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NONLOCAL BOUNDARY VALUE PROBLEMS WITH EVEN GAPS IN BOUNDARY CONDITIONS FOR THIRD ORDER DIFFERENTIAL EQUATIONS

XUEYAN LIU

Department of Mathematics, University of Tennessee at Chattanooga
Chattanooga, TN, 37403, USA E-mail: Xueyan-Liu@utc.edu

Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. We use solution matching to study the uniqueness and existence of solutions for the nonlocal boundary value problem for the third order differential equation, $y'''(x) = f(x, y(x))$, on an interval $[a, c]$ satisfying $y(a) - \int_a^b y(x)d\alpha(x) = y_1$, $y'(b) = y_2$, $\int_b^c y(x)d\beta(x) - y(c) = y_3$, where $\int_a^b y(x)d\alpha(x)$ and $\int_b^c y(x)d\beta(x)$ are Riemann-Stieltjes integrals with positive measures $d\alpha(x)$ and $d\beta(x)$, respectively. We match solutions on $[a, b]$ with solutions on $[b, c]$. Monotonicity conditions and some growth conditions on f are imposed.

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1. INTRODUCTION

Matching of solutions of boundary value problems is intimately involved with interface problems for which an intermediate boundary point corresponds to a point of interface [1, 20, 25, 27]. For such problems, as smooth as possible interfacing is desired. Otherwise, leakage or impulses in transfer rates occur. Most matching results deal with smoothing one possible break in some order derivative. When gaps in the derivatives at the interface point involve several successive derivatives, there is great difficulty in transfer across the interface. Especially when the gap is even, as discussed in this paper, instead of odd, it is even more difficult for smoothness, so the hypotheses for matching can be seemingly strong.

The solution-matching technique was first used by Bailey *et al.* [2]. They considered the solutions of two-point boundary value problems for the second order differential equation $y'' = f(x, y, y')$ by matching solutions of initial value problems. Then, in 1973, Barr and Sherman [3] first assumed monotonicity conditions on f and applied the solution-matching technique to third order equations and generalized to equations of arbitrary order. The boundedness of f was assumed. In 1978, Moorti [17] applied the monotonicity condition on f and solution matching method

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to the nth order boundary value problems. In 1981, Murthy *et al.* [18] and Rao *et al.* [23], in a certain sense, generalized the monotonicity of f of third order differential equations and introduced an auxiliary monotone function g. In 1983, Henderson [6] generalized to nth order BVP's and considered more general boundary conditions. In 1993, Taunton *et al.* [11] analyzed the properties of solutions of differential inequalities involved with the auxiliary monotone function g of the third order boundary value problems. In 2001, Henderson *et al.* [10] generated the solution method of nth order differential equations on time scales. Since then, a lot of work has been done on existence and uniqueness of certain BVP's for third order or higher order differential equations, differential systems or differential equations on time scales by matching solutions. We refer the readers to [4, 5, 12, 8, 7, 13, 14, 15, 9, 19, 21, 22, 24, 26], etc.

In the present paper, we study the uniqueness and existence of solutions for the nonlocal boundary value problem for third order differential equations (1.1), (1.2),

$$y'''(x) = f(x, y(x)), \quad x \in [a, c], \tag{1.1}$$

$$y(a) - \int_a^b y(x)d\alpha(x) = y_1, \quad y'(b) = y_2, \quad \int_b^c y(x)d\beta(x) - y(c) = y_3, \tag{1.2}$$

where $a < b < c$, $\int_a^b y(x)d\alpha(x)$ and $\int_b^c y(x)d\beta(x)$ are Riemann-Stieltjes integrals with positive measures $d\alpha(x)$ and $d\beta(x)$, respectively; $\int_a^b d\alpha(x) = \int_b^c d\beta(x) = 1$; $\alpha(x) \neq s + 1_{(a,b]}$ and $\beta \neq t + 1_{\{c\}}$ for any $s, t \in \mathbb{R}$; $y_1, y_2, y_3 \in \mathbb{R}$.

Examining the boundary conditions at b, we can see that the function value and the value of the second order derivative of solutions are missing. The difference of their order of derivatives (or gaps) is two, which is even. At the time of this work, no previous work has been done on the existence and uniqueness of solutions of this BVP with even gaps in boundary conditions at b based on the techniques of solution-matching. Liu [9] studied the multi-point n-th order problem in which the gap in the boundary conditions is odd. Liu [16] extended [9] and studied the even case. Here, we consider more general boundary conditions for the solution matching technique. Notice that if α and β are simple functions defined on $[a, b]$ and $[b, c]$, for example, $\alpha(x) = 1_{[\xi,b]}$ and $\beta = 1_{[\eta,c]}$, where $a < \xi < b < \eta < c$, then (1.1), (1.2) is a five-point boundary value problem.

In the spirit of solution matching, we consider the following list of four boundary conditions,

$$y(a) - \int_a^b y(x)d\alpha(x) = y_1, \quad y'(b) = y_2, \quad y(b) = m, \tag{1.3}$$

$$y(a) - \int_a^b y(x)d\alpha(x) = y_1, \quad y'(b) = y_2, \quad y''(b) = m, \tag{1.4}$$

$$y'(b) = y_2, \quad y(b) = m, \quad \int_b^c y(x)d\beta(x) - y(c) = y_3, \tag{1.5}$$

$$y'(b) = y_2, \quad y''(b) = m, \quad \int_b^c y(x)d\beta(x) - y(c) = y_3, \tag{1.6}$$

where $m \in \mathbb{R}$, and we are going to establish that (1.1), (1.2) has a unique solution on $[a, c]$ by matching solutions of the BVP's (1.1), (1.3) on $[a, b]$ with solutions of (1.1), (1.5) on $[b, c]$.

Throughout this paper, we assume that $f : [a, c] \times \mathbb{R} \to \mathbb{R}$ is continuous and that solutions of IVP's for (1.1) are unique and exist on all of $[a, c]$. The function f also satisfies the following monotone conditions:

(A) $f(x, v) - f(x, u) > 0$, when $x \in (a, b)$, $v < u$; or when $x \in (b, c)$, $v > u$.

The paper is organized as follows: in Section 2, some essential lemmas are shown. Section 3 involves the main theorem.

2. LEMMAS

First we give two lemmas on the measure functions α and β.

Lemma 2.1. *Given a strictly continuous function $h(x)$ on $[a, b]$ such that $h(x) > 0$ for $x \in (a, b]$, it follows that $\int_a^b h(x)d\alpha(x) > 0$.*

Proof. Obviously, $\int_a^b h(x)d\alpha(x) \geq 0$ since $d\alpha(x)$ is a positive measure. We prove the strict inequality by contradiction. Suppose $\int_a^b h(x)d\alpha(x) = 0$. Then, for any $\xi \in (a, b)$,

$$\begin{aligned}
0 &= \int_a^b h(x)d\alpha(x) \\
&= \int_a^\xi h(x)d\alpha(x) + \int_\xi^b h(x)d\alpha(x) \\
&\geq \int_\xi^b h(x)d\alpha(x) \\
&= h(\xi) \int_\xi^b d\alpha(x) \\
&= h(\xi)(\alpha(b) - \alpha(\xi)) \\
&\geq 0.
\end{aligned}$$

Therefore, $\alpha(b) = \alpha(\xi)$ for any $\xi \in (a, b)$, which is a contradiction to $\alpha(x) \neq s + 1_{(a,b]}$ for any $s \in \mathbb{R}$. □

Lemma 2.2. *Given a strictly decreasing continuous function $g(x)$ on $[b, c]$ satisfying that $g(x) > 0$ for $x \in [b, c)$, it follows that $\int_b^c g(x)d\beta(x) > 0$.*

Proof. Similar to the proof of Lemma 2.1, we prove the strict inequality by contradiction. Suppose $\int_b^c g(x)d\beta(x) = 0$. Then, for any $\eta \in (b, c)$,

$$\begin{aligned} 0 &= \int_b^c g(x)d\beta(x) \\ &= \int_b^\eta g(x)d\beta(x) + \int_\eta^c g(x)d\beta(x) \\ &\geq \int_b^\eta g(x)d\beta(x) \\ &= g(\eta)\int_b^\eta d\beta(x) \\ &= g(\eta)(\beta(\eta) - \beta(b)) \\ &\geq 0. \end{aligned}$$

So, $\beta(\eta) = \beta(b)$ for any $\eta \in (b, c)$, which is a contradiction to $\beta(x) \neq t + 1_{\{c\}}$ for any $t \in \mathbb{R}$. □

The next two lemmas are fundamental for our main theorem, which describe the monotonicity relations between the changes in the function values and changes in the values of the second derivative of solutions of (1.1), (1.2) at b. These lemmas are proved by contradiction.

Lemma 2.3. *Assume the condition (A) is satisfied. Suppose p and q are solutions of (1.1) satisfying $y(a) - \int_a^b y(x)d\alpha(x) = y_1$, $y'(b) = y_2$ on $[a, b]$, and let $w = p - q$. Then, $w(b) = 0$ if and only if $w''(b) = 0$, and $w(b) > 0$ if and only if $w''(b) < 0$.*

Proof. First, w satisfies

$$\begin{aligned} &w'''(x) = f(x, p(x)) - f(x, q(x)), \quad x \in [a, b], \\ &w(a) - \int_a^b w(x)d\alpha(x) = 0 = w'(b). \end{aligned}$$

($\Rightarrow$) The necessity of equalities.

Suppose $w(b) = 0$ and $w''(b) \neq 0$. Without loss of generality, we assume $w''(b) > 0$. By $\int_a^b d\alpha(x) = 1$, we have $w(a) - \int_a^b w(x)d\alpha(x) = \int_a^b (w(a) - w(x))d\alpha(x) = 0$. We have that there is some point $x_0 \in [a, b)$ such that $w'(x_0) = 0$. If not, then $w'(x) < 0$ for $x \in [a, b)$ and $w(a) > w(x)$ for $x \in (a, b]$. By Lemma 2.1, we have $\int_a^b (w(a) - w(x))d\alpha(x) > 0$, a contradiction.

By $w'(x_0) = 0$, $w'(b) = 0$ and the Mean Value Theorem, there is some $x_1 \in (x_0, b)$ such that $w''(x_1) = 0$. From $w''(b) > 0$, there is some $x_2 \in [x_1, b)$ such that $w''(x_2) = 0$ and $w''(x) > 0$ for $x \in (x_2, b]$. By the Mean Value Theorem again, there is some $x_3 \in (x_2, b)$ such that $w'''(x_3) > 0$. However, from $w(b) = w'(b) = 0$, we have $w(x) > 0$ and $w'(x) < 0$ for $x \in [x_2, b)$, which by condition (A) imply $w'''(x) < 0$ for $x \in [x_2, b)$. This is a contradiction. Hence, $w''(b) = 0$.

($\Leftarrow$) The sufficiency of equalities.

Suppose $w''(b) = 0$ and $w(b) \neq 0$. Without loss of generality, we assume $w(b) > 0$. Therefore, in a left neighborhood of b, $w(x) > 0$. By condition (A), we know that $w'''(x) < 0$ in that deleted left neighborhood.

By $w'(b) = 0$, $w(b) > 0$, and $w''(b) = 0$, we have that in that deleted left neighborhood of b, $w''(x) > 0$, $w'(x) < 0$, $w(x) > 0$.

Similarly to the proof of necessity of equalities, by $w(a) - \int_a^b w(x)d\alpha(x) = \int_a^b (w(a) - w(x))d\alpha(x) = 0$, there is some point $x_0 \in [a, b)$ such that $w'(x_0) = 0$. Hence, there is some $x_1 \in [x_0, b)$ such that $w'(x_1) = 0$ and $w'(x) < 0$ for $x \in (x_1, b]$. So, $w(x) > 0$ for $x \in [x_1, b]$, which implies $w'''(x) < 0$ for $x \in [x_1, b)$ and $w''(x) > 0$ for $x \in [x_1, b)$. However, by $w'(x_1) = 0$ and $w'(b) = 0$, there is some $x_2 \in (x_1, b)$ such that $w''(x_2) = 0$, which leads to a contradiction. Hence, $w(b) = 0$.

($\Rightarrow$) The necessity of inequalities.

Assume $w(b) > 0$ and $w''(b) > 0$. Similarly to the proof of sufficiency of equalities, by $w(a) - \int_a^b w(x)d\alpha(x) = \int_a^b (w(a) - w(x))d\alpha(x) = 0$, there is some $x_0 \in [a, b)$ such that $w'(x_0) = 0$. By $w'(b) = 0$, there is some $x_1 \in (x_0, b)$ such that $w''(x_1) = 0$. Since $w''(b) > 0$, there is $x_2 \in [x_1, b)$ such that $w''(x) > 0$ for $x \in (x_2, b]$ and $w''(x_2) = 0$. From $w(b) > 0$ and $w'(b) = 0$, it follows that $w(x) > 0$ and $w'(x) < 0$ for $x \in [x_2, b)$. By the condition (A), $w'''(x) < 0$ for $x \in [x_2, b)$. However, from $w''(x_2) = 0$ and $w''(b) > 0$ and the Mean Value Theorem, there is some $x_3 \in (x_2, b)$ such that $w'''(x_3) > 0$. This is a contradiction. Therefore, if $w(b) > 0$, then $w''(b) < 0$.

($\Leftarrow$) The sufficiency of inequalities.

We assume that $w(b) < 0$ and $w''(b) < 0$. By substituting $-w$ for w, we are in the same situation as in the proof of the necessity of inequalities and we will arrive at a contradiction. Hence $w(b) > 0$, if $w''(b) < 0$. □

Lemma 2.4. *Assume the condition (A) is satisfied. Suppose p and q are solutions of (1.1) satisfying $y'(b) = y_2$ and $\int_b^c y(x)d\beta(x) - y(c) = y_3$ on $[b, c]$, and let $w = p - q$. Then $w(b) = 0$ if and only if $w''(b) = 0$, and $w(b) > 0$ if and only if $w''(b) < 0$.*

Proof. First, w satisfies

$$w'''(x) = f(x, p(x)) - f(x, q(x)), \quad x \in [b, c],$$
$$w'(b) = 0 = \int_a^b w(x)d\beta(x) - w(c).$$

($\Rightarrow$) The necessity of equalities.

Suppose $w(b) = 0$ and $w''(b) \neq 0$. Without loss of generality, we suppose $w''(b) > 0$. By $\int_b^c d\beta(x) = 1$, we have $\int_b^c w(x)d\beta(x) - w(c) = \int_b^c (w(x) - w(c))d\beta(x) = 0$. We have that there is some point $x_0 \in (b, c]$ such that $w'(x_0) = 0$. If not, then $w'(x) > 0$

for $x \in (b, c]$ and $w(c) > w(x)$ for $x \in [b, c)$. By Lemma 2.2, $\int_b^c (w(x) - w(c)) d\beta(x) < 0$, a contradiction.

The Mean Value Theorem and $w'(b) = w'(x_0) = 0$ imply that there is some $x_1 \in (b, x_0)$ such that $w''(x_1) = 0$. By $w''(b) > 0$, there is some $x_2 \in (b, x_1]$ such that $w''(x_2) = 0$ and $w''(x) > 0$ for $x \in [b, x_2)$. $w(b) = 0$ implies that $w(x) > 0$ for $x \in (b, x_2]$, which by condition (A) implies that $w'''(x) > 0$ for $x \in (b, x_2]$. However, by $w''(b) > 0$ and $w''(x_2) = 0$ and the Mean Value Theorem, there is some $x_3 \in (b, x_2)$ such that $w'''(x_3) < 0$, which is again a contradiction. Hence, $w''(b) = 0$.

($\Leftarrow$) The sufficiency of equalities.

Suppose $w''(b) = 0$ and $w(b) \neq 0$. Without loss of generality, we suppose $w(b) > 0$. Since w is continuous on $[b, c]$, w is positive in a right neighborhood of b. From condition (A), we have that $w''' > 0$ in that deleted neighborhood. The rest of the proof is very similar to the proof of the necessity of equalities. Hence, we will finally have a contradiction. Therefore, if $w''(b) = 0$, then $w(b) = 0$.

($\Rightarrow$) The necessity of inequalities.

Assume $w(b) > 0$ and $w''(b) > 0$. The proof is similar to that of the necessity of equalities. A contradiction yields that $w''(b) < 0$, if $w(b) > 0$.

($\Leftarrow$) The sufficiency of inequalities.

We assume that $w(b) < 0$ and $w''(b) < 0$. Then, we have the same situation as the proof of necessity of inequalities with opposite sign of w, which leads to a contradiction, as well. Hence $w(b) > 0$, if $w''(b) < 0$. □

With the above two fundamental lemmas, we are in a position to establish some more lemmas helpful in our matching ideas.

Lemma 2.5. *Let $y_1, y_2, y_3 \in \mathbb{R}$ be given and assume condition (A) is satisfied. Then, given $m \in \mathbb{R}$, each of the BVP's for* (1.1) *satisfying any of conditions* (1.3), (1.4), (1.5), *or* (1.6) *has at most one solution.*

Proof. Here we prove the uniqueness of solutions of (1.1), (1.3) for any $m \in \mathbb{R}$. The other cases are very similar based on Lemma 2.3 or Lemma 2.4.

Suppose there are two solutions p and q of (1.1) satisfying (1.3). Let $w = p - q$. Then, we can see that w satisfies

$$w'''(x) = f(x, p(x)) - f(x, q(x)), \quad x \in [a, b],$$

$$w(a) - \int_a^b w(x) d\alpha(x) = w(b) = w'(b) = 0.$$

By Lemma 2.3, we have $w''(b) = 0$. From the assumption that solutions of IVP's for (1.1) are unique and exist on all of $[a, c]$, it follows that $p \equiv q$ on $[a, c]$. Therefore, solutions of (1.1), (1.3) are unique for any $m \in \mathbb{R}$. □

Now we show that solutions of (1.1) satisfying each of (1.3), (1.4), (1.5), or (1.6), respectively, are monotone functions of m at b. For notation purposes, given any $m \in \mathbb{R}$, let $\alpha(x, m)$, $u(x, m)$, $\beta(x, m)$, $v(x, m)$ denote the solutions, when they exist, of the boundary value problems of (1.1) satisfying (1.3), (1.4), (1.5), or (1.6), respectively.

Lemma 2.6. *Suppose that condition (A) is satisfied and that for each $m \in \mathbb{R}$, there exist solutions of* (1.1) *satisfying each of the conditions* (1.3)*,* (1.4)*,* (1.5)*,* (1.6)*, respectively. Then, all of $\alpha''(b, m)$, $\beta''(b, m)$, $u(b, m)$ and $v(b, m)$ are strictly decreasing functions of m with ranges all of $\mathbb{R}$.*

Proof. First, by using Lemmas 2.3 and 2.4, $\alpha''(b, m)$, $\beta''(b, m)$, $u(b, m)$ and $v(b, m)$ are all strictly decreasing functions of m.

Next, we prove the range of $\alpha''(b, m)$ as a function of m is all of $\mathbb{R}$. The proofs of other three cases are very similar. It suffices to show $\{\alpha''(b, m) | m \in \mathbb{R}\} = \mathbb{R}$. Given any $l \in \mathbb{R}$. Consider the solution $u(x, l)$ of (1.1) satisfying (1.4) and the solution $\alpha(x, u''(b, l))$ of (1.1) satisfying (1.3). Then, both $u(x, l)$ and $\alpha(x, u(b, l))$ satisfy (1.1) and the boundary conditions $y(a) - \int_a^b y(x) d\alpha(x) = y_1$, $y'(b) = y_2$ and $\alpha(b, u(b, l)) = u(b, l)$. By Lemma 2.3, $\alpha(x, u(b, l)) \equiv u(x, l)$ for $x \in [a, b]$. Hence, $\alpha''(b, u(b, l)) = u''(b, l) = l$. Therefore, $l \in \{\alpha''(b, m) | m \in \mathbb{R}\}$, that is, $\{\alpha''(b, m) | m \in \mathbb{R}\} = \mathbb{R}$. □

Next, under certain Lipschitz and inverse Lipschitz conditions of f, we obtain some bounds to the rate of change of the second order derivative of solutions of (1.1) at b with respect to $m \in \mathbb{R}$.

Lemma 2.7. *Assume condition (A) is satisfied, and suppose there is some $M_1 > 0$, such that*

$$f(x, v) - f(x, u) \geq -M_1(v - u), \quad \forall\, x \in (a, b), \quad \forall\, v \geq u \in \mathbb{R}. \tag{2.1}$$

Assume for each $m \in \mathbb{R}$, there exists a solution $\alpha(x, m)$ of (1.1) *satisfying* (1.3)*. Let $m_1, m_2 \in \mathbb{R}$ with $m_1 < m_2$. Then,*

$$\alpha''(b, m_2) - \alpha''(b, m_1) > -M_1(b - a)(m_2 - m_1). \tag{2.2}$$

Proof. Let $m_1, m_2 \in \mathbb{R}$ with $m_1 < m_2$ be fixed. We denote $\Phi(x) = \frac{\alpha(x,m_2)-\alpha(x,m_1)}{m_2-m_1}$. Then, it is easy to see that $\Phi(x)$ satisfies

$$\Phi'''(x) = \frac{f(x, \alpha(x, m_2)) - f(x, \alpha(x, m_1))}{m_2 - m_1}, \quad x \in [a, b],$$

$$\Phi(b) = 1, \quad \Phi'(b) = 0 = \Phi(a) - \int_a^b \Phi(x) d\alpha(x),$$

and by Lemma 2.6, $\Phi''(b) < 0$. It suffices to show that $\Phi''(b) > -M_1(b - a)$.

Since $\Phi(a) - \int_a^b \Phi(x)d\alpha(x) = \int_a^b(\Phi(a) - \Phi(x))d\alpha(x) = 0$. We can see that there is some $x_0 \in [a, b)$ such that $\Phi'(x_0) = 0$. If not, by $\Phi'(b) = 0$ and $\Phi''(b) < 0$, we have $\Phi'(x) > 0$ for $x \in [a, b)$. Hence, $\Phi(a) - \Phi(x)$ is strictly decreasing and $\Phi(a) - \Phi(x) < 0$ for $x \in [a, b)$. By Lemma 2.3, $\int_a^b(\Phi(a) - \Phi(x))d\alpha(x) < 0$. This is a contradiction.

From $\Phi'(b) = 0$, $\Phi'(x_0) = 0$, and the Mean Value Theorem, there is some $x_1 \in (x_0, b)$ such that $\Phi''(x_1) = 0$. Since $\Phi''(b) < 0$, there is some $x_2 \in [x_1, b)$ such that $\Phi''(x_2) = 0$ and $\Phi''(x) < 0$ for $x \in (x_2, b]$. From $\Phi'(b) = 0$, we can see that $\Phi'(x) > 0$ for $x \in [x_2, b)$. Since $\Phi'(x_0) = 0$, there is some $x_3 \in [x_0, x_2)$ such that $\Phi'(x_3) = 0$ and $\Phi'(x) > 0$ for $x \in (x_3, b)$.

Next, we show $\Phi(x) > 0$ for $x \in [x_2, b]$. If not, then, from $\Phi'(x_3) = 0$ and $\Phi'(x) > 0$ for $x \in (x_3, b)$ and $x_3 \in [x_0, x_2)$, we have that $\Phi(x_2) \leq 0$ and $\Phi(x) < 0$ for $x \in [x_3, x_2)$. From condition (A), $\Phi'''(x) > 0$ for $x \in [x_3, x_2)$. However, from $\Phi'(x_3) = 0$ and $\Phi'(x_2) > 0$, there is some $x_4 \in (x_3, x_2)$ such that $\Phi''(x_4) > 0$. Also from $\Phi''(x_2) = 0$, there is some $x_5 \in (x_4, x_2) \subset (x_3, x_2)$ such that $\Phi'''(x_5) < 0$. This is a contradiction to $\Phi'''(x) > 0$ for $x \in [x_3, x_2)$. Therefore, $\Phi(x) > 0$ for $x \in [x_2, b]$.

Now from $\Phi(x) > 0$ for $x \in [x_2, b]$ and $\Phi'(x) > 0$ for $x \in [x_2, b)$, it is easy to see that $0 < \Phi(x) < 1$ for $x \in [x_2, b)$. Then, by (2.1), for $x \in [x_2, b)$,

$$\begin{aligned}
\Phi'''(x) &= \frac{f(x, \alpha(x, m_2)) - f(x, \alpha(x, m_1))}{m_2 - m_1} \\
&\geq \frac{-M_1(\alpha(x, m_2) - \alpha(x, m_1))}{m_2 - m_1} \\
&= -M_1\Phi(x) \\
&> -M_1.
\end{aligned}$$

Next we show that $\Phi''(b) > -M_1(b - a)$. If not, then, $\Phi''(b) \leq -M_1(b - a)$. By $\Phi''(x_2) = 0$, there is some $x_5 \in (x_2, b)$ such that

$$\Phi'''(x_5) = \frac{\Phi''(b) - \Phi''(x_2)}{b - x_2} \leq \frac{-M_1(b - a)}{b - x_2} = -M_1\frac{(b - a)}{b - x_2} < -M_1,$$

which is a contradiction to $\Phi'''(x) > -M_1$ for $x \in [x_2, b)$. Therefore, $\Phi''(b) > -M_1(b - a)$. □

Lemma 2.8. *Suppose f satisfies the condition (A) and there is a continuous function $M_1(x)$ for $x \in [a, b]$, such that*

$$f(x, v) - f(x, u) \leq -M_1(x)(v - u), \quad \forall\, x \in (a, b), \quad \forall\, v \geq u \in \mathbb{R}, \tag{2.3}$$

where $M_1(x) > 0$, for $x \in [a, b)$, and

$$\frac{\int_a^b \left(\int_a^x \int_l^b \int_r^b M_1(e)de\,dr\,dl\right) d\alpha(x)}{\frac{(b-a)^2}{2} - \int_a^b \frac{(b-x)^2}{2}d\alpha(x) + \int_a^b \left(\int_a^x \int_l^b \int_r^b M_1(e)\frac{(b-e)^2}{2}de\,dr\,dl\right) d\alpha(x)} \geq \frac{2}{(b - a)^2}. \tag{2.4}$$

Assume for each $m \in \mathbb{R}$, there exists a solution $\alpha(x, m)$ of (1.1) *satisfying* (1.3)*. Let $m_1 < m_2 \in \mathbb{R}$. Then,*

$$\alpha''(b, m_2) - \alpha''(b, m_1) < -\frac{2(m_2 - m_1)}{(b-a)^2}. \tag{2.5}$$

Proof. Let $m_1, m_2 \in \mathbb{R}$ be fixed such that $m_1 < m_2$. We denote

$$\Phi(x) = \frac{\alpha(x, m_2) - \alpha(x, m_1)}{m_2 - m_1}.$$

Then, $\Phi(x)$ satisfies

$$\Phi'''(x) = \frac{f(x, \alpha(x, m_2)) - f(x, \alpha(x, m_1))}{m_2 - m_1}, \quad x \in [a, b],$$

$$\Phi(b) = 1, \quad \Phi'(b) = 0 = \Phi(a) - \int_a^b \Phi(x) d\alpha(x),$$

and by Lemma 2.6, $\Phi''(b) < 0$. It suffices to show that $\Phi''(b) < -\frac{2}{(b-a)^2}$. Suppose this is not true. Then, $\Phi''(b) \geq -\frac{2}{(b-a)^2}$.

By $\Phi(b) = 1$, $\Phi'(b) = 0$, and $\Phi''(b) \geq -\frac{2}{(b-a)^2}$, we have that

$$\begin{aligned}
\Phi(x) &= \Phi(b) - \int_x^b \Phi'(l) dl \\
&= \Phi(b) + \int_x^b \int_l^b \Phi''(r) dr\, dl \\
&= \Phi(b) + \int_x^b \int_l^b \left(\Phi''(b) - \int_r^b \Phi'''(e) de \right) dr\, dl \\
&= 1 + \Phi''(b) \cdot \frac{(b-x)^2}{2} - \int_x^b \int_l^b \int_r^b \Phi'''(e) de\, dr\, dl.
\end{aligned}$$

Next, we show $\Phi(x) > 0$ for $x \in [a, b]$. Assume this is not true. Let $x_0 \in [a, b)$ such that $\Phi(x_0) = 0$ and $\Phi(x) > 0$ for $x \in (x_0, b]$. Then, by (2.3),

$$\Phi'''(x) = \frac{f(x, \alpha(x, m_2)) - f(x, \alpha(x, m_1))}{m_2 - m_1} \leq -M_1(x)\Phi(x), \quad \forall x \in (x_0, b].$$

Hence, by $\Phi''(b) \geq -\frac{2}{(b-a)^2}$,

$$\begin{aligned}
\Phi(x_0) &= 1 + \Phi''(b) \cdot \frac{(b-x_0)^2}{2} - \int_{x_0}^b \int_l^b \int_r^b \Phi'''(e) de\, dr\, dl \\
&\geq 1 + \Phi''(b) \cdot \frac{(b-x_0)^2}{2} + \int_{x_0}^b \int_l^b \int_r^b M_1(e)\Phi(e) de\, dr\, dl \\
&> 1 + \Phi''(b) \cdot \frac{(b-x_0)^2}{2} \\
&\geq 1 - \frac{(b-x_0)^2}{(b-a)^2} \\
&\geq 0,
\end{aligned}$$

which is a contradiction to $\Phi(x_0) = 0$.

From $\Phi(x) > 0$ for $x \in [a, b]$, we have that $\Phi'''(x) \le 0$ for $x \in [a, b]$. Hence, by (2.3), $\Phi'''(x) \le -M_1(x)\Phi(x)$ for $x \in [a, b]$. Therefore,

$$\begin{aligned}\Phi(x) &= 1 + \Phi''(b) \cdot \frac{(b-x)^2}{2} - \int_x^b \int_l^b \int_r^b \Phi'''(e)de\,dr\,dl \\ &\ge 1 + \Phi''(b) \cdot \frac{(b-x)^2}{2} + \int_x^b \int_l^b \int_r^b M_1(e)\Phi(e)de\,dr\,dl \\ &> 1 + \Phi''(b) \cdot \frac{(b-x)^2}{2}.\end{aligned}$$

Now, we use the expression

$$\Phi(x) = 1 + \Phi''(b) \cdot \frac{(b-x)^2}{2} - \int_x^b \int_l^b \int_e^b \Phi'''(e)de\,dr\,dl.$$

From $\Phi(a) - \int_a^b \Phi(x)d\alpha(x) = 0$, we have that

$$\begin{aligned}&1 + \Phi''(b) \cdot \frac{(b-a)^2}{2} - \int_a^b \int_l^b \int_r^b \Phi'''(e)de\,dr\,dl \\ &= \int_a^b \left(1 + \Phi''(b) \cdot \frac{(b-x)^2}{2} - \int_x^b \int_l^b \int_r^b \Phi'''(e)de\,dr\,dl\right) d\alpha(x),\end{aligned}$$

that is,

$$\begin{aligned}&\Phi''(b) \cdot \left[\frac{(b-a)^2}{2} - \int_a^b \frac{(b-x)^2}{2} d\alpha(x)\right] \\ &= \int_a^b \left(\int_a^b \int_l^b \int_r^b \Phi'''(e)de\,dr\,dl - \int_x^b \int_l^b \int_r^b \Phi'''(e)de\,dr\,dl\right) d\alpha(x) \\ &= \int_a^b \left(\int_a^x \int_l^b \int_r^b \Phi'''(e)de\,dr\,dl\right) d\alpha(x).\end{aligned}$$

By $\Phi'''(x) \le -M_1(x)\Phi(x)$ for $x \in [a, b]$, $\Phi(x) > 1 + \Phi''(b) \cdot \frac{(b-x)^2}{2}$, and Lemma 2.1, we have that

$$\begin{aligned}&\Phi''(b) \cdot \left[\frac{(b-a)^2}{2} - \int_a^b \frac{(b-x)^2}{2} d\alpha(x)\right] \\ &= \int_a^b \left(\int_a^x \int_l^b \int_r^b \Phi'''(e)de\,dr\,dl\right) d\alpha(x) \\ &\le -\int_a^b \left(\int_a^x \int_l^b \int_r^b M_1(e)\Phi(e)de\,dr\,dl\right) d\alpha(x) \\ &< -\int_a^b \left(\int_a^x \int_l^b \int_r^b M_1(e) \left(1 + \Phi''(b) \cdot \frac{(b-e)^2}{2}\right) de\,dr\,dl\right) d\alpha(x) \\ &= -\int_a^b \left(\int_a^x \int_l^b \int_r^b M_1(e)de\,dr\,dl\right) d\alpha(x) \\ &\quad - \int_a^b \left(\int_a^x \int_l^b \int_r^b M_1(e)\Phi''(b) \cdot \frac{(b-e)^2}{2} de\,dr\,dl\right) d\alpha(x),\end{aligned}$$

which gives that

$$-\Phi''(b) > \frac{\int_a^b \left(\int_a^x \int_l^b \int_r^b M_1(e)de\,dr\,dl\right) d\alpha(x)}{\frac{(b-a)^2}{2} - \int_a^b \frac{(b-x)^2}{2} d\alpha(x) + \int_a^b \left(\int_a^x \int_l^b \int_r^b M_1(e)\frac{(b-e)^2}{2} de\,dr\,dl\right) d\alpha(x)}.$$

By (2.4), we have

$$-\Phi''(b) > \frac{2}{(b-a)^2},$$

which is a contradiction to the assumption $-\Phi''(b) \leq \frac{2}{(b-a)^2}$. Therefore, $\Phi''(b) < -\frac{2}{(b-a)^2}$. □

The next two lemmas are the corresponding versions of Lemmas 2.7 and 2.8 on $[b, c]$. Proofs are very similar and omitted here.

Lemma 2.9. *Suppose f satisfies the condition (A) and there is some $M_2 > 0$, such that*

$$f(x, v) - f(x, u) \leq M_2(v - u), \quad \forall\, x \in (b, c), \quad \forall\, v \geq u \in \mathbb{R}. \tag{2.6}$$

Assume for each $m \in \mathbb{R}$, there exists a solution $\beta(x, m)$ of (1.1) *satisfying* (1.5)*. Let $m_1 < m_2 \in \mathbb{R}$. Then,*

$$\beta''(b, m_2) - \beta''(b, m_1) > -M_2(c - b)(m_2 - m_1). \tag{2.7}$$

Lemma 2.10. *Suppose f satisfies the condition (A) and there is a continuous function $M_2(x)$ for $x \in [b, c]$, such that*

$$f(x, v) - f(x, u) \geq M_2(x)(v - u), \quad \forall\, x \in (b, c), \quad \forall\, v \geq u \in \mathbb{R}, \tag{2.8}$$

where $M_2(x) > 0$, for $x \in (b, c]$, and

$$\frac{\int_b^c \left(\int_x^c \int_b^l \int_b^r M_2(e)de\,dr\,dl\right) d\beta(x)}{\frac{(c-b)^2}{2} - \int_b^c \frac{(x-b)^2}{2} d\beta(x) + \int_b^c \left(\int_x^c \int_b^l \int_b^r M_2(e)\frac{(e-b)^2}{2} de\,dr\,dl\right) d\beta(x)} \geq \frac{2}{(c-b)^2}. \tag{2.9}$$

Assume for each $m \in \mathbb{R}$, there exists a solution $\beta(x, m)$ of (1.1) *satisfying* (1.5)*. Let $m_1 < m_2 \in \mathbb{R}$. Then,*

$$\beta''(b, m_2) - \beta''(b, m_1) < -\frac{2(m_2 - m_1)}{(c - b)^2}. \tag{2.10}$$

The next lemma is about the existence and uniqueness of an intersection point of two continuous and strictly decreasing functions with ranges all of $\mathbb{R}$. The proof is based on calculus and omitted here.

Lemma 2.11. *Assume $\mu(x), \omega(x) \in C(\mathbb{R})$ and both are strictly decreasing functions and range all of $\mathbb{R}$. Suppose there exist $\sigma_1 < \sigma_2 < 0$ such that*

$$\mu(x_2) - \mu(x_1) \leq \sigma_1(x_2 - x_1), \quad \omega(x_2) - \omega(x_1) \geq \sigma_2(x_2 - x_1), \quad \forall\, x_1 < x_2.$$

Then, there exists a unique $x_0 \in \mathbb{R}$ such that $\mu(x_0) = \omega(x_0)$.

3. MAIN RESULTS

Now, we are in the position to show our main results.

Theorem 3.1. *Suppose that f satisfies condition (A) and that for each $m \in \mathbb{R}$, there exist solutions $\alpha(x,m)$, $u(x,m)$, $\beta(x,m)$, $v(x,m)$ of* (1.1) *satisfying each of the conditions* (1.3), (1.4), (1.5), (1.6), *respectively. Suppose f satisfies one of the following:*

(H1): there is some $M_1 > 0$ and a continuous function $M_2(x)$ for $x \in [b,c]$ with $M_2(x) > 0$ for $x \in (b,c]$, such that

$$0 > f(x,v) - f(x,u) \geq -M_1(v-u), \quad \forall\, x \in (a,b), \quad \forall\, v > u \in \mathbb{R},$$
$$f(x,v) - f(x,u) \geq M_2(x)(v-u), \quad \forall\, x \in (b,c), \quad \forall\, v > u \in \mathbb{R},$$

where

$$M_1(b-a) < \frac{2}{(c-b)^2},$$

and

$$\frac{\int_b^c \left(\int_x^c \int_b^l \int_b^r M_2(e)de\,dr\,dl\right) d\beta(x)}{\frac{(c-b)^2}{2} - \int_b^c \frac{(x-b)^2}{2} d\beta(x) + \int_b^c \left(\int_x^c \int_b^l \int_b^r M_2(e)\frac{(e-b)^2}{2} de\,dr\,dl\right) d\beta(x)} \geq \frac{2}{(c-b)^2};$$

or

(H2): there is some $M_2 > 0$ and a continuous function $M_1(x)$ for $x \in [a,b]$ with $M_1(x) > 0$ for $x \in [a,b)$, such that

$$f(x,v) - f(x,u) \leq -M_1(x)(v-u), \quad \forall\, x \in (a,b), \quad \forall\, v > u \in \mathbb{R},$$
$$0 < f(x,v) - f(x,u) \leq M_2(v-u), \quad \forall\, x \in (b,c), \quad \forall\, v > u \in \mathbb{R},$$

where

$$\frac{\int_a^b \left(\int_a^x \int_l^b \int_r^b M_1(e)de\,dr\,dl\right) d\alpha(x)}{\frac{(b-a)^2}{2} - \int_a^b \frac{(b-x)^2}{2} d\alpha(x) + \int_a^b \left(\int_a^x \int_l^b \int_r^b M_1(e)\frac{(b-e)^2}{2} de\,dr\,dl\right) d\alpha(x)} \geq \frac{2}{(b-a)^2}.$$

and

$$\frac{2}{(b-a)^2} > M_2(c-b).$$

Then the BVP (1.1), (1.2) *has a unique solution.*

Proof. We show the proof for the case that f satisfies (H1). The proof for the other case is very similar and omitted.

First, we prove the existence of solutions of the BVP (1.1), (1.3). Since for any $m \in \mathbb{R}$, there exist solutions $\alpha(x,m)$, $u(x,m)$, $\beta(x,m)$, $v(x,m)$ of (1.1) satisfying each of the conditions (1.3), (1.4), (1.5), (1.6), we consider $\alpha''(b,m)$, $u(b,m)$, $\beta''(b,m)$, $v(b,m)$ as functions of m. By Lemma 2.6, they are all strictly decreasing continuous functions.

For any $m_1 < m_2 \in \mathbb{R}$, from Lemma 2.7, we have $\alpha''(b, m_2) - \alpha''(b, m_1) > -M_1(b-a)(m_2 - m_1)$; and from Lemma 2.10, we have $\beta''(b, m_2) - \beta''(b, m_1) < -\frac{2(m_2-m_1)}{(c-b)^2}$. Notice, $-M_1(b-a) > -\frac{2}{(c-b)^2}$. By Lemma 2.11, there is a unique $m_0 \in \mathbb{R}$ such that $\alpha''(b, m_0) = \beta''(b, m_0)$. Then the piecewise defined function

$$y(x) = \begin{cases} \alpha(x, m_0), & x \in [a, b], \\ \beta(x, m_0), & x \in [b, c], \end{cases}$$

is a solution of (1.1), (1.2).

Second, we prove the uniqueness. Suppose there are two solutions $y_1(x)$ and $y_2(x)$ of (1.1), (1.2). Then, we have some $m_1 = y_1(b)$ and $m_2 = y_2(b)$ such that $\alpha(x, m_1) = y_1(x)$ for $x \in [a, b]$, $\beta(x, m_1) = y_1(x)$ for $x \in [b, c]$, $\alpha(x, m_2) = y_2(x)$ for $x \in [a, b]$, and $\beta(x, m_2) = y_2(x)$ for $x \in [b, c]$. By Lemma 2.5, $m_1 \neq m_2$. Without loss of generality, we suppose $m_2 > m_1$. Then by (H1) and Lemmas 2.7 and 2.10, we have that $\beta''(b, m_2) - \beta''(b, m_1) < -\frac{2(m_2-m_1)}{(c-b)^2}$, and $\alpha''(b, m_2) - \alpha''(b, m_1) > -M_1(b-a)(m_2 - m_1)$, that is,

$$\begin{aligned} -M_1(b-a)(m_2 - m_1) &< \alpha''(b, m_2) - \alpha''(b, m_1) \\ &= \beta''(b, m_2) - \beta''(b, m_1) < -\frac{2(m_2 - m_1)}{(c-b)^2}, \end{aligned}$$

which is a contradiction to $-M_1(b-a) > -\frac{2}{(c-b)^2}$.

Therefore, $y(x)$ defined as above is the unique solution of (1.1), (1.2). □

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