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ESTIMATES OF EIGENVALUES OF LINEAR OPERATORS ASSOCIATED WITH NONLINEAR BOUNDARY VALUE PROBLEMS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. Nonlinear boundary value problems are often written as equivalent integral equations by utilizing the Green's function. The spectral radius of an associated linear operator can be used in some fixed point index results and leads to some sharp existence criteria for positive solutions of the nonlinear problem. It is of interest to estimate the spectral radius in concrete cases. We discuss two methods of approximating the spectral radius of such compact linear operators.

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1. INTRODUCTION

In the study of existence of positive solutions of boundary value problems (BVPs) for ordinary differential equations the BVP is often written as an equivalent integral equation by utilizing the Green's function. The problem then becomes one of showing that an integral operator has a fixed point in some cone of positive functions. Some existence results depend on the principal eigenvalue of an associated linear problem. There are many papers using the eigenvalue, we mention only a few [2, 4, 5, 6, 8, 9, 16, 17, 23, 33, 34]. A general approach based on the integral equation was given in [31] which was illustrated with applications to some multipoint BVPs; a question left open in [31] was answered in [28].

A standard situation is when the nonlinear integral operator N acts in the space $C[0,1]$ and has the form

$$Nu(t) = \int_0^1 G(t,s) f(u(s))\, ds.$$

The associated linear operator L is then given by

$$Lu(t) = \int_0^1 G(t,s) u(s)\, ds.$$

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Let $r(L)$ denote the spectral radius of L. Some existence results depend on a comparison of the behaviour of $f(u)/u$ for u near 0 and for u near ∞ with $\mu(L) = 1/r(L)$, the principal characteristic value of L, (also called the principal eigenvalue of the differential equation), it is the characteristic value of L for which there is a positive eigenfunction (that is, belongs to some cone).

When f depends only on u, an example of an existence theorem where the conditions are sharp is: there is at least one positive solution if "the nonlinearity crosses the eigenvalue", that is,

$$\text{either } \limsup_{u\to 0+} f(u)/u < \mu(L) \text{ and } \liminf_{u\to\infty} f(u)/u > \mu(L),$$
$$\text{or } \liminf_{u\to 0+} f(u)/u > \mu(L) \text{ and } \limsup_{u\to\infty} f(u)/u < \mu(L).$$

This result applies quite generally, to BVPs where the boundary conditions (BCs) are local (depending only on the endpoints 0 and 1), also to nonlocal BCs such as multipoint problems (depending also on points in the interior of $(0, 1)$). It also applies to similar types of problems for some fractional differential equations that can be similarly written as integral equations.

It is therefore of interest to be able to calculate $\mu(L)$ or at least find good upper and lower estimates. In general the calculation of $\mu(L)$ is a problem in numerical analysis. However, in some cases the eigenvalue can be found from the differential equation. For example, if the problem is

$$u''(t) + f(u(t)) = 0,\ t \in (0, 1),\quad u'(0) = 0,\ u(1) + u'(1) = 0,$$

the eigenvalue is the smallest $\mu > 0$ such that the problem

$$u'' + \mu u = 0,\quad u'(0) = 0,\ u(1) + u'(1) = 0,$$

has a positive solution φ; φ is called a corresponding eigenfunction. In this case $\mu(L) = \omega^2$ with eigenfunction a multiple of $\cos(\omega t)$ where, from the BC at 1, ω is the smallest positive root of $\cos(\omega) - \omega \sin(\omega) = 0$. Using Maple we determine that $\mu(L) \approx 0.740174$.

However, if we consider a problem such as

$$u''(t) + g(t)f(u(t)) = 0,\ t \in (0, 1),\quad u'(0) = 0,\ u(1) + u'(1) = 0,$$

where g is some given function, it is, in general, no longer possible to find the eigenvalue by this method. The corresponding integral operator is

$$Lu(t) = \int_0^1 G(t, s)g(s)u(s)\, ds,$$

which is of the same type as before if we replace $G(t, s)$ by $\widetilde{G}(t, s) = G(t, s)g(s)$, so the same theory applies to this case. The eigenvalue must then be found by numerical methods in general.

There are two main types of fixed point index results. One type compares the behaviour $f(u)$ on certain intervals with constants that have been called m, $M = M(a,b)$ (see for example [15, 29, 31]) which are defined by

$$1/m = \|L\| = \max_{t\in[0,1]} \int_0^1 G(t,s)\,ds, \tag{1.1}$$

and for a subinterval $[a,b]$ of $[0,1]$,

$$1/M(a,b) = \min_{t\in[a,b]} \int_a^b G(t,s)\,ds. \tag{1.2}$$

A second type compares the behaviour $f(u)$ on certain intervals with $\mu(L)u$, as mentioned above, see for example [31]. It was shown in [31] that

$$m \le \mu(L) \le M(a,b).$$

However, the results using these different types of hypotheses are in general complementary, neither includes the other. The numbers m, M are not primarily estimates of $\mu(L)$, they are particulary useful in proving existence of multiple positive solutions, as for example in [13, 15, 29, 30].

Two examples show that m, M are often not good approximations to $\mu(L)$.

Example 1.1. For the problem $u'' + \mu u = 0,\ t \in (0,1),\ u(0) = 0,\ u(1) = 0$, it is well known that $\mu = \pi^2$, and

$$G(t,s) = \begin{cases} s(1-t), & \text{if } 0 \le s \le t \le 1, \\ t(1-s), & \text{if } 0 \le t < s \le 1. \end{cases}$$

By routine calculations one obtains $m = 8$ and $M(1/4, 3/4) = 16$, and $[1/4, 3/4]$ is the interval which gives the smallest value of $M(a,b)$ for $[a,b]$ a subinterval of $[0,1]$. Thus m, M are not good approximations to $\mu(L)$, the percentage error is quite large.

Example 1.2. For the 4th order conjugate problem

$$u^{(4)}(t) = \mu u,\ t \in (0,1);\ \ u(0) = 0,\ u'(0) = 0,\ u''(0) = 0,\ u(1) = 0,$$

it was shown in [24] that $m = 2048/9 \approx 227.556$, and the smallest value of $M(a,b)$ is very close to $M(0.4286, 0.9786) \approx 2859.530$. By a numerical calculation, using a C program that runs on a pc which was written by the author's colleague Prof. K. A. Lindsay, it was given in [24] that $\mu(L) \approx 950.884$. However, in this case it is possible to find the eigenfunction from the differential equation and hence obtain $\mu(L) = \omega^4$, where $\omega = r\sqrt{2}$ and r is the smallest positive root of $\tan(x) = \tanh(x)$; thus, with the aid of Maple, $r \approx 3.926602312$, $\mu(L) \approx 950.884270$, so the C program works rather well on this example. Again m, M are far from good approximations to $\mu(L)$.

The constants m, M have been considered as approximations to $\mu(L)$, for example by Lan [14] and by Bo Yang [32]. Since these are often not good approximations, in the interesting paper [32] Bo Yang investigated some other approximations for the n-th order boundary value problem

$$u^{(n)}(t) + \mu u(t) = 0,\ t \in (0,1), \tag{1.3}$$

with the $(n-1,1)$ conjugate BCs

$$u^{(k)}(0) = 0,\ 0 \le k \le n-2,\ \ u(1) = 0. \tag{1.4}$$

The Green's function is known and is given by the formula

$$G_0(t,s) := \frac{t^{n-1}(1-s)^{n-1}}{(n-1)!} - \frac{(t-s)^{n-1}}{(n-1)!} H(t-s), \tag{1.5}$$

where $H(t) := \begin{cases} 1, & \text{if } t \ge 0, \\ 0, & \text{if } t < 0, \end{cases}$ is the Heaviside function.

Bo Yang first showed that any solution of the problem (1.3), (1.4) satisfies the bounds

$$w_1(t)\|u\| \le u(t) \le w_2(t)\|u\|, \tag{1.6}$$

for explicitly found functions w_i. He then gave an iteration process using the functions w_1, w_2 to give constants m_n, M_n and showed that they satisfied $m_n \le \mu(L) \le M_n$. He gave a few numerical calculations for the Example 1.2, ending with $m_4 \approx 751.6$, $M_4 \approx 1260.9$. Thus they improve on $227.556 \approx m \le \mu(L) \le M \approx 2859.530$.

An obvious question that was not answered in [32] is whether or not these sequence converge to $\mu(L)$. One of our purposes is to answer this question in the affirmative here.

In spite of the name, M_n is not related to $M = M(a,b)$. It turns out that the sequences $\{m_n\}$, $\{M_n\}$ are related to the concept of *local spectral radius*. The local spectral radius was studied in a number of papers, some of the first were Vrbová [22] and Daneš [1]. Some of the properties were extended in [20] and some inequalities for local spectral radii of sums and products of operators were investigated in [35, 36]. Other work that is important to us has been done by Forster, Nagy [3] and Marek [18]. In a general situation the local spectral radius does not equal the spectral radius but in many cases we do have equality; this will occur in the case we study.

We also rediscovered another iteration process that gives approximations to the spectral radius. Again it turns out that it is related to the local spectral radius and was studied by Forster, Nagy [3] and Marek [18]. In this paper we therefore do not prove convergence, as it is already known.

We believe this method is not familiar to people working on nonlinear problems so we recall known facts. We also illustrate the method with numerical examples. These illustrate that the first approximation method (as in [32]) converges more slowly than the second method.

2. A CLASS OF POSITIVE OPERATORS

A subset K of a Banach space X is called a cone if K is closed and $x, y \in K$ and $\alpha \geq 0$ imply that $x + y \in K$ and $\alpha x \in K$, and $K \cap (-K) = \{0\}$. We always suppose that $K \neq \{0\}$. A cone defines a partial order by $x \preceq_K y \iff y - x \in K$. A cone is said to be *reproducing* if $X = K - K$ and to be *total* if $X = \overline{K - K}$. A cone is said to be *normal* with normality constant γ if $0 \preceq_K x \preceq_K y$ implies $\|x\| \leq \gamma \|y\|$.

A useful concept due to Krasnosel'skiĭ, [7, 10, 12] is that of a u_0-positive linear operator on a cone.

In a recent paper [25], we gave a modification of this definition. We suppose that we have two cones in a Banach space X, $K_0 \subset K_1$ and we let $\preceq$ denote the partial order defined by the larger cone K_1, that is, $x \preceq y \iff y - x \in K_1$. We say that L is *positive* if $L(K_1) \subset K_1$,

Our modified definition reads as follows.

Definition 2.1. Let $K_0 \subset K_1$ be cones as above. A positive bounded linear operator $L : X \to X$ is said to be *u_0-positive relative to the cones* (K_0, K_1), if there exists $u_0 \in K_1 \setminus \{0\}$, such that for every $u \in K_0 \setminus \{0\}$ there are constants $k_2(u) \geq k_1(u) > 0$ such that

$$k_1(u)u_0 \preceq Lu \preceq k_2(u)u_0.$$

When $K_0 = K_1$ we recover the original definition in [7, 12]. This is stronger than requiring that L is positive and is satisfied if L is u_0-positive on K_1 according to the original definition.

Krasnosel'skiĭ, [10], has a more general definition (in the case $K_0 = K_1$) which supposes an inequality of the form

$$k_1(u)u_0 \preceq L^n u \preceq k_2(u)u_0,$$

holds for some positive integer n, where n may depend on u. We showed in [25] that if $L : K_1 \to K_0$ and L is u_0-positive relative to the cones (K_0, K_1) then L^2 is u_0-positive so L is u_0-positive in the sense of Krasnosel'skiĭ.

In the recent paper [25], we proved a comparison theorem which is similar to one given by Keener and Travis [7], which was itself a sharpening of some results of Krasnosel'skiĭ [10], § 2.5.5.

Theorem 2.2 ([25])**.** *Let $K_0 \subset K_1$ be cones in a Banach space X, and let $\preceq$ denote the partial order of K_1. Suppose that L_1, L_2 are bounded linear operators and that at* *least one is u_0-positive relative to (K_0, K_1). If there exist*

$$u_1 \in K_0 \setminus \{0\},\ \lambda_1 > 0, \ \text{such that} \ \lambda_1 u_1 \preceq L_1 u_1, \quad \text{and}$$
$$u_2 \in K_0 \setminus \{0\},\ \lambda_2 > 0, \ \text{such that} \ \lambda_2 u_2 \succeq L_2 u_2, \tag{2.1}$$

and $L_1 u_j \preceq L_2 u_j$ for $j = 1, 2$, then $\lambda_1 \leq \lambda_2$. If, in addition, $L_j\,(K_1 \setminus \{0\}) \subset K_0 \setminus \{0\}$ and if $\lambda_1 = \lambda_2$ in (2.1)*, then it follows that u_1 is a (positive) scalar multiple of u_2.*

This is most often applied when one of u_j is an eigenfunction of L_j corresponding to a positive eigenvalue λ_j.

3. BASIC SET-UP

Some of our results hold in a Banach space X with a suitable cone K, but unless specified otherwise we work in the space $C[0,1]$ with the usual supremum norm. Let

$$P := \{u \in C[0,1] : u(t) \geq 0 \text{ for } 0 \leq t \leq 1\}$$

be the standard cone of non-negative continuous functions. We write $\preceq$ for the ordering induced by P. It is well-known that P is a reproducing, normal cone with normality constant 1.

We will study linear operators of the form $Lu(t) = \int_0^1 G(t,s)u(s)\,ds$ which arise naturally when studying solutions of BVPs as fixed points of the nonlinear operator $Nu(t) = \int_0^1 G(t,s)f(s,u(s))\,ds$, where G is the Green's function.

The conditions we impose on G are similar to ones in the papers [29, 30, 31].

(C_1) The kernel $G \geq 0$ is measurable, and for every $\tau \in [0,1]$ we have

$$\lim_{t \to \tau} |G(t,s) - G(\tau,s)| = 0 \ \text{ for almost every (a. e.) } \ s \in [0,1].$$

(C_2) There exist a non-negative function $\Phi \in L^1$ with $\Phi(s) > 0$ for a.e. $s \in (0,1)$, and $c \in P \setminus \{0\}$ such that

$$c(t)\Phi(s) \leq G(t,s) \leq \Phi(s), \ \text{ for } \ 0 \leq t, s \leq 1. \tag{3.1}$$

It is well known, using the Arzelà-Ascoli theorem, that L is a compact operator in $C[0,1]$, that is L is continuous and the image of each bounded set is relatively compact (often termed 'completely continuous'), and so also is N if f satisfies Carathéodory conditions.

For a subinterval $J = [a,b] \subseteq [0,1]$ such that $c_J := \min\{c(t) : t \in J\} > 0$, we define cones K_c, K_J by

$$K_c := \{u \in P : u(t) \geq c(t)\|u\|,\ t \in [0,1]\}, \tag{3.2}$$

$$K_J := \{u \in P : u(t) \geq c_J\|u\|,\ t \in J\}. \tag{3.3}$$

It is clear that $K_c \subset K_J$. When we consider the cone K_J we will always suppose that $c_J > 0$.

These cones fit the hypotheses $(C_1),(C_2)$, in fact, under those conditions both N and L map P into K_c, the routine arguments have been given many times, see, for example, [15, 26, 30].

We would like to know when L is u_0-positive on P or relative to (K_c, P). It has not been possible to prove that L itself is u_0-positive without some assumptions in addition to (C_1)–(C_2). A simple additional assumption is either of the 'symmetry' assumptions

$$G(t,s) = G(s,t), \text{ or } G(t,s) = G(1-s, 1-t), \text{ for all } t,\ s \in [0,1],$$

as shown in [28] and Corollary 7.5 of [30]. However, a related operator is u_0-positive relative to two cones, and this was an important motivation for our introducing the concept in [25]. We recall the result here.

Theorem 3.1. *Let G satisfy (C_1)–(C_2) and let $J = [a,b]$ and $c_J = \min\{c(t) : t \in J\}$ and suppose $c_J > 0$. Let L_J be defined on $C[0,1]$ by $L_J u(t) = \int_a^b G(t,s)u(s)\,ds$. Then L_J is u_0-positive relative to (K_c, P) for $u_0(t) := \int_a^b G(t,s)\,ds$.*

This was essentially first proved in [25] with a small refinement in [26], see also [27].

With the above concept we give a short proof of the inequality $\mu(L) \leq M(a,b)$. Let $\hat{1}$ be the constant function with value 1; then $\hat{1} \in K_c$. For a subinterval $J = [a,b]$ and $t \in [0,1]$,

$$L_J\hat{1}(t) = \int_a^b G(t,s)\,ds \geq \min_{t\in[a,b]} \int_a^b G(t,s)\,ds = (1/M(a,b))\hat{1}(t).$$

If $r(L) > 0$ is an eigenvalue of L with eigenfunction $\varphi \in P$, hence also in K_c, by the comparison theorem, Theorem 2.2, it follows that $r(L) \geq 1/M(a,b)$, hence $\mu(L) \leq M(a,b)$. Note also that since $1/m = \|L\|$ we have $1/m \geq r(L)$ so $m \leq \mu(L)$.

4. LOCAL SPECTRAL RADIUS

We first recall some results from the paper by Daneš [1].

Definition 4.1. Let X be a (real or complex) Banach space, and let $L : X \to X$ be a bounded linear operator. For $x \in X$ define

$$r(L,x) = \limsup_{n\to\infty} \|L^n x\|^{1/n}$$

and call it the *local spectral radius of L at x*.

It follows that $0 \leq r(L,x) \leq r(L)$ for every x in X, where $r(L)$ denotes the spectral radius given by $r(L) = \lim_{n\to\infty} \|L\|^{1/n} = \inf_{n\in\mathbb{N}} \|L\|^{1/n}$.

The main results of [1] are that $r(L,x) = r(L)$ for all 'almost all' x, that is for all x in a residual (2nd category) subset of X. However, the limit $\lim \|L^n x\|^{1/n}$ does not exist generally and the set of limits of all convergent subsequences of the sequence $\{\|L^n x\|^{1/n}\}$ can be the whole segment $[0, r(L)]$ for x in a dense subset of X. But, if $L : X \to X$ is a compact linear operator, then Daneš, [1, Corollary 1], proves that $r(L, x) = \lim \|L^n x\|^{1/n}$ for each x in X.

For bounded linear operators Müller [20] gave the following refinement:

Theorem 4.2 (Corollary 1.2, [20])**.** *The set* $\{x \in X : \limsup \|L^n x\|^{1/n} = r(L)\}$ *is residual. The set* $\{x \in X : \liminf \|L^n x\|^{1/n} = r(L)\}$ *is dense. In particular, there is a dense subset of points* $x \in X$ *with the property that the limit* $\lim \|L^n x\|^{1/n}$ *exists (and is equal to* $r(L)$*).*

In general it is not possible to replace the word 'dense' in this result by 'residual', see Example 1.3 of [20].

5. CONVERGENCE OF AN ITERATION PROCESS

In the paper [32], Bo Yang studied the problem (1.3), (1.4). He showed that all possible solutions satisfy the bounds

$$w_1(t)\|u\| \le u(t) \le w_2(t)\|u\|,$$

for explicit functions w_1, w_2. He then gave an iteration process using the functions w_1, w_2 to give constants m_n, M_n and showed that they satisfied $m_n \le \mu(L) \le M_n$. A question left open was whether or not these sequences converge to $\mu(L)$. We answer this affirmatively in this paper.

Condition (C_2) shows that all possible solutions satisfy the slightly different set of inequalities

$$c(t)\|u\| \le u(t) \le \|u\|,$$

which is of the same type as Bo Yang's, with a good lower bound and the trivial upper bound.

In fact, the sequences constructed by Bo Yang can be written

$$\frac{1}{m_n} = \|L^n w_2\|^{1/n}, \quad \frac{1}{M_n} = \|L^n w_1\|^{1/n}. \tag{5.1}$$

Thus they are special cases of local spectral radii. As mentioned above, the limits exist but need not equal the spectral radius in general but are equal for 'most' starting points. Here we have explicitly given starting points so convergence must be proved.

In fact, we give a general result which applies in normal cones for u_0-positive operators.

Theorem 5.1. *Let* K_0, K_1 *be cones in a Banach space* X *with* $K_0 \subset K_1$ *and let* K_1 *be a normal cone. Let* L *be* u_0*-positive relative to* (K_0, K_1) *and suppose that* $r(L) > 0$ *is an eigenvalue of* L *with eigenfunction* $\varphi \in K_0$*, normalised to have* $\|\varphi\| = 1$*. Then, for every* $v \in K_0 \setminus \{0\}$*,* $\|L^n v\|^{1/n} \to r(L)$ *as* $n \to \infty$*.*

Proof. Since $\varphi \in K_0$, it is easily shown ([25]) that L is φ-positive relative to (K_0, K_1). Let $v \in K_0 \setminus \{0\}$, then there exist constants $0 < k_1 \leq k_2$ (depending on v) such that $k_1\varphi \preceq Lv \preceq k_2\varphi$. We simply write r in place of $r(L)$. As L maps K_1 to itself and $L\varphi = r\varphi$, we have

$$k_1 r\varphi = L(k_1\varphi) \preceq L^2 v \preceq L(k_2\varphi) = k_2 r\varphi.$$

Hence we obtain, for $n \in \mathbb{N}$,

$$k_1 r^{n-1}\varphi \preceq L^n v \preceq k_2 r^{n-1}\varphi.$$

Since the cone is a normal cone there is a constant γ such that $0 \preceq u \preceq w$ implies $\|u\| \leq \gamma\|w\|$. Hence

$$\frac{k_1}{\gamma r} r^n \|\varphi\| \leq \|L^n v\| \leq \frac{\gamma k_2}{r} r^n \|\varphi\|.$$

Taking the n-th root we obtain

$$\Big(\frac{k_1\|\varphi\|}{\gamma r}\Big)^{1/n} r \leq \|L^n v\|^{1/n} \leq \Big(\frac{\gamma k_2\|\varphi\|}{r}\Big)^{1/n} r.$$

By the sandwich principle, $\lim_{n\to\infty} \|L^n v\|^{1/n} = r$, which concludes the proof. □

When we do not necessarily have the u_0-positivity property we can show that the conclusions of Bo Yang hold for problems in $C[0,1]$ that arise from many BVPs and we have convergence. The inequalities use the fact that P has normality constant 1.

Theorem 5.2. *If (C_1), (C_2) hold then $\|L^n c\|^{1/n} \leq r(L) \leq \|L^n \hat{1}\|^{1/n}$ for each n. The sequences $\{\|L^n \hat{1}\|^{1/n}\}$ and, if also $c(t) > 0$ for $t \in (0,1)$, $\{\|L^n c\|^{1/n}\}$ both converge to $r(L)$.*

Proof. Firstly we consider the case for $\hat{1}$. Clearly we have

$$\|L^n \hat{1}\| \leq \|L^n\|, \ \text{ hence } \ \limsup \|L^n \hat{1}\|^{1/n} \leq \limsup \|L^n\|^{1/n} = r(L).$$

Also, since we choose $\|\varphi\| = 1$, we have $\varphi \preceq \hat{1}$ which implies that

$$(r(L))^n \varphi = L^n \varphi \preceq L^n \hat{1},$$

and, since P is a normal cone with normality constant 1 it follows that

$$r(L) \leq \|L^n \hat{1}\|^{1/n} \ \text{ for each } n.$$

Thus we have $\liminf \|L^n \hat{1}\|^{1/n} \geq r(L)$. This proves $\|L^n \hat{1}\|^{1/n} \to r(L)$.

Now we consider the case for c. From (C_2) it follows that $L\varphi(t) \geq c(t)\|\varphi\|$, that is, $r(L)\varphi(t) \geq c(t)$. Hence we have

$$(r(L))^n \varphi \succeq L^n c.$$

Since P is a normal cone with normality constant 1 it follows that

(5.2)

$$(r(L))^n \geq \|L^n c\| \ \text{ for each } n, \ \text{ hence } \ r(L) \geq \|L^n c\|^{1/n}, \text{ and } r(L) \geq \limsup \|L^n c\|^{1/n}.$$

For convergence we will employ the operators L_J for $J = [a, b]$ which we write as $L_{a,b}$. A result of Nussbaum, Lemma 2 on page 226 of [21], shows that if L_n are compact linear operators and $L_n \to L$ in the operator norm then $r(L_n) \to r(L)$. Thus, given $\varepsilon > 0$ we choose $[a, b]$ so that we have $r(L_{a,b}) \le r(L) \le r(L_{a,b}) + \varepsilon$. Then, by Theorem 5.1, $\|L^n_{a,b}c\|^{1/n} \to r(L_{a,b})$. This proves that

$$r(L) \le \lim \|L^n_{a,b}c\|^{1/n} + \varepsilon \le \liminf \|L^n c\|^{1/n} + \varepsilon.$$

Since ε is arbitrary this and (5.2) prove that $\lim \|L^n c\|^{1/n} = r(L)$. □

Remark 5.3. The same result, with an almost identical proof, is valid with $\hat{1}$ replaced by a function w, with $\|w\| \le 1$, if we know that every solution of the BVP satisfies

$$c(t)\|u\| \le u(t) \le w(t)\|u\|.$$

This is the case studied by Bo Yang in [32].

6. SECOND ITERATION PROCESS

The comparison Theorem 2.2 is useful when L is u_0-positive but it is possible to deduce some inequalities in a more general case.

Theorem 6.1. *Let (C_1)–(C_2) be satisfied with $c(t) > 0$ for $t \in (0, 1)$.*

(i) Suppose there exist $v \in P \setminus \{0\}$ and $\lambda_1 > 0$ such that $\lambda_1 v \preceq Lv$ then $r(L) \ge \lambda_1$.

(ii) Suppose there exist $w \in P \setminus \{0\}$ and $\lambda_2 > 0$ such that $\lambda_2 w \succeq Lw$ then $r(L) \le \lambda_2$.

The above is proved in [26] and also recalled in [27]. As noted in those papers, part (i) is valid generally, but part (ii) requires some extra property. Our proof of (ii) uses the fact that $r(L)$ is the limit of $r(L_{a,b})$ where $a \to 0+, b \to 1-$, and the u_0-positivity of $L_{a,b}$ together with Theorem 2.2.

These inequalities led us to investigate the iteration process given below. Our research into the literature showed that we were rediscovering a known situation, which we now recall.

Let X be a partially ordered Banach space with positive cone K. For $x \in X$, define the following numbers:

(6.1) $\underline{r}(L, x) = \sup\{\rho \in \mathbb{R} : Lx - \rho x \in K\}, \;\; \overline{r}(L, x) = \inf\{\tau \in \mathbb{R} : \tau x - Lx \in K\}.$

For the case we considered above it follows from Theorem 6.1 that $\underline{r}(L, x) \le r(L) \le \overline{r}(L, x)$.

These numbers have been studied by several authors particularly Forster-Nagy [3] and Marek [18, 19]. They are called *upper and lower Collatz-Wielandt numbers* and are related to the local spectral radius $r(L, x)$.

For example it is shown in [3] that if the positive cone K of X is normal, then,

$$\underline{r}(L,x) \le r(L,x) \le \overline{r}(L,x), \text{ for all } x \in K \setminus \{0\}.$$

We now turn to our situation for the cone P in the space $C[0,1]$. It is natural to consider an iteration process. Suppose that there exist $v \in P \setminus \{0\}$ and $\lambda_1 > 0$ such that $\lambda_1 v_1 \preceq Lv_1$ and there exist $w_1 \in P \setminus \{0\}$ and $\Lambda_1 > 0$ such that $\Lambda_1 w_1 \succeq Lw_1$. Let $r_1 = \underline{r}(L, v_1),\ R_1 = \overline{r}(L, w_1)$, then we have $\lambda_1 \le r_1 \le r(L) \le R_1 \le \Lambda_1$.

Now let $v_2 = Lv_1,\ w_2 = Lw_1$ and let $r_2 = \underline{r}(L, v_2),\ R_2 = \overline{r}(L, w_2)$, and continue this process. The sequences obtained are monotone. Thus we are considering the sequences $\{\underline{r}(L, L^n x)\}$ and $\{\overline{r}(L, L^n x)\}$. At each stage we have lower and upper bounds so the question arises as to whether these sequences converge to $r(L)$.

Forster-Nagy [3] gave necessary and/or sufficient conditions for both sequences of Collatz-Wielandt numbers $\{\underline{r}(L, L^n x)\}$ and $\{\overline{r}(L, L^n x)\}$ to converge to $r(L,x)$. They pointed out that some previous papers discussing this topic had some errors. The result we will use is due to Marek [18], where some corrections were made to some of his earlier works.

It requires the cone to be normal.

Theorem 6.2 ([18])**.** *Let K be a normal cone in a Banach space X. Let L be a compact linear that is u_0-positive in the sense of Krasnosel'skiĭ. Then for any $x \in K$, $x \neq 0$, the sequences of Collatz-Wielandt numbers converge to $r(L)$.*

We can apply this result when L is u_0-positive relative to (K_c, P) and $L(P) \subset K_c$, but it remains an open question whether it holds when we only have the conditions (C_1),(C_2). In that case we can apply the result to the operator $L_{a,b}$ for a near 0 and b near 1 to get approximations, but see Remark 7.1 below.

For the $(n-1,1)$ conjugate problem studied by Webb [24] and Bo Yang [32], the Green's function satisfies $G(t,s) = G(1-s, 1-t)$ therefore L is u_0-positive on P as shown in [30]. Hence the above sequences converge to $r(L)$ in this case.

7. NUMERICAL ILLUSTRATIONS

We first consider the eigenvalue for the Dirichlet boundary value problem

$$u''(t) + \mu u(t) = 0,\ u(0) = 0,\ u(1) = 0. \tag{7.1}$$

It is very well-known that the smallest eigenvalue is $\mu = \pi^2 \approx 9.869604404$.

We first consider the iteration of Theorem 5.1 (as suggested by Bo Yang's paper) starting with $w_1 = t(1-t), w_2 = 1$. We obtain, using Maple, the following iterates (we omit several):

$$m_1 = 8,\ \ M_1 \approx 38.40000010$$
$$m_5 \approx 9.404121039,\ \ M_5 \approx 12.94111587$$
$$m_{10} \approx 9.634046406,\ \ M_{10} \approx 11.30149008$$
$$m_{14} \approx 9.700769051,\ \ M_{14} \approx 10.87240029$$

At this stage the calculations became too complicated and Maple (or at least this user) gave up. The convergence is relatively slow.

Using the Collatz-Wielandt numbers we obtain (again with Maple's help) starting with $v = w = t(1-t)$ (a simple choice chosen to satisfy the BCs) we obtain (figures are truncated to fewer decimal places):

$$9.6 \le \mu \le 12$$
$$9.83606557 \le \mu \le 10$$
$$9.86570397 \le \mu \le 9.88235294$$
$$9.869163318 \le \mu \le 9.87096774$$
$$9.869555 \le \mu \le 9.8697539797$$
$$9.8695989 \le \mu \le 9.86962$$

The convergence is faster and we get good accuracy with a small number of iterations. Of course if we start with the 'lucky' choice of $v = w = \sin \pi t$ we get the exact result.

As a second example we take the 4-th order $(3,1)$-conjugate problem as in Example 1.2.

Using the Bo Yang iteration starting with the functions w_1, w_2 given in his paper [32], with Maple we calculated the following iterates (before Maple gave up)

$$m_1 \approx 437.137,\ \ M_1 \approx 2783.14$$
$$m_4 \approx 750.005,\ \ M_4 \approx 1256.87$$
$$m_8 \approx 844.467,\ \ M_8 \approx 1093.23$$
$$m_{12} \approx 878.546,\ \ M_{12} \approx 1043.56$$
$$m_{16} \approx 896.097,\ \ M_{16} \approx 1019.58$$
$$m_{17} \approx 899.231,\ \ M_{17} \approx 1015.40$$

We are still some way off the true value of ≈ 950.884270.

Using the Collatz-Wielandt numbers with Maple's help, we obtained starting with $v = w = t^3(1-t)$:

$$560 \le \mu \le 1680$$
$$857.3195876 \le \mu \le 978.3529412$$
$$937.5879619 \le \mu \le 953.2220796$$
$$949.4242658 \le \mu \le 951.1019276$$
$$950.7392026 \le \mu \le 950.9048935$$
$$950.8703010 \le \mu \le 950.8862317$$

Using the eigenfunction gives an exact result: $\mu_1 = \omega^4$, where $\omega = r\sqrt{2}$ and r is the smallest positive root of $\tan(x) = \tanh(x)$; thus $r \approx 3.926602312$, $\mu_1 \approx 950.884270$.

We see that the iterations using Collatz-Wielandt numbers give good accuracy with a small number of iterations.

Remark 7.1. Before we get carried away it should be remarked that although the method works well on a number of problems, for example we have also done some three-point boundary value problems, it can fail quite rapidly: it can be difficult or impossible to actually do the iterations using a tool such as Maple; an example is given in [26]. Therefore we are back to using some numerical analysis method if we want good approximations.

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