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STRONGLY POSITIVE OPERATORS AND u_0-POSITIVE OPERATORS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. We study u_0-positive operators and strongly positive operators in Banach spaces. A necessary and sufficient condition for an operator to be u_0-positive is given. The relations between the two types of operators are studied. Previous result only showed that when the cone is solid, a strongly positive operator is u_0-positive. We prove that if the cone is normal and solid and u_0 is an interior point of the cone, then the two classes of operators are equivalent.

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1. Introduction

Strongly positive operators and u_0-positive operators have been studied for example, in [1, 2, 3, 4, 5, 6, 7, 8]. It is well known that strongly positive operators defined on solid cones are u_0-positive, but in general, the converse may not be true. Under what additional conditions on the cones is the converse true? In this paper, we answer this question. We prove that if the cone is normal and solid and u_0 is an interior of the cone, then u_0-positive operators and strongly positive operators are equivalent (see Theorem 3.4). To do this, we provide a necessary and sufficient condition for an operator to be u_0-positive (see Theorem 3.1). To prove the latter result, we study the interior points of the cone P_{u_0} and provide a necessary and sufficient condition for an element to be in the interior of P_{u_0} (see Theorem 2.4). This enables us to prove that if P is a normal cone and $T : P \to P_{u_0}$ is u_0-positive relative to P, then $T : P_{u_0} \to P_{u_0}$ is strongly positive relative to P_{u_0}, where P is not required to be solid (see Theorem 3.2).

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2. Interior points of the cone P_{u_0}

Let X be a real Banach space. Recall that a nonempty closed convex set P in X is called a cone if $\lambda x \in P$ for $x \in P$ and $\lambda \geq 0$ and $P \cap (-P) = \{0\}$. A cone P induces a partial order $\leq$ in X:

$$x \leq y \quad \text{if and only if } y - x \in P. \tag{2.1}$$

Sometimes, we replace $\leq$ by $\leq_P$ to show that the order is induced by P. A cone P in X is called a normal cone if there exists $\delta > 0$ such that

$$0 \leq x \leq y \quad \text{implies } \|x\| \leq \delta \|y\|.$$

We denote by $\mathring{P}$ the interior of P. A cone P is said to be solid if $\mathring{P} \neq \emptyset$.

Let $u, v \in X$ with $u \leq v$. We denote by

$$[u, v] := \{x \in X : u \leq x \leq v\}$$

the order interval in X.

Let P be a cone in X with $P \neq \{0\}$. For each $u_0 \in P \setminus \{0\}$, we define the following linear subspace of X:

$$X_{u_0} = \{x \in X : \text{ there exist } t, s \geq 0 \text{ such that } -tu_0 \leq x \leq su_0\}, \tag{2.2}$$

see [5, page 21]. We define two functions $a, b : X_{u_0} \to \mathbb{R}_+$ by

$$a(x) = \inf\{\eta \in \mathbb{R}_+ : -\eta u_0 \leq x\} \quad \text{and} \quad b(x) = \inf\{\gamma \in \mathbb{R}_+ : x \leq \gamma u_0\}. \tag{2.3}$$

Noting that P is closed, by (2.2) we have $a(x), b(x) \in \mathbb{R}_+$ and

$$X_{u_0} = \{x \in X : x \in [-a(x)u_0, b(x)u_0]\}. \tag{2.4}$$

The following equivalent definition of X_{u_0} is used in [1]:

$$X_{u_0} = \cup\{\xi[-u_0, u_0] : \xi \geq 0\}. \tag{2.5}$$

We need the following known result (see [3, 1.3.1] or [1, Theorem 2.1]).

Lemma 2.1. *Let P be a solid cone in X with $P \neq \{0\}$. Then $u_0 \in \mathring{P}$ if and only if $u_0 \in P \setminus \{0\}$ and $X_{u_0} = X$.*

In the linear subspace X_{u_0}, one can define a norm called u_0-norm as follows:

$$\|x\|_{u_0} = \max\{a(x), b(x)\}, \tag{2.6}$$

where $a(x)$ and $b(x)$ are the same as in (2.3). It is known that if P is a solid cone in X with $P \neq \{0\}$ and $u_0 \in \mathring{P}$, then $X = X_{u_0}$ and there exists $\varepsilon > 0$ such that

$$\varepsilon \|x\|_{u_0} \leq \|x\| \quad \text{for } x \in X, \tag{2.7}$$

see [5, p. 26–27]. Moreover, P is normal if and only if there exists $b > 0$ such that

$$\|x\| \leq (b\|u_0\|)\|x\|_{u_0} \quad \text{for } x \in X_{u_0} \text{ and } u_0 \in P \setminus \{0\},$$

see [5, Theorem 1.1]. Hence, if P is a normal solid cone in X and $u_0 \in \mathring{P}$, then the two norms $\|x\|_{u_0}$ and $\|x\|$ in X_{u_0} are equivalent, that is, there exist $\varepsilon > 0$ and $\sigma > 0$ such that

$$\varepsilon\|x\|_{u_0} \le \|x\| \le \sigma\|x\|_{u_0} \quad \text{for } x \in X. \tag{2.8}$$

For a cone P with $P \neq \{0\}$ and $u_0 \in P \setminus \{0\}$, let

$$P_{u_0} = X_{u_0} \cap P. \tag{2.9}$$

The following known result provides properties of $\|\cdot\|_{u_0}$, X_{u_0} and P_{u_0}, see [1, Theorem 2.3]) and [1, Lemma 2.2]).

Lemma 2.2. *Let P be a normal cone in X and let $u_0 \in P \setminus \{0\}$. Then the following assertions hold.*

(i) X_{u_0} is a Banach space with the norm $\|\cdot\|_{u_0}$.

(ii) P_{u_0} is a normal and solid cone in X_{u_0} and $u_0 \in \mathring{P}_{u_0}$.

Note that if P is not normal, then X_{u_0} might not be complete with respect to the u_0-norm; see [5]. By (2.9) and Lemmas 2.1 and 2.2, we see that if P is a normal solid cone in X and $u_0 \in \mathring{P}$, then $X = X_{u_0}$, $P = P_{u_0}$ and by (2.8), we obtain

$$\mathring{P} = \mathring{P}_{u_0}, \tag{2.10}$$

where $\mathring{P}_{u_0}$ is the set of interior points of P_{u_0} in X_{u_0} with the u_0-norm defined in (2.6).

Now, we prove the following new result which gives the relations between two interior points of a solid cone.

Lemma 2.3. *Let P be a solid cone in X. Let $v \in P$ and $u_0 \in \mathring{P}$. Then $v \in \mathring{P}$ if and only if there exists $\sigma := \sigma(v, u_0) > 0$ such that*

$$v \ge \sigma u_0.$$

Proof. Assume that $v \in \mathring{P}$. Then there exists $\varepsilon := \varepsilon(v) > 0$ such that

$$\{z \in X : \|z - v\| < \varepsilon\} \subset \mathring{P}.$$

Let $\sigma := \sigma(v, u_0) \in (0, \varepsilon\|u_0\|^{-1})$ and $z = v - \sigma u_0$. Then

$$\|z - v\| = \sigma\|u_0\| < \varepsilon.$$

This implies that $z \in \mathring{P} \subset P$ and $\sigma u_0 \le v$. Conversely, assume that there exists $\sigma := \sigma(v, u_0) > 0$ such that $v \ge \sigma u_0$. Since $u_0 \in \mathring{P}$, by Lemma 2.1,

$$X = X_{u_0} = \cup\{\xi[-u_0, u_0] : \xi \ge 0\}.$$

Since $v \ge \sigma u_0$, we have $\sigma[-u_0, u_0] \subset [-v, v]$ and

$$X = X_{u_0} = \cup\{\xi\sigma[-u_0, u_0] : \xi \ge 0\} \subset \cup\{\xi[-v, v] : \xi \ge 0\} \subset X.$$

It follows from (2.5) that $X_v = \cup\{\xi[-v,v] : \xi \geq 0\} = X$. By Lemma 2.1, we obtain $v \in \overset{\circ}{P}$. □

Now, we are in a position to prove our main result of this section.

Theorem 2.4. *Let P be a normal cone in X. Let $u_0 \in P \setminus \{0\}$ and $u \in P_{u_0}$. Then $u \in \overset{\circ}{P}_{u_0}$ if and only if there exist $\sigma > 0$ and $\beta \in [\sigma, \infty)$ such that*

$$\sigma u_0 \leq u \leq \beta u_0. \tag{2.11}$$

Proof. Since P is normal, by Lemma 2.2, X_{u_0} is a Banach space with the norm $\|\cdot\|_{u_0}$, P_{u_0} is a normal and solid cone in X_{u_0}, and $u_0 \in \overset{\circ}{P}_{u_0}$. By Lemma 2.3 with $X = X_{u_0}$ and $P = P_{u_0}$, $u \in \overset{\circ}{P}_{u_0}$ if and only if there exists $\sigma > 0$ such that $u \geq_{P_{u_0}} \sigma u_0$ if and only if $u - \sigma u_0 \in P_{u_0} = X_{u_0} \cap P$ if and only if $u - \sigma u_0 \in P$ and

$$u - \sigma u_0 \in X_{u_0} = \cup\{\xi[-u_0, u_0]_P : \xi \geq 0\}$$

if and only if there exists $\xi \geq 0$ such that $-\xi u_0 \leq u - \sigma u_0 \leq \xi u_0$ and $0 \leq u - \sigma u_0$ if and only if there exists $\xi \geq 0$ such that $0 \leq u - \sigma u_0 \leq \xi u_0$. This implies

$$\sigma u_0 \leq u \leq (\xi + \sigma) u_0 := \beta u_0$$

and (2.11) holds. □

3. **Strongly positive and u_0-positive operators**

Let P be a cone in X and let $u_0 \in P \setminus \{0\}$. Recall that an operator $T : P \to P$ is said to be u_0-positive relative to P if for each $x \in P \setminus \{0\}$, there exist $\alpha(x) > 0$, $\beta(x) > 0$ and $n(x) \in \mathbb{N}$ such that

$$\alpha(x) u_0 \leq T^{n(x)} x \leq \beta(x) u_0, \tag{3.1}$$

see [5]. If P is a solid cone in X, then $T : P \to P$ is said to be strongly positive relative to P if for each $x \in P \setminus \{0\}$ there exists $n(x) \in \mathbb{N}$ such that $T^{n(x)} x \in \overset{\circ}{P}$, see [5, page 59-60].

By Theorem 2.4, we obtain the following result which gives a necessary and sufficient condition for T to be u_0-positive.

Theorem 3.1. *Let P be a normal cone in X. Let $u_0 \in P \setminus \{0\}$. Then $T : P \to P$ is u_0-positive relative to P if and only if for each $x \in P \setminus \{0\}$ there exists $n(x) \in \mathbb{N}$ such that $T^{n(x)} x \in \overset{\circ}{P}_{u_0}$.*

Proof. By Theorem 2.4, we see that $T^{n(x)} x \in \overset{\circ}{P}_{u_0}$ if and only if (3.1) holds. The result follows. □

The following new result shows that u_0-positive operators relative to P are strongly positive relative to P_{u_0}.

Theorem 3.2. *Let P be a normal cone in X and $u_0 \in P \setminus \{0\}$. Assume that $T : P \to P_{u_0}$ is u_0-positive relative to P. Then $T : P_{u_0} \to P_{u_0}$ is strongly positive relative to P_{u_0}.*

Proof. Since $T : P \to P_{u_0}$ is u_0-positive relative to P, it follows that for each $x \in P_{u_0} \setminus \{0\} \subset P \setminus \{0\}$ there exist $\alpha(x) > 0$, $\beta(x) > 0$ and $n(x) \in \mathbb{N}$ such that (3.1) holds. Since $T(P) \subset P_{u_0} \subset P$, $T^{n(x)}x \in P_{u_0}$. By Theorem 2.4, $T^{n(x)}x \in \mathring{P}_{u_0}$. The result follows. □

Theorem 3.2 requires the cone P to be normal, which may not be solid. The following known result requires the cone P to be solid, which may not be normal. We provide a proof which is different from that given in [5, page 60].

Lemma 3.3. *Let P be a solid cone in X. Assume that $T : P \to P$ is strongly positive relative to P. Then T is u_0-positive relative to P for each $u_0 \in \mathring{P}$.*

Proof. Let $u_0 \in \mathring{P}$. Since T is strongly positive relative to P, for each $x \in P \setminus \{0\}$ there exists $n(x) \in \mathbb{N}$ such that $T^{n(x)}x \in \mathring{P}$. Noting that both u_0 and $T^{n(x)}x$ are interior points of P and applying Lemma 2.3 twice, we see that there exist $\alpha(x) > 0$ and $\beta(x) > 0$ such that (3.1) holds. The result follows. □

The following new result shows that if the cone in Lemma 3.3 is normal, then the inverse of Lemma 3.3 is true.

Theorem 3.4. *Let P be a normal solid cone in X and let $T : P \to P$ be an operator. Then the following assertions are equivalent.*

(1) *T is u_0-positive for some $u_0 \in \mathring{P}$.*

(2) *T is strongly positive relative to P.*

(3) *T is u_0-positive for every $u_0 \in \mathring{P}$.*

Proof. By Lemma 3.3, we see that (2) implies (3). It is obvious that (3) implies (1). We prove that (1) implies (2). Assume that T is u_0-positive for some $u_0 \in \mathring{P}$. By Theorem 3.1, for each $x \in P \setminus \{0\}$ there exists $n(x) \in \mathbb{N}$ such that $T^{n(x)}x \in \mathring{P}_{u_0}$. By (2.10), $\mathring{P}_{u_0} = \mathring{P}$. It follows that $T^{n(x)}x \in \mathring{P}$ for each $x \in P \setminus \{0\}$ and T is strongly positive relative to P. □

The following example gives applications of Theorem 3.4 and shows that u_0-positive operators may not be linear.

Example 3.5. Let $P = \{x \in C[0,1] : x(t) \geq 0 \quad \text{for } t \in [0,1]\}$ be the standard positive cone in the Banach space $C[0,1]$ with the maximum norm. Let $p > 0$. We define the following two operators from P to P:

(1) $T_1x(t) = x^p(t) + 1$ for $t \in [0,1]$ and $x \in P$.

(2) $T_2x(t) = x^p(t)$ for $t \in [0,1]$ and $x \in P$.

Then T_1 is u_0-positive relative to P for any $u_0 \in \mathring{P}$ and T_2 is not u_0-positive for each $u_0 \in \mathring{P}$.

Proof. (1) Let $x \in P \setminus \{0\}$. Then $T_1x(t) = x^p(t) + 1 > 0$ for $t \in [0,1]$. Noting that $\mathring{P} = \{x \in C[0,1] : x(t) > 0 \quad \text{for } t \in [0,1]\}$, we have $T_1x \in \mathring{P}$. Hence T_1 is strongly positive relative to P. The result follows from Theorem 3.4 (2) and (3).

(2) Let $x_1(t) = t^2$ for $t \in [0,1]$. Then $T_2x_1(0) = 0$. This implies $T_2x_1 \notin \mathring{P}$ and T_2 is not strongly positive relative to P. Since P is a normal solid cone in $C[0,1]$, it follows from Theorem 3.4 (1) and (2) that T_2 is not u_0-positive for each $u_0 \in \mathring{P}$. □

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