Dynamic Systems and Applications 23 (2014) 479-492

GENERALIZED MONOTONE ITERATIVE METHOD FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH ANTI-PERIODIC BOUNDARY CONDITIONS

J. D. RAMÍREZ

Department of Mathematics, Lamar University, Beaumont, TX 77710, USA

Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. The purpose of this work is to develop a Monotone Method for the anti-periodic boundary value problem with $0 < q < 1$ on $J = [0, T]$,

$$^cD^q u(t) = f(t, u(t)) + g(t, u(t)),$$
$$u(0) = -u(T),$$

where $f(t, u)$ is increasing in u and $g(t, u)$ is decreasing in u.

We will define coupled lower and upper solutions $v_0(t)$ and $w_0(t)$. Next we will construct two sequences $\{v_n(t)\}, \{w_n(t)\}$ which converge uniformly and monotonically to coupled minimal and maximal solutions ρ and r, respectively; i.e. ρ and r satisfy the system

$$^cD^q \rho(t) = f(t, \rho(t)) + g(t, r(t)), \qquad \rho(0) = -r(T),$$
$$^cD^q r(t) = f(t, r(t)) + g(t, \rho(t)), \qquad r(0) = -\rho(T).$$

Our iterates are solutions of initial value problems.

AMS (MOS) Subject Classification. 26A33, 34A08, 34B15

1. INTRODUCTION

The study of fractional differential equations has become a popular subject in recent years because they frequently represent more appropriate models than their counterpart with integer derivatives, see [3, 4, 6, 12, 13] for more information. A useful technique for solving ordinary differential equations is the study on the existence of solutions by using upper and lower solutions, which is well established in [5]. These methods have now been applied to fractional differential equations, see the book [6] and the papers [1, 2, 7, 8, 9, 10, 11, 14, 15, 16] for recent work.

In this paper we recall a comparison theorem from [6] for a Caputo fractional differential equation of order q, $0 < q < 1$, with initial condition. We will define and use coupled lower and upper solutions combined with a generalized monotone method

Received August 2013

of initial value problems to prove the existence of coupled minimal and maximal anti-periodic solutions. This monotone method was first introduced in [17] for ordinary differential equations with anti-periodic boundary conditions. The monotone iterative techniques presented in [5] for boundary value problems, where the iterates are solutions of linear equations with boundary conditions, do not apply for anti-periodic boundary conditions. Another advantage of our method is that it does not require the Mittag-Leffler function in our computations. The result developed provides natural sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of the anti-periodic boundary value problem.

2. PRELIMINARIES

In this section we state the definitions and results concerning the Caputo derivative of fractional order that are required to prove our main result.

Consider the initial value problem of the form

$$\begin{aligned} {}^cD^q u(t) &= f(t,u(t)), \\ u(0) &= u_0. \end{aligned} \tag{2.1}$$

Here, ${}^cD^q u(t)$ is the Caputo derivative of order $n-1 < q \le n$ for $t \in [a,b]$, where n is a positive integer, which is defined in [3, 4, 6, 13] as

$${}^cD^q u(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} u^{(n)}(s) ds.$$

Throughout this work we will consider the Caputo derivative of order q, where $0 < q < 1$.

We recall the following definitions.

Definition 2.1. Let $0 < q < 1$ and $p = 1 - q$. If G is an open set in $\mathbb{R}$, then we denote by $C_p([a,b],G)$ the function space

$$C_p([a,b],G) = \left\{ u \in C((a,b],G) \,\middle|\, (t-a)^p u(t) \in C([a,b],G) \right\}.$$

Remark 2.2. In [4] it is shown that if $0 < q < 1$, G is an open set of $\mathbb{R}$, and $f : (a,b] \times G \to \mathbb{R}$ is such that for any $u \in G$, $f \in C_p$, then u satisfies (2.1) if and only if it satisfies the Volterra fractional integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s,u(s)) ds. \tag{2.2}$$

In particular, this relationship is true if $f : [a,b] \times G \to \mathbb{R}$ is continuous.

Definition 2.3. The two parameter **Mittag-Leffler function** is defined as

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)},$$

and the one parameter Mittag-Leffler function is defined as

$$E_\alpha(t) = E_{\alpha,1}(t).$$

Remark 2.4. $E_1(t) = e^t$.

In [6], it was shown that the solution to (2.1) for $f(t, u(t)) = Mu(t) + f(t)$ where M is a real number and $f \in C([0,T], \mathbb{R})$, i.e. a non homogeneous linear fractional differential equation, is given by

$$u(t) = u_0 E_q(Mt^q) + \int_0^t (t-s)^{q-1} E_{q,q}(M(t-s)^q) f(s) ds \quad t \in [0,T], \tag{2.3}$$

where $E_q(t)$ and $E_{q,q}(t)$ are the one parameter and two parameter Mittag-Leffler functions, respectively.

Consider now the non homogeneous linear problem with anti-periodic boundary conditions,

$$\begin{aligned} {}^cD^q u(t) &= Mu(t) + f(t), \\ u(0) &= -u(T). \end{aligned} \tag{2.4}$$

We begin by stating the solution of (2.1) given by (2.3).

Setting $t = T$, and $u(0) = -u(T) = u_0$, we get

$$u(T) = u_0 E_q(MT^q) + \int_0^T (T-s)^{q-1} E_{q,q}(M(T-s)^q) f(s) ds.$$

Hence, $u(0) = -u(T) = u_0$ implies that

$$u_0 = -u_0 E_q(MT^q) - \int_0^T (T-s)^{q-1} E_{q,q}(M(T-s)^q) f(s) ds,$$

and consequently,

$$u_0 (1 + E_q(MT^q)) = - \int_0^T (T-s)^{q-1} E_{q,q}(M(T-s)^q) f(s) ds.$$

Thus,

$$u_0 = -\frac{1}{1 + E_q(MT^q)} \int_0^T (T-s)^{q-1} E_{q,q}(M(T-s)^q) f(s) ds,$$

and the solution to the linear boundary value problem (2.4) is given by

$$\begin{aligned} u(t) &= -\frac{E_q(MT^q)}{1 + E_q(MT^q)} \int_0^T (T-s)^{q-1} E_{q,q}(M(T-s)^q) f(s) ds \\ &\quad + \int_0^t (t-s)^{q-1} E_{q,q}(M(t-s)^q) f(s) ds \quad t \in [0,T]. \end{aligned} \tag{2.5}$$

We show numerical results for two linear anti-periodic boundary value problems of the form (2.4), with $q = \frac{1}{2}$ and using the representation found in (2.5).

We approximate the solution by using

$$E_{\frac{1}{2}}(t) = e^{t^2} erfc(-t),$$

where

$$erfc(t) = \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-s^2} ds,$$

Using this expression for $E_{\frac{1}{2}}(t)$, we can derive

$$E_{\frac{1}{2},\frac{1}{2}}(t) = \frac{1}{\sqrt{\pi}} + te^{t^2} erfc(-t).$$

Example 2.5. If $q = \frac{1}{2}$, $M = \frac{1}{2}$ and $f(t) = t^2$, we obtain on the interval $[0, 2\pi]$ the following graph.

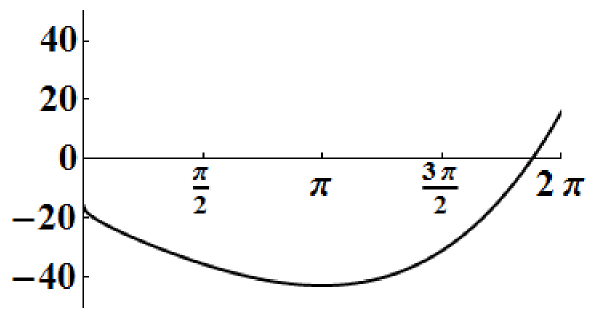

FIGURE 1. $u(0) = -u(2\pi) = -15.9419$

Example 2.6. Solution for $q = \frac{1}{2}$, $M = \frac{1}{2}$ and $h(t) = e^t$ on $[0, 2\pi]$.

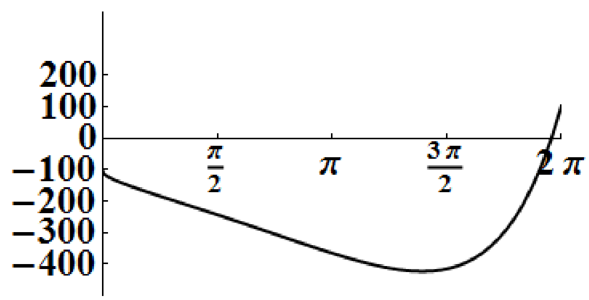

FIGURE 2. $u(0) = -u(2\pi) = -103.818$

Now we are ready to state some comparison results relative to initial value problems with the Caputo derivative.

Lemma 2.7. *Let $m(t) \in C^1([0, T], \mathbb{R})$. If there exists $t_1 \in [0, T]$ such that $m(t_1) = 0$ and $m(t) \leq 0$ on $[0, t_1]$, then it follows that*

$$^cD^q m(t_1) \geq 0.$$

Proof. Let $t_1 \in [0, T]$, then using the relation between the Caputo derivative and Riemann-Liouville derivative given by

$$^cD^q u(t) = D^q \left[u(s) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!} (s-a)^k \right] (t),$$

where D^q denotes the Riemann-Liouville derivative, we have that

$$^cD^q m(t_1) = D^q m(t_1) - \frac{m(0)}{\Gamma(1-q)} t^{-q} \geq D^q m(t_1).$$

Since this lemma was proven in [2] for the Riemann-Liouville derivative, we have that $D^q m(t_1) \geq 0$ implies $^cD^q m(t_1) \geq 0$, and the proof is complete. □

Remark 2.8. In [6] they proved the above result by assuming that $m(t)$ is Hölder continuous of order $\lambda > q$. Although the proof is correct, it is not useful in the monotone method or any iterative method because we will not be able to prove that each of those iterates are Hölder continuous of order $\lambda > q$.

The above result will allow us to prove the following comparison theorem.

Theorem 2.9. *Let $J = [0, T]$, $f \in C[J \times \mathbb{R}, \mathbb{R}]$, $v, w \in C^1[J, \mathbb{R}]$, and for $t \in J$ the following inequalities hold true,*

$$\begin{aligned} {}^cD^q v(t) &\leq f(t, v(t)), \quad v(0) \leq u_0, \\ {}^cD^q w(t) &\geq f(t, w(t)), \quad w(0) \geq u_0. \end{aligned} \tag{2.6}$$

Suppose further that $f(t, u)$ satisfies the following Lipschitz condition,

$$f(t, x) - f(t, y) \leq L(x - y), \text{ for } x \geq y \text{ and } L > 0, \tag{2.7}$$

then $v(0) \leq w(0)$ implies that

$$v(t) \leq w(t), \quad \text{for } 0 \leq t \leq T.$$

Proof. Assume first without loss of generality that one of the inequalities in (2.6) is strict, say $^cD^q v(t) < f(t, v(t))$, and $v_0 < w_0$ where $v(0) = v_0$ and $w(0) = w_0$. We will show that $v(t) < w(t)$ for $t \in J$.

Suppose, to the contrary, that there exists t_1 such that $0 < t_1 \leq T$ for which

$$v(t_1) = w(t_1), \quad \text{and} \quad v(t) \leq w(t), \quad , \text{for } t < t_1.$$

Setting $m(t) = v(t) - w(t)$ it follows that $m(t_1) = 0$ and $m(t) \leq 0$ for $t < t_1$. Then by Lemma 2.7 we have that $^cD^q m(t_1) \geq 0$. Thus

$$f(t_1, v(t_1)) > {}^cDv(t_1) \geq {}^cDw(t_1) \geq f(t_1, w(t_1)),$$

which is a contradiction to the assumption $v(t_1) = w(t_1)$. Therefore $v(t) < w(t)$.

Now assume that the inequalities (2.6) are non strict. We will show that $v(t) \leq w(t)$.

Set $w_\epsilon(t) = w(t) + \epsilon\lambda(t)$, where $\epsilon > 0$ and $\lambda(t) = E_q[2Lt^q]$, where E_q is the one parameter Mittag-Leffler function. This implies that $w_\epsilon(0) = w_0 + \epsilon > w_0$ and $w_\epsilon(t) > w(t)$.

Using (2.6) and the Lipschitz condition (2.7), we find that

$$\begin{aligned} {}^cD^q w_\epsilon(t) &= {}^cD^q w(t) + \epsilon {}^cD^q \lambda(t) \\ &\geq f\left(t, w(t)\right) + 2\epsilon L\lambda(t) \\ &\geq f\left(t, w_\epsilon(t)\right) - \epsilon L\lambda(t) + 2\epsilon L\lambda(t) \\ &= f(t, w_\epsilon(t)) + \epsilon L\lambda(t) \\ &> f\left(t, w_\epsilon(t)\right), \; 0 < t \leq T. \end{aligned}$$

Here we have utilized the fact that $\lambda(t)$ is the solution of the Initial Value Problem

$${}^cD^q\lambda(t) = 2L\lambda(t), \quad \lambda(0) = 1 > 0.$$

Clearly there is no assumption on the growth of $L > 0$. Applying now the result for strict inequalities to $v(t), w_\epsilon(t)$, we get that $v(t) < w_\epsilon(t)$ for $t \in J$, for every $\epsilon > 0$ and consequently making $\epsilon \to 0$, we get that $v(t) \leq w(t)$ for $t \in J$. □

The following corollary will be useful in our main results.

Corollary 2.10. *Let $m \in C^1[J, \mathbb{R}]$ be such that*

$$\begin{aligned} {}^cD^q m(t) &\leq Lm(t), \\ m(0) &= m_0. \end{aligned}$$

Then we have from the previous theorem the estimate

$$m(t) \leq m_0 E_q(Lt^q), \qquad \text{for } 0 \leq t \leq T \text{ and } L > 0.$$

The result of the above corollary is still true even if $L = 0$, which we state separately and prove it.

Corollary 2.11. *Let ${}^cD^q m(t) \leq 0$ on $[0, T]$. Then $m(t) \leq 0$, if $m(0) \leq 0$.*

Proof. By definition of ${}^cD^q m(t)$ and by hypothesis,

$${}^cD^q m(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} m'(s) ds \leq 0,$$

which implies that $m'(t) \leq 0$, and consequently $m(t)$ is decreasing, on $[0, T]$. Therefore, $m(t) \leq m(0) \leq 0$ on $[0, T]$. □

Note that the above result may not be true for the Riemann-Liouville derivative.

We recall a comparison result similar to the one given in [6] for periodic boundary conditions. As in Theorem 2.9, the proof does not require Hölder's continuity.

Theorem 2.12. *Let* $J = [0, T]$, $F \in C[J \times \mathbb{R}, \mathbb{R}]$, $v, w \in C^1[J, \mathbb{R}]$, *and for* $0 < t \leq T$,

$$^cD^q v(t) \leq F(t, v(t)), \quad v(0) \leq -w(T), \quad \text{and}$$
$$^cD^q w(t) \geq F(t, w(t)), \quad w(0) \geq -v(T).$$

Suppose further that $F(t, u(t))$ *is decreasing in* u, *then* $v(0) \leq w(0)$ *implies that*

$$v(t) \leq w(t), \quad \text{for } 0 \leq t \leq T.$$

Proof. Suppose, to the contrary, that there exists $t_0 \in J$ such that

$$v(t_0) = w(t_0) + \varepsilon \quad \text{and} \quad v(t) \leq w(t) + \varepsilon,$$

for $0 \leq t \leq t_0 \leq T$, and let $m(t) = v(t) - w(t) - \varepsilon$.

If $t_0 \in (0, T]$, then $m(t_0) = 0$ and, consequently, $m(t) \leq 0$ for $0 \leq t \leq t_0$. By Lemma 2.7 we obtain that $^cD^q m(t_0) \geq 0$ and $^cD^q v(t_0) \geq {}^cD^q w(t_0)$. By hypothesis we get that

$$F(t_0, v(t_0)) \geq {}^cD^q v(t_0) \geq {}^cD^q w(t_0) \geq F(t_0, w(t_0))$$

But this is a contradiction to the hypothesis that F is decreasing in u because we had assumed that $v(t_0) > w(t_0)$.

Now assume that $t_0 = 0$, then we have that

$$-w(T) \geq v(0) = w(0) + \varepsilon \geq -v(T) + \varepsilon.$$

Thus, $v(T) \geq w(T) + \varepsilon$ and $v(T) > w(T)$. Proceeding as in the previous part of the proof, we get a contradiction and the proof is complete. □

Two important cases of this theorem are the following, which are useful to develop a monotone method for the Caputo fractional differential equation with anti-periodic boundary conditions by using solutions of linear anti-periodic boundary value problems.

Corollary 2.13. *Let* $m \in C^1[J, \mathbb{R}]$ *be such that*

$$^cD^q m(t) \leq -Mm(t),$$
$$m(0) \leq m(T),$$

for $0 \leq t \leq T$ *and* $M > 0$. *Then* $m(t) \leq 0$ *for* $0 \leq t \leq T$.

Similarly, if

$$^cD^q m(t) \geq -Mm(t),$$
$$m(0) \geq m(T),$$

for $0 \leq t \leq T$ *and* $M > 0$. *Then* $m(t) \geq 0$ *for* $0 \leq t \leq T$.

Remark 2.14. It is to be noted that in the proof of these equivalent results from [6], we use Lemma 2.7 which does not require the Hölder's continuity assumption.

3. MAIN RESULTS

In this section we will develop a generalized monotone method for the nonlinear anti-periodic boundary value problem (3.1), given below, by using coupled upper and lower solutions and the corresponding initial value problem (2.1) where f does not depend on u, and $u(t)$ can be represented uniquely by

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds.$$

For that purpose consider the nonlinear anti-periodic boundary value problem of the form

$$\begin{aligned} {}^cD^q u(t) &= f(t, u(t)) + g(t, u(t)), \\ u(0) &= -u(T), \end{aligned} \tag{3.1}$$

where $f, g \in C[J \times \mathbb{R}, \mathbb{R}]$ and $u \in C^1[J \times \mathbb{R}]$.

If $u \in C^1[0, T]$ satisfies the fractional differential equation

$${}^cD^q u(t) = f(t, u(t)) + g(t, u(t)),$$

and u is such that $u(0) = -u(T)$ for $t \in J$, then u is an **anti-periodic solution** of (3.1).

Throughout the rest of this paper, we will assume that f is increasing in u and g is decreasing in u for $t \in J$.

Here below we provide the definition of coupled lower and upper solutions of (3.1).

Definition 3.1. Let $v_0, w_0 \in C^1[J, \mathbb{R}]$. Then v_0 and w_0 are said to be **coupled lower and upper solutions** for (3.1), respectively, if

$$\begin{aligned} {}^cD^q v_0(t) &\leq f(t, v_0(t)) + g(t, w_0(t)),\ v_0(0) \leq -w_0(T), \\ {}^cD^q w_0(t) &\geq f(t, w_0(t)) + g(t, v_0(t)),\ w_0(0) \geq -v_0(T). \end{aligned} \tag{3.2}$$

We will develop the generalized monotone method for the anti-periodic boundary value problem via the initial value problem approach; that is, a method introduced first in [17] where the iterates are solutions to initial value problems. Observe that, since the iterates are not solutions of linear boundary value problems of the form (2.5), there is no need to compute the Mittag-Leffler function on each iterate. We obtain natural sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of (3.1).

We will state the following theorem related to coupled lower and upper solutions of the form (3.2).

Theorem 3.2. *Assume that*

(A1) v_0, w_0 *are coupled lower and upper solutions for* (3.1) *with* $v_0(t) \leq w_0(t)$ *in* J*; and*

(A2) $f, g \in C[J \times \mathbb{R}, \mathbb{R}]$*, where* $f(t, u(t))$ *is increasing in* u *and* $g(t, u(t))$ *is decreasing in* u*.*

If u *is a solution of* (3.1) *such that* $v_0(t) \leq u(t) \leq w_0(t)$ *in* J*, then the sequences defined by*

$$\begin{aligned} {}^cD^q v_{n+1}(t) &= f\big(t, v_n(t)\big) + g\big(t, w_n(t)\big), \\ v_{n+1}(0) &= -w_n(T), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} {}^cD^q w_{n+1}(t) &= f\big(t, w_n(t)\big) + g\big(t, v_n(t)\big), \\ w_{n+1}(0) &= -v_n(T). \end{aligned} \tag{3.4}$$

are such that

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq v_{n+1} \leq u \leq w_{n+1} \leq w_n \leq \cdots \leq w_1 \leq w_0,$$

where $v_n(t) \to \rho(t)$ *and* $w_n(t) \to r(t)$ *uniformly and monotonically in* $C^1[J, \mathbb{R}]$*, and* ρ, r *are coupled minimal and maximal solutions of (3.1), respectively; i.e.,* ρ *and* r *satisfy the coupled system*

$$\begin{aligned} {}^cD^q \rho(t) &= f(t, \rho(t)) + g(t, r(t)), \\ \rho(0) &= -r(T) \text{ on } J, \end{aligned}$$

and

$$\begin{aligned} {}^cD^q r(t) &= f(t, r(t)) + g(t, \rho(t)), \\ r(0) &= -\rho(T) \text{ on } J, \end{aligned}$$

with $\rho \leq u \leq r$*.*

Proof. By hypothesis, $v_0 \leq u \leq w_0$. We will show that $v_0 \leq v_1 \leq u \leq w_1 \leq w_0$.

It follows from (3.2) that

$$\begin{aligned} {}^cD^q v_0(t) &\leq f(t, v_0(t)) + g(t, w_0(t)),\ v_0(0) \leq -w_0(T), \\ {}^cD^q w_0(t) &\geq f(t, w_0(t)) + g(t, v_0(t)),\ w_0(0) \geq -v_0(T), \end{aligned}$$

and by (3.3), we get that

$$\begin{aligned} {}^cD^q v_1 &= f(t, v_0) + g(t, w_0), \\ v_1(0) &= -w_0(T). \end{aligned}$$

Therefore, $v_0(0) \leq -w_0(T) = v_1(0)$. If we let $p = v_0 - v_1$, then $p(0) \leq 0$ and,

$${}^cD^q p = {}^cD^q v_0 - {}^cD^q v_1$$

$$\leq f(t, v_0) + g(t, w_0) - f(t, v_0) - g(t, w_0)$$
$$= 0.$$

Since ${}^cD^q p \leq 0$ and $p(0) \leq 0$, by an application of Corollary 2.11 we have that $p(t) \leq 0$ and, consequently, $v_0(t) \leq v_1(t)$ on J. By a similar argument we can show that $v_1(t) \leq u$, $u \leq w_1(t)$ and $w_1(t) \leq w_0(t)$. Thus, $v_0 \leq v_1 \leq u \leq w_1 \leq w_0$.

Now we will show that $v_k \leq v_{k+1}$ for $k \geq 1$.

Assume that

$$v_{k-1} \leq v_k \leq u \leq w_k \leq w_{k-1},$$

for $k > 1$.

Let $p = v_k - v_{k+1}$. Then

$$v_k(0) = -w_{k-1}(T) \leq -w_k(T) = v_{k+1}(0),$$

so $p(0) \leq 0$. By the increasing nature of f and the decreasing nature of g it follows that

$$\begin{aligned} {}^cD^q p &= {}^cD^q v_k - {}^cD^q v_{k+1} \\ &= f(t, v_{k-1}) + g(t, w_{k-1}) - f(t, v_k) - g(t, w_k) \\ &\leq 0. \end{aligned}$$

Similarly, by Corollary 2.11 we have that $p(t) \leq 0$ and consequently $v_k(t) \leq v_{k+1}(t)$.

By a similar argument we can show that $w_{k+1} \leq w_k$. Using the hypothesis that $v_0(t) \leq u(t) \leq w_0(t)$ on J, the above argument and induction we can show that $v_{k+1} \leq u \leq w_{k+1}$. Therefore for $n > 0$,

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq u \leq w_n \leq \cdots \leq w_2 \leq w_1 \leq w_0.$$

Now we have to show that the sequences converge uniformly. We will use the Arzela-Ascoli Theorem by showing that the sequences are uniformly bounded and equicontinuous.

First we show uniform boundedness. By hypothesis both $v_0(t)$ and $w_0(t)$ are bounded on $[0, T]$, then there exists $M > 0$ such that for any $t \in [0, T]$, $|v_0(t)| \leq M$ and $|w_0(t)| \leq M$. Since $v_0(t) \leq v_n(t) \leq w_0(t)$ for each $n > 0$, it follows that

$$0 \leq v_n(t) - v_0(t) \leq w_0(t) - v_0(t),$$

and consequently $\{v_n(t)\}$ is uniformly bounded. By a similar argument $\{w_n(t)\}$ is also uniformly bounded.

To prove that $\{v_n(t)\}$ is equicontinuous, let $0 \le t_1 \le t_2 \le T$. Then for $n > 0$,

$$\begin{aligned}
&|v_n(t_1) - v_n(t_2)| = \\
&= \Bigg| -w_{n-1}(T) + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} \left[f\left(s, v_{n-1}(s)\right) + g\left(s, w_{n-1}(s)\right) \right] ds \\
&\quad + w_{n-1}(T) - \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} \left[f\left(s, v_{n-1}(s)\right) + g\left(s, w_{n-1}(s)\right) \right] ds \Bigg| \\
&= \Bigg| \frac{1}{\Gamma(q)} \int_0^{t_1} \left[(t_1 - s)^{q-1} - (t_2 - s)^{q-1} \right] \left[f\left(s, v_{n-1}(s)\right) + g\left(s, w_{n-1}(s)\right) \right] ds \\
&\quad - \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \left[f\left(s, v_{n-1}(s)\right) + g\left(t, w_{n-1}(t)\right) \right] ds \Bigg| \\
&\le \frac{1}{\Gamma(q)} \int_0^{t_1} \Big| \left[(t_1 - s)^{q-1} - (t_2 - s)^{q-1} \right] \left[f\left(s, v_{n-1}(s)\right) + g\left(s, w_{n-1}(s)\right) \right] \Big| ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \Big| \left[f\left(s, v_{n-1}(s)\right) + g\left(t, w_{n-1}(t)\right) \right] \Big| ds.
\end{aligned}$$

Since $\{v_n(t)\}$ and $\{w_n(t)\}$ are uniformly bounded and $f\left(t, u(t)\right)$ and $g\left(t, u(t)\right)$ are continuous on $[0, T]$, there exists $\bar{M}$ independent of n such that

$$\begin{aligned}
&\frac{1}{\Gamma(q)} \int_0^{t_1} \Big| \left[(t_1 - s)^{q-1} - (t_2 - s)^{q-1} \right] \left[f\left(s, v_{n-1}(s)\right) + g\left(s, w_{n-1}(s)\right) \right] \Big| ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \Big| \left[f\left(s, v_{n-1}(s)\right) + g\left(t, w_{n-1}(t)\right) \right] \Big| ds \\
&\le \frac{\bar{M}}{\Gamma(q)} \int_0^{t_1} [(t_1 - s)^{q-1} - (t_2 - s)^{q-1}] ds + \frac{\bar{M}}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds \\
&= -\frac{\bar{M}}{q\Gamma(q)} (t_1 - s)^q \Big|_0^{t_1} + \frac{\bar{M}}{q\Gamma(q)} (t_2 - s)^q \Big|_0^{t_1} - \frac{\bar{M}}{q\Gamma(q)} (t_2 - s)^q \Big|_{t_1}^{t_2} \\
&= \frac{\bar{M}}{\Gamma(q+1)} t_1^q + \frac{\bar{M}}{\Gamma(q+1)} (t_2 - t_1)^q - \frac{\bar{M}}{\Gamma(q+1)} t_2^q + \frac{\bar{M}}{\Gamma(q+1)} (t_2 - t_1)^q \\
&\le \frac{2\bar{M}}{\Gamma(q+1)} (t_2 - t_1)^q = \frac{2\bar{M}}{\Gamma(q+1)} |t_1 - t_2|^q.
\end{aligned}$$

Thus, for any $\varepsilon > 0$ there exists $\delta > 0$ independent of n such that for each n,

$$|v_n(t_1) - v_n(t_2)| < \varepsilon,$$

provided that $|t_1 - t_2| < \delta$.

Similarly we can prove that $\{w_n(t)\}$ is equicontinuous.

We have obtained that $\{v_n(t)\}$ and $\{w_n(t)\}$ are uniformly bounded and equicontinuous on $[0, T]$. Hence by the Arzela-Ascoli Theorem there exist subsequences $\{v_{n_k}(t)\}$ and $\{w_{n_k}(t)\}$ which converge uniformly to $\rho(t)$ and $r(t)$, respectively. Since the sequences are monotone, the entire sequences converge uniformly.

We have shown that the sequences converge in $C[0,T]$. In order to show that they converge in $C^1[0,T]$, observe that since each v_n is constructed as follows

$$\begin{aligned} {}^cD^q v_n &= f(t, v_{n-1}) + g(t, w_{n-1}),\\ v_n(0) &= -w_{n-1}(T), \end{aligned}$$

and we get that

$$v_n(t) = -w_{n-1}(T) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[f(s, v_{n-1}(s)) + g(s, w_{n-1}(s))]ds.$$

Taking limits when $n \to \infty$, we obtain by the Lebesgue Dominated Convergence theorem that

$$\rho(t) = -r(T) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[f(s, \rho(s)) + g(s, r(s))]ds.$$

Hence $v_n(t) \to \rho(t)$ in $C^1[0,T]$. Furthermore, the above expression is equivalent to

$$\begin{aligned} {}^cD^q \rho &= f(t, \rho) + g(t, r) \text{ on } J,\\ \rho(0) &= -r(T). \end{aligned}$$

By a similar argument $w_n(t) \to r(t)$ in $C^1[0,T]$ and it can be shown that

$$\begin{aligned} {}^cD^q r &= f(t, r) + g(t, \rho) \text{ on } J,\\ r(0) &= -\rho(T). \end{aligned}$$

Since $v_n \le u \le w_n$ on $[0,T]$ for all n, we get that $\rho \le u \le r$ on $[0,T]$ which shows that ρ and r are coupled minimal and maximal solutions of (3.1), respectively. This completes the proof. □

4. NUMERICAL RESULTS

In this section we present an example that illustrates the result from Theorem 3.2.

Example 4.1. Consider the following anti-periodic boundary value problem

$$\begin{aligned} {}^cD^q u &= u^4 - u^2 \text{ on } J = [0,T], \text{ for any } T > 0\\ u(0) &= -u(T). \end{aligned} \tag{4.1}$$

The function $h(u) = u^4 - u^2$ is increasing for $u > \frac{1}{\sqrt{2}}$ and decreasing for $u < -\frac{1}{\sqrt{2}}$. Then $v_0 \equiv -1$ and $w_0 \equiv 1$ are coupled lower and upper solutions that satisfy (3.2), in fact

$$\begin{aligned} 0 = {}^cD^q v_0 &\le v_0^2 - v_0^4 = 1 - 1 = 0\\ -1 = v_0(0) &\le -w_0(T) = -1, \end{aligned}$$

and

$$0 = {}^cD^q w_0 \geq v_0^2 - v_0^4 = 1 - 1 = 0$$
$$1 = w_0(0) \geq -v_0(T) = 1,$$

We construct the sequences according to Theorem 3.2 and obtain for all $n > 0$ that

$$v_0 = v_1 = v_2 = \cdots = v_n = \cdots = -1 \text{ and } w_0 = w_1 = w_2 = \cdots = w_n = \cdots = 1.$$

Thus $\rho = -1$ and $r = 1$ are coupled minimal and maximal solutions of (4.1). Observe that $u \equiv 0$ is a solution of (4.1), where $v_0 = \rho < u < r = w_0$.

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