta_i t}\bigg[(-a+a_1\eta_i)\|u(t,x)\|_2^2 + a_2\eta_i\int_{t-\tau_i}^{t}\|u(s,x)\|_2^2 ds\\ &\quad + a_3\eta_i\int_{-\delta_i}^{0}\int_{t+\theta}^{t}\|u(s,x)\|_2^2 ds\, d\theta\bigg].\end{aligned}$$

Noting that

$$\int_{-\delta_i}^{0}\int_{t+\theta}^{t}\|u(s,x)\|_2^2 ds\, d\theta = \int_{t-\delta_i}^{t}(s-t+\delta_i)\|u(s,x)\|_2^2 ds \le \delta_i\int_{t-\delta_i}^{t}\|u(s,x)\|_2^2 ds.$$

Hence

$$\begin{aligned}\mathcal{L}[e^{\eta_i t}V(t,u(t,x),i)] &\le e^{\eta_i t}[(-a+a_1\eta_i)\|u(t,x)\|_2^2\\ &\quad + a_2\eta_i\int_{t-\tau_i}^{t}\|u(s,x)\|_2^2 ds + a_3\eta_i\delta_i\int_{t-\delta_i}^{t}\|u(s,x)\|_2^2 ds].\end{aligned}$$

By Itô formula, we have

$$V(t,u(t,x),i) = V(0,\varphi,\gamma_0) + \int_0^t \mathcal{L}V(s,u(s,x),i)ds.$$

Take the mathematical expectation of both side of above equation, we obtain

$$\begin{aligned}&\mathbb{E}[e^{\eta_i t}V(t,u(t,x),i)] = \mathbb{E}V(0,\varphi,\gamma_0) + \mathbb{E}\left[\int_0^t \mathcal{L}[e^{\eta_i s}V(s,u(s,x),i)]ds\right]\\ &\le \mathbb{E}V(0,\varphi,\gamma_0) + \mathbb{E}[(-a+a_1\eta_i)\int_0^t e^{\eta_i s}\|u(s,x)\|_2^2 ds\\ &\quad + a_2\eta_i\int_0^t\int_{t-\tau_i}^{t} e^{\eta_i s}\|u(\theta,x)\|_2^2 d\theta ds + a_3\eta_i\delta_i\int_0^t\int_{t-\delta_i}^{t} e^{\eta_i s}\|u(\theta,x)\|_2^2 d\theta ds].\end{aligned}$$

Noting that

$$\begin{aligned}\int_0^t \int_{t-\tau_i}^t e^{\eta_i s}\|u(\theta,x)\|_2^2 d\theta ds &= \int_{-\tau_i}^0 \int_0^{\theta+\tau_i} e^{\eta_i s}\|u(\theta,x)\|_2^2 ds\, d\theta \\ &+ \int_0^{t-\tau_i} \int_\theta^{\theta+\tau_i} e^{\eta_i s}\|u(\theta,x)\|_2^2 ds\, d\theta + \int_{t-\tau_i}^t \int_\theta^t e^{\eta_i s}\|u(\theta,x)\|_2^2 ds\, d\theta \\ &\le \tau_i \int_{-\tau_i}^0 e^{\eta_i(s+\tau_i)}\|u(s,x)\|_2^2 ds + \tau_i \int_0^t e^{\eta_i(s+\tau_i)}\|u(s,x)\|_2^2 ds.\end{aligned}$$

and

$$\begin{aligned}\int_0^t \int_{t-\delta_i}^t e^{\eta_i s}\|u(\theta,x)\|_2^2 d\theta ds &= \int_{-\delta_i}^0 \int_0^{\theta+\delta_i} e^{\eta_i s}\|u(\theta,x)\|_2^2 ds\, d\theta \\ &+ \int_0^{t-\delta_i} \int_\theta^{\theta+\delta i} e^{\eta_i s}\|u(\theta,x)\|_2^2 ds\, d\theta + \int_{t-\delta_i}^t \int_\theta^t e^{\eta_i s}\|u(\theta,x)\|_2^2 ds\, d\theta \\ &\le \delta_i \int_{-\delta_i}^0 e^{\eta_i(s+\delta_i)}\|u(s,x)\|_2^2 ds + \delta_i \int_0^t e^{\eta_i(s+\delta_i)}\|u(s,x)\|_2^2 ds.\end{aligned}$$

Then

$$\begin{aligned}&\mathbb{E}[e^{\eta_i t}V(t,u(t,x),i)] \le \mathbb{E}V(0,\varphi,\gamma_0) \\ &\quad + \mathbb{E}\bigg\{ \left[-a + a_1\eta_i + a_2\eta_i\tau_i e^{\eta_i\tau_i} + a_3\eta_i\delta_i^2 e^{\eta_i\delta_i}\right] \int_0^t e^{\eta_i s}\|u(s,x)\|_2^2 ds \\ &\quad + a_2\eta_i\tau_i e^{\eta_i\tau_i} \int_{-\tau_i}^0 e^{\eta_i s}\|u(s,x)\|_2^2 ds + a_3\eta_i\delta_i^2 e^{\eta_i\delta_i} \int_{-\delta_i}^0 e^{\eta_i s}\|u(s,x)\|_2^2 ds \bigg\} \\ &= \mathbb{E}V(0,\varphi,\gamma_0) + \mathbb{E}\bigg\{ a_2\eta_i\tau_i e^{\eta_i\tau_i} \int_{-\tau_i}^0 e^{\eta_i s}\|u(s,x)\|_2^2 ds \\ &\quad + a_3\eta_i\delta_i^2 e^{\eta_i\delta_i} \int_{-\delta_i}^0 e^{\eta_i s}\|u(s,x)\|_2^2 ds \bigg\} \\ &\le \mathbb{E}V(0,\varphi,\gamma_0) + \bigg\{ a_2\eta_i\tau_i e^{\eta_i\tau_i} \int_{-\tau_i}^0 e^{\eta_i s} ds \sup_{-\tau_0\le\theta\le 0} \mathbb{E}\|\varphi(\theta,x)\|_2^2 \\ &\quad + a_3\eta_i\delta_i^2 e^{\eta_i\delta_i} \int_{-\delta_i}^0 e^{\eta_i s} ds \sup_{-\tau_0\le\theta\le 0} \mathbb{E}\|\varphi(\theta,x)\|_2^2 \bigg\} \\ &\le [a_1 + a_2\tau_i e^{\eta_i\tau_i} + a_3\delta_i^2 e^{\eta_i\delta_i}] \sup_{-\tau_0\le\theta\le 0} \mathbb{E}\|\varphi(\theta,x)\|_2^2 \\ &= [a_1 + (a - a_1\eta_i)\eta_i^{-1}] \sup_{-\tau_0\le\theta\le 0} \mathbb{E}\|\varphi(\theta,x)\|_2^2.\end{aligned}$$

Considering the definition of $V(t,u(t,x),i)$, we have

$$V(t,u(t,x),i) \ge \min\{\beta_i\}\|u(t,x)\|_2^2.$$

Therefore, we obtain

$$E\|u(t,\varphi)\|_2^2 \le Me^{-\eta_i t} \sup_{-\tau_0\le\theta\le 0} E\|\varphi(\theta,x)\|_2^2.$$

where $M = \frac{a_1+(a-a_1\eta_i)/\eta_i}{\min\{\beta_i\}}$. By the shur's complement, it's easy to see that $\Xi_i < 0$ is equivalent to the following LMI:

$$\mathbf{\Gamma}_i = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ * & \Gamma_{22} & 0 & \Gamma_{24} \\ * & * & \Gamma_{33} & \Gamma_{34} \\ * & * & * & \Gamma_{44} \end{bmatrix} < 0.$$

hold. The theorem is completed. □

When $D(t,x,y) = 0$, the system (2.5) can be transformed into:

$$\begin{cases} du(t,x) = \Big[-(A_i + \triangle A_i)u(t,x) + (W_{0i} + \triangle W_{0i})F_0(u(t,x)) \\ +(W_{1i} + \triangle W_{1i})F_1(u(t-\tau_i,x)) \\ +(W_{2i} + \triangle W_{2i}) \displaystyle\int_{t-\delta_i}^{t} F_2(u(s,x))ds \Big] dt \\ + \Big[(W_{3i} + \triangle W_{3i})u(t,x) + (B_i + \triangle B_i)u(t-\tau_i,x) \\ +(C_i + \triangle C_i) \displaystyle\int_{t-\delta_i}^{t} F_2(u(s,x))ds \Big] d\omega(t), \\ \dfrac{\partial u}{\partial n} = 0, \quad t \geq t_0 \geq 0, \quad x \in \partial\Omega, \\ u(t_0+\theta, x, \gamma_0) = \varphi(\theta, x), \quad -\tau_0 \leq \theta \leq 0, \quad x \in \Omega, \gamma_0 \in \mathbb{S}. \end{cases} \tag{3.15}$$

where $u \in R^n$, $\varphi(\theta,x) = \xi(\theta,x) - y^*$, $F_k(u) = G_k(u+y^*) - G_k(y^*)$, $k = 0,1,2$. Then, we have the following corollary.

Corollary 3.2. *The null solution of system* (3.15) *is robust stability on norm* $\|\cdot\|_2$ *in the mean square for any time-varying delays* τ_i *and* δ_i *satisfying* $p_i \leq 0$ *and* $h_i \leq 0$, *if there exist a sequence of positive scalars* β_i, $(i \in \mathbb{S})$ *and positive definite matrices* $Q > 0$ *and* $R > 0$, *such that the following linear matrix inequalities*

$$\mathbf{\Gamma}_i = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ * & \Gamma_{22} & 0 & \Gamma_{24} \\ * & * & \Gamma_{33} & \Gamma_{34} \\ * & * & * & \Gamma_{44} \end{bmatrix} < 0. \tag{3.16}$$

hold, where

$$\begin{aligned} \Gamma_{11} &= -\beta_i A_i - \beta_i A_i^{\intercal} + N_{1i}^{\intercal} N_{1i} + S_0^{\intercal} S_0 + S_0^{\intercal} N_{2i}^{\intercal} N_{2i} S_0 \\ &\quad + \beta_i W_{3i}^{\intercal} W_{3i} + (2\beta_i^2 + \beta_i + 1) N_{5i}^{\intercal} N_{5i} + \sum_{j=1}^{N} \pi_{ij}\beta_j I + Q + \delta_i R, \\ \Gamma_{12} &= W_{3i}^{\intercal} B_i + N_{5i}^{\intercal} N_{6i}, \\ \Gamma_{13} &= W_{3i}^{\intercal} C_i S_2 + B_i^{\intercal} C_i S_2 + N_{5i}^{\intercal} N_{7i} S_2 + N_{6i}^{\intercal} N_{7i} S_2, \\ \Gamma_{22} &= S_1^{\intercal} S_1 + S_1^{\intercal} N_{3i}^{\intercal} N_{3i} S_1 + (\beta_i + 1) B_i^{\intercal} B_i + (\beta_i + 2) N_{6i}^{\intercal} N_{6i} - (1 - p_i) Q, \end{aligned}$$

$$\Gamma_{33} = -(1-h_i)R + S_2^{\intercal}S_2 + S_2^{\intercal}N_{4i}^{\intercal}N_{4i}S_2 + (\beta_i+2)S_2^{\intercal}C_i^{\intercal}C_iS_2 + (\beta_i+3)S_2^{\intercal}N_{7i}^{\intercal}N_{7i}S_2,$$
$$\Gamma_{14} = [2\beta_iM_i, \beta_iW_{0i}, \beta_iW_{1i}, \beta_iW_{2i}, \beta_iB_i^{\intercal}, \beta_iW_{3i}^{\intercal}, \beta_iN_{5i}^{\intercal}, \beta_iN_{6i}^{\intercal}],$$
$$\Gamma_{24} = [\beta_iB_i^{\intercal}, 0, 0, 0, 0, 0, 0, 0],$$
$$\Gamma_{34} = [\beta_iS_2^{\intercal}C_i^{\intercal}, 0, 0, 0, 0, 0, 0, 0],$$
$$\Gamma_{44} = diag\left[-I, -I, -I, -I, -I, -\frac{1}{3}I, -\frac{1}{2}I, -I\right].$$

4. EXAMPLE

Consider a two-neuron Markovian jumping neural networks with one mode (3.15), and the parameters are given as follows:

$$\prod = \begin{bmatrix} -2.3 & 2.3 \\ 1.9 & -1.9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 9.0 & 8.4 \\ 7.9 & 5.6 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.3 & 0.4 \\ 0.9 & 0.6 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 3.0 & 4.4 \\ 1.9 & 2.6 \end{bmatrix}, \quad W_{01} = \begin{bmatrix} 2.9 & 5.4 \\ 6.9 & 5.6 \end{bmatrix}, W_{11} = \begin{bmatrix} 3.7 & 1.4 \\ 4.9 & 0.6 \end{bmatrix}, W_{21} = \begin{bmatrix} 9.0 & 8.4 \\ 7.9 & 5.6 \end{bmatrix},$$

$S_0 = S_1 = S_2 = 0.8I$, $N_{k1} = 0.03I$, $k = 1, \dots, 7$, $\tau = \delta = 0.5$. By using the Matlab LMI toolbox, we solve the LMIs (33), and obtain

$$P_1 = \begin{pmatrix} 19.7711 & 3.0843 \\ 3.0843 & 16.4831 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 20.2092 & 5.2305 \\ 5.2305 & 18.0242 \end{pmatrix}$$
$$Q_1 = \begin{pmatrix} 11.7744 & 1.7144 \\ 1.7144 & 10.2777 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 12.0053 & 1.7571 \\ 1.7571 & 10.4559 \end{pmatrix}, \quad \beta_i = 0.0085.$$

Therefore, it follows from Corollary 3.2 that the solution is robust exponentially stable on norm $\|\cdot\|_2$ in the mean square.

5. CONCLUSIONS

This paper is devoted to investigating global robust exponential stability for a class of delayed stochastic reaction-diffusion recurrent neural networks. The network parameters are governed by a continuous-time discrete-state Markov process which takes values in a finite set. By employing a Lyapunov-Krasovskii functional and some inequalities, some easy-to-test criteria on global exponential stability for this kind of stochastic neural networks are established in the form of linear matrix inequalities. An example is presented to illustrate the effectiveness of the theoretical results.

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