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ASYMPTOTIC BEHAVIOR OF THE EIGENVALUES OF TOEPLITZ INTEGRAL OPERATORS ASSOCIATED WITH THE HANKEL TRANSFORM

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. We prove results concerning the asymptotic behavior of eigenvalues of finite section Toeplitz integral operators associated with the Hankel transform. We also make conjectures about the corresponding problem for the Jacobi transform.

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1. Introduction

Let ν be a positive constant and let

$$J(x) = x^{1/2-\nu} J_{\nu-1/2}(x), \quad 0 \le x < \infty,$$

where

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\alpha+1)} \left(\frac{x}{2}\right)^{\alpha+2k}$$

is the usual Bessel function of order α. For a given real function F belonging to $L^1(0,\infty)$, we define for $u, v > 0$

$$\rho(u,v) = \int_0^\infty F(t) J(ut) J(vt) t^{2\nu} dt.$$

For $A \ge 1$, then

$$(T_A h)(x) = \int_0^A h(y)\rho(x,y) y^{2\nu} dy, \quad 0 < x \le A, \tag{1.1}$$

is the (finite section) Toeplitz integral operator. Using Lemma 6 below, $\rho(u,v)$ exists and $x^\nu y^\nu \rho(x,y)$ is bounded on $(0,\infty)$. It is easy then to see that $T_A : L^2(0,A;x^{2\nu}) \to L^2(0,A;x^{2\nu})$. The Hankel transform does not have a standard definition; we will define it on $L^1(0,\infty;x^{2\nu})$ by

$$\hat{f}(x) = \int_0^\infty f(t) J(xt) t^{2\nu}\, dt$$

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which differs only slightly from other sources, including [7], [12], [15]. In particular, our definition differs from [7] by a multiplicative constant.

We are interested in formulating conditions on $F(t)$ which allow the determination of the asymptotic behavior of the eigenvalues of T_A as $A \to \infty$.

This problem was previously considered by J. R. Davis in [3] and [4]. Although our definition of T_A differs from that of Davis, so does our definition of $\rho(x,y)$ and $J(x)$, and in fact our T_A is the same as his. Using methods in complex analysis, he found in [3] that for $F(t)$ satisfying the following three conditions:

C1: $F(t)$ is a bounded, continuous, real function defined on $[0,\infty)$ such that $t^2F(t)$ is bounded as $t \to \infty$;
C2: $F(0) = M$ and $F(t) < M$ for $t > 0$;
C3: $F(t)$ is twice continuously differentiable in a one sided neighborhood of 0, $F'(0) = 0$, and $F''(0) = -2\sigma^2$ for some constant σ.

then the k^{th} largest eigenvalue $\lambda_{k,A}$ of the associated Toeplitz integral operator satisfies

$$\lim_{A\to\infty} A^2(M - \lambda_{k,A}) = \sigma^2 z_k^2, \tag{1.2}$$

where $0 < z_1 < z_2 < \cdots$ are the positive zeros of the Bessel function $J_{\nu-1/2}(z)$. In his later paper [4], he extended his results for functions satisfying these more general conditions:

C4: $F(t)$ is a bounded, real function in $L^1(0,\infty;t^{2\nu})$;
C5: $F(0) = M$ is the unique maximum and $\limsup_{t\to\infty} F(t) < M$;
C6: $\lim_{t\to 0^+} \frac{F(0)-F(t)}{t^\omega} = \sigma^2$ for some constant $\omega > 0$ and $\sigma > 0$.

In this case he was able to describe an abstractly defined operator, with positive eigenvalues $0 < \mu_1 \le \mu_2 \le \cdots$, so that the k^{th} largest eigenvalue $\lambda_{k,A}$ of the Toeplitz integral operator satisfies:

$$\lim_{A\to\infty} A^\omega(F(0) - \lambda_{k,A}) = \sigma^2 \mu_k. \tag{1.3}$$

Note that C4 and C5 imply $F(0) = M > 0$, while C1 and C2 only imply $M \ge 0$. We suspect that the proof in [3] also requires $M > 0$ and that its omission was an oversight.

In our treatment of these theorems below, we will assume that $\sigma > 0$ so that $\sigma^2 > 0$. We suspect this is assumed in [3], without emphasis. We shall also assume that $M > 0$.

Ideally, the result in [4] should give the result in [3] in the case $\omega = 2$. Although C6 is more general than C3 in this case, the comparison of the integrability condition of C6, which varies with ν, with C4 is curious. For example, for $\nu > 1$, C6 is more restrictive than C3. Also, it is not clear how the result in [4] is related to the zeros of the Bessel function $J_{\nu-1/2}(z)$.

Our first goal has been to understand these results from a different point of view and to obtain more specific information about the abstract operator whose eigenvalues μ_k appear in the result in [4]. Our results apply only to the case that $\omega = 2n$ is an even integer and are then only partially successful, providing upper bounds on a limsup for the cases $n > 1$. In these cases, we will exhibit a specific ordinary differential operator $\tilde{L}_n$ in a specific Hilbert space and provide conditions on $F(t)$ for which

$$\limsup_{A\to\infty} A^{2n}(F(0) - \lambda_{k,A}) \leq \sigma^2 \Lambda_k,$$

where Λ_k are the eigenvalues of $\tilde{L}_n$ arranged in nondecreasing order with repetitions for multiple eigenvalues. Our conjecture is that equality holds in this result, but we have been able to prove this only for the case $n = 1$. The key result is given in the first three lemmas in Section 4 and we have been unable to find appropriate extensions to the case $n > 1$. Thus, in the case $n = 1$, we will recover the result of [3] about the zeros of the Bessel functions, under the following hypotheses. For $F_S(t) = (1 + t^2)^{-1}$,

H1: $F(t)$ is bounded and absolutely integrable on $(0, \infty)$;
H2: $F(0) - F(t) \geq q^2(1 - F_S(t))$ for $t > 0$, for some constant $q > 0$;
H3: $\lim_{t\to 0^+} \frac{F(0)-F(t)}{t^2} = \sigma^2$ for some constant σ.

It is easy to see that C1 implies H1, C3 implies H3, and if $F(0) = M > 0$, then C1 and C2 imply H2. Even for our inequality result in the case $\omega = 2n$, we require only the integrability hypothesis H1; the integrability requirement C4 with respect to the weight function $t^{2\nu}$ remains a mystery for us.

We follow closely the methods used in [14], which proved similar results for Toeplitz operators associated with the Fourier transform. These methods were previously used on a discrete form of the problem in the case of Fourier series by Parter in [10] and [11] and later by Baxley in [1], in a case involving orthogonal polynomials.

Our second goal was born in an effort, suggested by R. A. Askey, to solve a similar problem involving the Jacobi functions. For $\alpha, \beta \geq -1/2$ and for $0 \leq x, t < \infty$, the Jacobi functions are defined by

$$\phi_t(x) = \phi_t^{(\alpha,\beta)}(x) = {}_2F_1\left(\frac{1}{2}(\rho + it), \frac{1}{2}(\rho - it); \alpha + 1; -(\sinh x)^2\right), \tag{1.4}$$

where $\rho = \alpha + \beta + 1$ and ${}_2F_1$ denotes the usual hypergeometric function. We recall some facts from [6]. The Jacobi functions satisfy the differential equation

$$-\Delta(x)^{-1}(\Delta(x)\phi_t'(x))' = (\rho^2 + t^2)\phi_t(x), \tag{1.5}$$

where

$$\Delta(x) = 2^{2\rho}(\sinh x)^{2\alpha+1}(\cosh x)^{2\beta+1}. \tag{1.6}$$

The Jacobi transform is given by

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x)\phi_t(x)\Delta(x)dx, \tag{1.7}$$

for suitable functions f. The inverse transform is then

$$f(x) = \int_0^\infty \hat{f}(t)\phi_t(x)d\nu(t), \tag{1.8}$$

where

$$d\nu(t) = \frac{1}{\sqrt{2\pi}}|c(t)|^{-2}dt \tag{1.9}$$

and

$$c(t) = \frac{2^{\rho-it}\Gamma(it)\Gamma(\alpha+1)}{\Gamma(\frac{1}{2}(\rho+it))\gamma(\frac{1}{2}(\rho+it)-\beta)}. \tag{1.10}$$

For the present purpose, it is convenient to change these formulas slightly. We put

$$\omega_t(x) = (\phi_0(x))^{-1}\phi_t(x). \tag{1.11}$$

Using (1.5), it is easy to verify that

$$-(p(x))^{-1}(p(x)\omega_t'(x))' = t^2\omega_t(x), \tag{1.12}$$

where

$$p(x) = \phi_0^2(x)\Delta(x). \tag{1.13}$$

We put

$$d\mu(x) = \frac{1}{\sqrt{2\pi}}p(x)dx$$

and replace (1.7) by

$$\hat{f}(t) = \int_0^\infty f(x)\omega_t(x)d\mu(x). \tag{1.14}$$

Then the inverse transform takes the new form

$$f(x) = \int_0^\infty \hat{f}(t)\omega_t(x)d\nu(t). \tag{1.15}$$

For an appropriate real function $F(t)$, the finite Toeplitz integral operator of Jacobi type is

$$(T_A h)(x) = \int_0^A h(y)\rho(x,y)d\mu(y), \quad 0 < x \leq A, \tag{1.16}$$

where

$$\rho(x,y) = \int_0^\infty F(t)\omega_t(x)\omega_t(y)d\nu(t).$$

Although a large part of the work below in the Hankel case can be successfully implemented for the Jacobi case, there is a central difficulty which we have not overcome.

We will make some concluding comments about this difficulty in the final section below.

2. **Preliminary Results**

We first establish basic notation. We will be using three particular Hilbert spaces, namely $L^2(0,1;x^{2\nu})$, $L^2(0,A;x^{2\nu})$ for $A>0$, and $L^2(0,\infty;x^{2\nu})$. We will use $(\cdot\,,\cdot)$ and $\|\cdot\|$ to denote the inner product and norm of $L^2(0,1;x^{2\nu})$, $(\cdot\,,\cdot)_A$ and $\|\cdot\|_A$ to denote the inner product and norm of $L^2(0,A;x^{2\nu})$, and $\langle\cdot\,,\cdot\rangle$ to denote the inner product of $L^2(0,\infty;x^{2\nu})$.

As usual, $C^\infty(a,b)$ will denote the set of infinitely differentiable functions on the interval (a,b) and $C_0^\infty(a,b)$ will denote those functions of $C^\infty(a,b)$ having compact support in (a,b).

Elementary properties of Bessel functions will also be needed. In particular we will use the following identities, which can be found in [8] and [13].

Lemma 2.1.

$$\frac{d}{dx}[x^\alpha J_\alpha(x)] = x^\alpha J_{\alpha-1}(x),$$

$$\frac{d}{dx}[x^{-\alpha} J_\alpha(x)] = -x^{-\alpha} J_{\alpha+1}(x),$$

$$\frac{d}{dx}[x^{-\alpha} K_\alpha(x)] = -x^{-\alpha} K_{\alpha+1}(x).$$

We will need information on the asymptotic behavior of Bessel functions. The following lemma comes from [8, pg. 122–123], [13, pg. 199–202].

Lemma 2.2. *$\sqrt{x}J_\alpha(x)$ is bounded as $x\to\infty$, and $\lim_{x\to\infty} e^x\sqrt{x}K_\alpha(x) = \sqrt{\frac{\pi}{2}}$.*

We will use the following two facts, first proved in [7]. The first lemma is a fairly simple application of the Bessel identities from Lemma 2.1.

Lemma 2.3. *Let $(\tau f)(x) = -\frac{1}{x^{2\nu}}(x^{2\nu}f'(x))'$. Then $(\tau J)(xt) = t^2 J(xt)$.*

To avoid confusion, we will sometimes write τ_x rather than τ to indicate that derivatives are with respect to x. The next lemma is the inversion theorem for the Hankel transform from [7].

Lemma 2.4. *If $f,\hat{f}\in L^1(0,\infty;x^{2\nu})$, then f may be redefined on a set of measure zero so that it is continuous on $(0,\infty)$ and then*

$$f(x) = \int_0^\infty \hat{f}(t)J(xt)t^{2\nu}dt.$$

Next is the Parseval Theorem for Hankel transforms. The proof is an application of Fubini's Theorem and the inversion theorem and can be found in [15]. However, some translation is required due to the differences in the definition of the Hankel transform.

Theorem 2.5. *If $f, \hat{g} \in L^1(0,\infty; x^{2\nu})$, then*

$$\int_0^\infty f(x)\overline{g(x)}x^{2\nu}dx = \int_0^\infty \hat{f}(x)\overline{\hat{g}(x)}x^{2\nu}dx.$$

A critical role will be played by the Friedrichs extension of an unbounded symmetric operator. The following statement concerning the Friedrichs extension is found in [5, Section XII.5].

Theorem 2.6. *If an operator T, with a dense domain, is symmetric and semi-bounded, then there exists a particular self-adjoint extension which preserves the lower bound, called the Friedrichs extension and denoted $\tilde{T}$. If $g \in D(\tilde{T})$, there exists a sequence $\{g_n\} \subset D(T)$ with $\|g_n - g\| \to 0$ and $(Tg_n, g_n) \to (\tilde{T}g, g)$ as $n \to \infty$.*

The ordinary differential operator which is central in our result is described in detail in [2]. It is obtained by first defining $L_n u = \tau^n u$ for $u \in D(L_n)$ where

$$D(L_n) = \big\{u(x) \in C^\infty(0,1) : u = 0 \text{ near } x = 1, , (\tau^{n-1}u)' = 0 \text{ near } x = 0,$$
$$\text{and for } n \geq 2,\ x^{2\nu}u', x^{2\nu}(\tau u)', \ldots, x^{2\nu}(\tau^{n-2}u)' \to 0 \text{ as } x \to 0^+\big\},$$

and then obtaining $\tilde{L}_n$ as the Friedrichs extension of L_n in the Hilbert space $L^2(0,1;x^{2\nu})$. Then $\tilde{L}_n$ has a compact inverse, its spectrum consists only of positive eigenvalues of finite multiplicity, and the boundary condition description of this unbounded selfadjoint operator is given by

Theorem 2.7. *Let ${L_n}^*$ be the Hilbert space adjoint of L_n. Then $D(\tilde{L_n}) = \{u \in D({L_n}^*) : u^{(i)}(1) = 0$ for $i = 0, 1, \ldots, n-1\}$*

In particular, the eigenfunctions and eigenvalues of $\tilde{L_1}$ are known:

Theorem 2.8. *The eigenvalues Λ_k of $\tilde{L_1}$ have multiplicity one and $\Lambda_k = z_k^2$, where z_k is the k^{th} positive zero of the Bessel function $J_{\nu-1/2}$. An eigenfunction corresponding to Λ_k is $x^{1/2-\nu}J_{\nu-1/2}(z_k x)$.*

We will also need a result about Rayleigh quotients which guarantees the existence of positive eigenvalues for compact self-adjoint operators and provides a "maximin" characterization of these eigenvalues. Since we could find a precise statement of this result in the literature only for the finite dimensional case concerning symmetric matrices, we also provide a proof. The result easily found in the literature for compact self-adjoint operators is a complementary "minimax" statement.

Theorem 2.9. *Suppose T is a compact self-adjoint operator in a Hilbert space H, and let n be a positive integer. If there exists a subspace M of dimension n for which*

$$\inf_{f\in M}\left\{\frac{(Tf,f)}{(f,f)}\right\}>0,$$

then T has a least n positive eigenvalues, counting multiplicities. Moreover, if $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n>0$ are the largest n eigenvalues of T arranged in nondecreasing order, with repetitions for multiple eigenvalues, then

$$\lambda_n=\sup_M\left[\inf_{f\in M}\left\{\frac{(Tf,f)}{(f,f)}\right\}\right],$$

where M ranges over all n dimensional subspaces of H.

Proof. We prove the contrapositive of the first assertion. If T does not have at least n positive eigenvalues, then by the Riesz theorem, we can decompose H into three mutually orthogonal subspaces H_1, H_2, H_3 with H_1 being the span of the eigenvectors of T corresponding to positive eigenvalues, H_3 being the span of the eigenvectors of T corresponding to the negative eigenvalues, and H_2 the null space of T. Then by our assumption, the dimension of H_1 is less than n. Let $u_1,u_2,\ldots,u_n$ be an orthonormal basis for the subspace M of our hypothesis. By the projection theorem, each u_k may be decomposed as a sum $u_k=v_k+w_k$, with $v_k\in H_1$, w_k orthogonal to H_1. Then $v_1,v_2,\ldots,v_n$ are n vectors in a subspace H_1 of dimension less than n. Hence there is a nontrivial linear combination

$$\sum_{k=1}^{n}c_kv_k=0.$$

Since $u_1,u_2,\ldots,u_n$ are linearly independent, then u defined as

$$u=\sum_{k=1}^{n}c_ku_k=\sum_{k=1}^{n}c_kv_k+\sum_{k=1}^{n}c_kw_k=\sum_{k=1}^{n}c_kw_k$$

is a nonzero vector in M orthogonal to H_1. Thus $u\in H_2\oplus H_3$ and so $\frac{(Tu,u)}{(u,u)}\leq 0$.

To prove the second assertion, let u_k be a normalized eigenvector of T corresponding to λ_k. Choosing M_1 as the n dimensional subspace of H spanned by $u_1,u_2,\ldots,u_n$, a straightforward calculation shows that $\frac{(Tf,f)}{(f,f)}\geq\lambda_n$, for any $f\in M_1$. Also, $\frac{(Tu_n,u_n)}{(u_n,u_n)}=\lambda_n$. Thus, for M_1, the relevant infimum equals λ_n. To complete the proof, we need only show that for any n dimensional M, the relevant infimum is less than or equal λ_n. Let S be the span of $u_1,u_2,\ldots,u_{n-1}$. We can choose an arbitrary orthonormal basis $v_1,v_2,\ldots,v_n$ of M and use the projection theorem to decompose each $v_k=w_k+x_k$ with $w_k\in S$ and x_k orthogonal to S. Then $w_1,w_2,\ldots,w_n$ all belong to a subspace of dimension less than n and some nontrivial linear combination

$$\sum_{k=1}^{n}c_kw_k=0.$$

Thus v defined as

$$v = \sum_{k=1}^{n} c_k v_k = \sum_{k=1}^{n} c_k w_k + \sum_{k=1}^{n} c_k x_k = \sum_{k=1}^{n} c_k x_k$$

is a nonzero vector in M orthogonal to S. We can decompose $v = y_1 + y_2 + y_3$ with $y_1 \in H_1$, $y_2 \in H_2$, $y_3 \in H_3$, and y_1 orthogonal to S. Thus we have $v \in M$ so that

$$\frac{(Tv,v)}{(v,v)} = \frac{(Ty_1,y_1) + (Ty_2,y_2) + (Ty_3,y_3)}{(v,v)} \le \frac{(Ty_1,y_1)}{(v,v)} \le \frac{\lambda_n(y_1,y_1)}{(v,v)} \le \lambda_n.$$

□

Lemma 2.10. *$x^\nu J(x)$, $J(x)$, and $J'(x)$ are all bounded for $x \ge 0$.*

Proof. From the definition of $J(x)$ and the series expansion of $J_{\nu-1/2}(x)$, we have

$$x^\nu J(x) = \sqrt{x} J_{\nu-1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \sqrt{2}}{\Gamma(k+1)\Gamma(k+\nu+1/2)} \left(\frac{x}{2}\right)^{\nu+2k}$$

so $x^\nu J(x)$ is a continuous function for $x \ge 0$. Further, from Lemma 2.2, $J_\nu(x) \le Mx^{-1/2}$ for large x and some constant M, independent of ν. Thus $x^\nu J(x) = \sqrt{x} J_{\nu-1/2}(x)$ is bounded as $x \to \infty$. Therefore, $x^\nu J(x)$ is bounded for $x \ge 0$. Since $J(0) = \frac{1}{\Gamma(\nu+1/2)}$, $J(x)$ is also bounded for $x \ge 0$. Using the Bessel identity from Lemma 2.1, we have $J'(x) = -x^{1/2-\nu} J_{\nu+1/2}(x)$. Since the asymptotic behavior of $J_\nu(x)$ is independent of ν, $J'(x)$ behaves similarly to $J(x)$ as $x \to \infty$ (in absolute value). Since $J'(0) = 0$, then by continuity $J'(x)$ is bounded for $x \ge 0$. □

Lemma 2.11. *For $F \in L^1(0,\infty)$, the integral defining $\rho(u,v)$ converges for all $u, v > 0$ and $\rho(u,v)$ is a continuous function of (u,v). Also, $(uv)^\nu |\rho(u,v)|$ is bounded.*

Proof. From Lemma 2.10, for $u, v > 0$,

$$\left| \int_0^\infty F(t)J(ut)J(vt)t^{2\nu} dt \right| \le \int_0^\infty \left| F(t)(ut)^\nu J(ut)(vt)^\nu J(vt)(uv)^{-\nu} \right| dt$$
$$\le C(uv)^{-\nu} \int_0^\infty |F(t)| dt$$

for some constant C. Since $F(t) \in L_1(0,\infty)$, the integral defining $\rho(u,v)$ converges for all $u, v > 0$. Note that this implies $(uv)^\nu |\rho(u,v)|$ is bounded.

For $u, v > 0$, let $(u_n, v_n) \to (u,v)$. Then

$$\lim_{n\to\infty} \rho(u_n, v_n) = \int_0^\infty \lim_{n\to\infty} F(t)J(u_n t)J(v_n t)t^{2\nu} dt$$

by the generalized Lebesgue Dominated Convergence Theorem since the absolute value of the integrand is dominated by the integrable function $C|F(t)|(u_n v_n)^{-\nu}$ from above. Then since $J(x)$ is a continuous function, $J(u_n t) \to J(ut)$ and $J(v_n t) \to J(vt)$ and we have $\rho(u_n, v_n) \to \rho(u,v)$, so $\rho(u,v)$ is a continuous function for $u, v > 0$. □

Define

$$\mathcal{G} = \{f(x) \in C^\infty(0,\infty) :\ x^{2\nu}(\tau^k f)' \to 0 \text{ as } x \to 0^+, \tau^k f \text{ is bounded on } (0,\infty),$$
$$\text{and } (\tau^k f)' \text{ and } \tau^k f \text{ are rapidly decreasing on } (0,\infty) \text{ for all } k \geq 0\}.$$

The following lemma is proved in [2] for a class of functions larger than $\mathcal{G}$.

Lemma 2.12. *If $u \in \mathcal{G} \subset L^2(0,\infty; x^{2\nu})$, the following conditions hold:*

1. $\tau u \in \mathcal{G}$
2. *For $i + j = k + l$, $\langle \tau^i u, \tau^j u\rangle = \langle \tau^k u, \tau^l u\rangle$*
3. *For $0 < x_1 < x_2 < \infty$, $|u(x_2) - u(x_1)|^2 \leq \langle \tau u, u\rangle \int_{x_1}^{x_2} \frac{1}{t^{2\nu}}\, dt$*
4. *For $0 < x_1 < x_2 < \infty$, $|{x_2}^{2\nu} u'(x_2) - {x_1}^{2\nu} u'(x_1)|^2 \leq \langle \tau^2 u, u\rangle \int_{x_1}^{x_2} t^{2\nu}\, dt$*

Lemma 2.13. *If $f \in \mathcal{G}$, then $x^{2k}\hat{f}(x) = \widehat{\tau^k f}(x)$ for $k = 1, 2, \ldots$ and $\hat{f}$ is rapidly decreasing on $(0,\infty)$.*

Proof. Given the formula for the Hankel transform,

$$\hat{f}(x) = \int_0^\infty f(t) J(xt) t^{2\nu} dt,$$

we multiply the equation by x^2, apply Lemma 2.3, and integrate by parts:

$$\begin{aligned} x^2\hat{f}(x) &= \int_0^\infty f(t) x^2 J(xt) t^{2\nu} dt = \int_0^\infty f(t) \tau_t J(xt) t^{2\nu} dt \\ &= -\int_0^\infty f(t)(t^{2\nu} J'(xt))' dt = -f(t) t^{2\nu} J'(xt)\big|_0^\infty + \int_0^\infty f'(t) J'(xt) t^{2\nu} dt \\ &= \int_0^\infty t^{2\nu} f'(t) J'(xt) dt. \end{aligned}$$

Since $J'(xt)$ is bounded on $(0,\infty)$ from Lemma 2.10 and $f(t) \in \mathcal{G}$, the boundary term vanishes at infinity because f is rapidly decreasing and vanishes at 0 because f is bounded. Integrating by parts again we have

$$\int_0^\infty t^{2\nu} f'(t) J'(xt) dt = t^{2\nu} f'(t) J(xt)\big|_0^\infty - \int_0^\infty (t^{2\nu} f'(t))' J(xt) dt = \int_0^\infty \tau_t f(t) J(xt) t^{2\nu} dt$$

Here the boundary term vanishes because $J(xt)$ is bounded from Lemma 2.10 and since $f(t) \in \mathcal{G}$, f' is rapidly decreasing and $t^{2\nu} f'(t) \to 0$ as $t \to 0^+$. Thus, we have the relation

$$x^2\hat{f}(x) = \int_0^\infty \tau f(t) J(xt) t^{2\nu} dt = \widehat{\tau f}(x).$$

From Lemma 2.12(i), $\tau f \in \mathcal{G}$, so we can iterate this process to generate

$$x^{2k}\hat{f}(x) = \int_0^\infty \tau^k f(t) J(xt) t^{2\nu} dt = \widehat{\tau^k f}(x)$$

and since $\tau^k f$ is rapidly decreasing and J is bounded, the integral will converge for all k, yielding

$$x^{2k}\hat{f}(x) \leq M_k \quad \text{for } k = 0, 1, \ldots$$

and thus $\hat{f}$ is rapidly decreasing. □

The proof of the next lemma is a standard exercise.

Lemma 2.14. *For a given A, the integral operator $T_A : L^2(0, A; x^{2\nu}) \to L^2(0, A; x^{2\nu})$ is a compact self-adjoint operator.*

Since we would prefer to study our operators on a fixed interval of integration we replace x with Ax and y with Ay in Equation (1.1) to obtain

$$(T_A h)(Ax) = A^{2\nu+1} \int_0^1 h(Ay)\rho(Ax, Ay)y^{2\nu} dy$$

for $0 < x \le 1$, $A \ge 1$. If we now let $f(y) = h(Ay)$ we may now view the operator T_A defined for $f \in L_2(0, 1; x^{2\nu})$ by the equation

$$(T_A f)(x) = A^{2\nu+1} \int_0^1 f(y)\rho(Ax, Ay)y^{2\nu} dy. \tag{2.1}$$

The restriction of T_A to

$$C_1^\infty(0, 1) = \{v \in C^\infty(0, 1) : v \text{ is bounded and } v(x) \equiv 0 \text{ for } x \text{ near } 1\}$$

has a useful alternative formula to Equation (2.1), which requires that we extend any $f \in C_1^\infty(0, 1)$ to $(0, \infty)$ by defining $f(x) = 0$ for $x \ge 1$ (we will often assume, without comment, that such an extension has been made for functions in $C_1^\infty(0, 1)$). For such f we have

$$\begin{aligned}(T_A f)(x) &= A^{2\nu+1} \int_0^1 f(y)\rho(Ax, Ay)y^{2\nu} dy \\ &= A^{2\nu+1} \int_0^1 \int_0^\infty f(y)F(t)J(Ayt)J(Axt)t^{2\nu}y^{2\nu} dt\, dy.\end{aligned}$$

Using Lemma 2.10, we can apply Fubini's theorem to get

$$(T_A f)(x) = A^{2\nu+1} \int_0^\infty F(t)\hat{f}(At)J(Axt)t^{2\nu} dt.$$

With a change of variable, replacing t with t/A, we have

$$(T_A f)(x) = \int_0^\infty F(t/A)\hat{f}(t)J(xt)t^{2\nu} dt. \tag{2.2}$$

Since T_A is a compact self-adjoint operator, then every nonzero element in the spectrum of T_A is an eigenvalue of finite multiplicity. The next lemma gives further information about these eigenvalues.

Lemma 2.15. *Let $m \le 0$ and $M > 0$. Suppose $m \le F(t) \le M$ for $-\infty < t < \infty$. Then every eigenvalue λ of the operator T_A satisfies $m \le \lambda \le M$.*

Proof. Let λ be an eigenvalue of T_A, and f a corresponding normalized eigenfunction. Let $0 < \epsilon < 1$. Then there exists an $h \in C_0^\infty(0,1)$ such that $\|f-h\| < \epsilon$ and $\|h\| = 1$. Using the boundedness property of T_A and the Schwarz inequality, we have

$$\begin{aligned}\lambda = (T_A f, f) &= (T_A(f-h),(f-h)) + (T_A(f-h),h) + (T_A h,(f-h)) + (T_A h,h)\\ &\leq 3\|T_A\|\epsilon + (T_A h,h)\end{aligned}$$

and similarly $(T_A h,h) \leq 3\|T_A\|\epsilon + (T_A f,f)$, so

$$(T_A h,h) - 3\|T_A\|\epsilon \leq \lambda \leq (T_A h,h) + 3\|T_A\|\epsilon.$$

Since $h(t) = 0$ for $t \geq 1$, Theorem 2.5 implies $(T_A h,h) = \langle T_A h,h\rangle = \langle F(t/A)\hat{h},\hat{h}\rangle$. Thus

$$m\langle \hat{h},\hat{h}\rangle \leq \langle F(t/A)\hat{h},\hat{h}\rangle \leq M\langle \hat{h},\hat{h}\rangle$$

and $\langle \hat{h},\hat{h}\rangle = \langle h,h\rangle = (h,h) = 1$ together imply $m \leq (T_A h,h) \leq M$. Thus,

$$m - 3\epsilon\|T_A\| \leq \lambda \leq M + 3\epsilon\|T_A\|$$

and letting $\epsilon \to 0$, we have $m \leq \lambda \leq M$. □

We now wish to introduce a special case where $F(t)$ is chosen as $F_S(t) = (1+t^2)^{-1}$ with the corresponding

$$\rho_S(u,v) = \int_0^\infty F_S(t)J(ut)J(vt)t^{2\nu}\,dt.$$

Then for $f \in L^2(0,1;x^{2\nu})$, the corresponding Toeplitz integral operator is

$$(S_A f)(x) = A^{2\nu+1}\int_0^1 f(y)\rho_S(Ax,Ay)y^{2\nu}dy. \tag{2.3}$$

and for $f \in C_1^\infty(0,1)$,

$$(S_A f)(x) = \int_0^\infty \frac{1}{1+(t/A)^2}\hat{f}(t)J(xt)t^{2\nu}dt \tag{2.4}$$

Lemma 2.16. *Let $f \in C_1^\infty(0,1) \cap \mathcal{G}$, and let $g = \left(I + \frac{\tau}{A^2}\right)^n f$. Then*

$${S_A}^i g = \left(I + \frac{\tau}{A^2}\right)^{n-i} f \qquad \textit{for } i = 1,2,\ldots,n.$$

Proof. With

$$g = \left(I + \frac{\tau}{A^2}\right)^n f = \sum_{j=0}^n \binom{n}{j}\left(\frac{\tau}{A^2}\right)^j f,$$

since $f \in \mathcal{G}$ from Lemma 2.13 we have $\widehat{\tau^j f} = t^{2j}\hat{f}$, so

$$\hat{g} = \sum_{j=0}^n \binom{n}{j}\left(\frac{t^2}{A^2}\right)^j \hat{f} = \left(1 + \frac{t^2}{A^2}\right)^n \hat{f}.$$

Since $f \in C_1^\infty(0,1) \cap \mathcal{G}$, then from Lemma 2.12($i$) we can conclude $g \in C_1^\infty(0,1)$. Thus from Equation (2.4),

$$S_A g(x) = \int_0^\infty \left(1 + \frac{t^2}{A^2}\right)^{n-1} \hat{f}(t) J(xt) t^{2\nu}\, dt$$

which is the Hankel transform of $\left(1 + \frac{t^2}{A^2}\right)^{n-1} \hat{f}$. From Lemma 2.13 and using Lemma 2.4,

$$S_A g = \left(I + \frac{\mathcal{T}}{A^2}\right)^{n-1} f.$$

Iterating this process, we have our result. □

Lemma 2.17. *Let $f \in C_1^\infty(0,1) \cap \mathcal{G}$. Then $(S_A f)(x)$ and $(S_A f)'(x)$ are both bounded on $x \in [0,1]$ as $A \to \infty$.*

Proof. From Equation (2.4),

$$(S_A f)(x) = \int_0^\infty F_S(t/A) \hat{f}(t) J(xt) t^{2\nu} dt$$

and since $0 < F_S(t/A) \leq 1$, both J and J' are bounded from Lemma 2.10, and $\hat{f}$ is rapidly decreasing from Lemma 2.13, we can differentiate to obtain

$$(S_A f)'(x) = \int_0^\infty F_S(t/A) \hat{f}(t) J'(xt) t^{2\nu+1} dt.$$

Thus $(S_A f)(x)$ and $(S_A f)'(x)$ are integrals which converge for $x \in (0,1)$ independent of A. □

3. **First Main Theorem**

Recall the special function $F_S(t) = (1+t^2)^{-1}$. Let n be a fixed positive integer and $F(t)$ be a function satisfying the following conditions

H4: $F(t)$ is bounded and absolutely integrable on $(0,\infty)$,
H5: $F(t)$ has a positive absolute maximum M at $t = 0$,
H6: $\lim_{t\to 0^+} \frac{F(0)-F(t)}{t^{2n}} = \sigma^2$ for some constant σ.

Note that H1–H3 are a special case of H4–H6, where $n = 1$. Let $M = F(0)$. From H6:

$$\lim_{t\to 0^+} \frac{M - F(t)}{(1 - F_S(t))^n} = \sigma^2$$

and so given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \frac{M - F(t)}{(1 - F_S(t))^n} - \sigma^2 \right| < \epsilon \quad \text{for } 0 \leq t \leq \delta. \tag{3.1}$$

Define

$$G(t) = \begin{cases} F(t), & 0 \leq t < \delta, \\ M - (\sigma^2 + \gamma)(1 - F_S(t))^n, & t \geq \delta. \end{cases}$$

where $\gamma = \frac{M - F(\delta)}{(1-F_S(\delta))^n} - \sigma^2$.

Lemma 3.1. *Suppose $F(t)$ satisfies* H4–H6 *and let $G(t)$ be defined as above for an arbitrary $\epsilon > 0$. Then for $t \geq 0$,*

$$M - F(t) < (\sigma^2 + \epsilon)(1 - F_S(t))^n + |G(t) - F(t)|$$

and the quantity $|G(t) - F(t)|$ is bounded for $t \geq 0$ and vanishes for $0 \leq t < \delta$.

Proof. We decompose $M - F(t)$ into three terms:

$$M - F(t) = \sigma^2(1 - F_S(t))^n + \left(M - G(t) - \sigma^2(1 - F_S(t))^n\right) + (G(t) - F(t))\,.$$

We consider the second term. For $0 \leq t < \delta$, $G(t) = F(t)$ so (3.1) implies

$$\left|M - G(t) - \sigma^2(1 - F_S(t))^n\right| < \epsilon(1 - F_S(t))^n, \quad \text{for } 0 \leq t < \delta.$$

For $t \geq \delta$, $G(t) = M - (\sigma^2 + \gamma)(1 - F_S(t))^n$ and so

$$M - G(t) - \sigma^2(1 - F_S(t))^n = \gamma(1 - F_S(t))^n$$

but (3.1) and the definition of γ imply $|\gamma| < \epsilon$. Thus for all $t \geq 0$,

$$\left|M - G(t) - \sigma^2(1 - F_S(t))^n\right| < \epsilon(1 - F_S(t))^n,$$

and the three term decomposition gives

$$M - F(t) < (\sigma^2 + \epsilon)(1 - F_S(t))^n + |G(t) - F(t)|\,, \quad \text{for } t \geq 0,$$

and the remaining statements are obvious. □

The next two lemmas are straightforward.

Lemma 3.2. *Let $\{f_j\}_{j=1}^k \subset L^2(0,1;x^{2\nu})$ be a set of orthonormal functions. For $0 < \epsilon < 1/k$, let $h_j \in L^2(0,1;x^{2\nu})$ be defined such that $\|f_j - h_j\| < \epsilon$ for $j = 1, 2, \ldots, k$. Then the h_j's are linearly independent.*

Lemma 3.3. *Let $\mathcal{L} = span\{g_1, \ldots, g_n\}$ be an n-dimensional subspace of a Hilbert space $\mathcal{H}$. Then the Rayleigh quotient for a bounded operator T*

$$Q(g) = \frac{(Tg, g)}{(g, g)}$$

achieves a minimum on $\mathcal{L}$, and the minimum is achieved for some $g_0 = \sum_{i=1}^n c_i g_i$ such that

$$\sum_{i=1}^n |c_i|^2 = 1.$$

We come to our first main theorem.

Theorem 3.4. *Let k be a positive integer. Then for A sufficiently large, the integral operator T_A, with $F(t)$ satisfying conditions* H4–H6*, will have at least k positive eigenvalues with $M \geq \lambda_{1,A} \geq \lambda_{2,A} \geq \cdots \geq \lambda_{k,A} > 0$, allowing repetitions for multiple eigenvalues, and we have*

$$\limsup_{A\to\infty} A^{2n}(M - \lambda_{j,A}) \leq \sigma^2 \Lambda_j \quad \text{for } j = 1, 2, \ldots, k,$$

where $\{\Lambda_j\}$ are the eigenvalues of $\tilde{L}_n$ arranged in nondecreasing order with repetitions for multiple eigenvalues.

Proof. Let $1/k > \epsilon > 0$ be given. Let f_j be the normalized eigenfunction of $\tilde{L_n}$ corresponding to Λ_j for $j = 1, 2, \ldots, k$. From Theorem 2.6, there exists $h_j \in D(L_n)$ such that $\|h_j\| = 1$ and

$$\|f_j - h_j\| < \epsilon \quad \text{for } j = 1, 2, \ldots, k$$

and for $i, j = 1, 2, \ldots, k$,

$$|(h_i, h_j) - (f_i, f_j)| < \epsilon$$
$$|(L_n h_i, h_j) - (\tilde{L_n} f_i, f_j)| < \epsilon.$$

From Lemma 3.2, the h_j's are linearly independent. Now define

$$g_{j,A} = \left(I + \frac{\tau}{A^2}\right)^n h_j.$$

From Lemma 2.16 we have ${S_A}^n g_{j,A} = h_j$, and so the $g_{j,A}$'s are also linearly independent.

Let $\mathcal{M}_k$ denote any k-dimensional subspace of $L^2(0, 1; x^{2\nu})$. We shall show that for A sufficiently large,

$$\sup_{M_k} \inf_{g\in\mathcal{M}_k} \left[\frac{(T_A g, g)}{(g, g)}\right] > 0.$$

It will then follow from Theorem 2.9 that T_A has at least k positive eigenvalues.

Let $\mathcal{L}_k$ be the span of $g_{j,A}$. Since the $g_{j,A}$'s are linearly independent, $\mathcal{L}_k$ is a particular k-dimensional subspace of $L^2(0, 1; x^{2\nu})$. Now define $\mu_{k,A}$ as follows:

$$\mu_{k,A} = \sup_{g\in\mathcal{L}_k} \left[M - \frac{(T_A g, g)}{(g, g)}\right] = M - \inf_{g\in\mathcal{L}_k} \left[\frac{(T_A g, g)}{(g, g)}\right].$$

From Lemma 3.3, the Rayleigh quotient $Q(g) = \frac{(T_A g, g)}{(g, g)}$ achieves a minimum on $\mathcal{L}_k$ and there exists $g_A \in \mathcal{L}_k$ such that $g_A = \sum_{j=1}^k c_j g_{j,A}$ minimizes the Rayleigh quotient and $\sum_{j=1}^k |c_j|^2 = 1$. For this particular g_A, we have

$$\mu_{k,A} = \frac{((MI - T_A)g_A, g_A)}{(g_A, g_A)}.$$

We will first consider the numerator $((MI - T_A)g_A, g_A)$. Note that

$$((MI - T_A)g_A)(x) = \int_0^\infty (M - F(t/A))\hat{g_A}(t) J(xt) t^{2\nu} dt$$

is the Hankel transform of $(M - F(t/A))\hat{g_A}(t)$, and that $g_A \in C_1^\infty(0,1)$. So

$$((MI - T_A)g_A, g_A) = \langle (MI - T_A)g_A, g_A\rangle$$

and by Theorem 2.5,

$$\langle (MI - T_A)g_A, g_A\rangle = \langle (M - F(t/A))\hat{g_A}, \hat{g_A}\rangle = \int_0^\infty (M - F(t/A))|\hat{g_A}(t)|^2 t^{2\nu} dt.$$

From Lemma 3.1,

$$\int_0^\infty (M - F(t/A))|\hat{g_A}(t)|^2 t^{2\nu} dt < (\sigma^2 + \epsilon)\mathcal{I}_1 + \mathcal{I}_2,$$

where

$$\mathcal{I}_1 = \int_0^\infty (1 - F_S(t/A))^n |\hat{g_A}(t)|^2 t^{2\nu} dt \text{ and}$$

$$\mathcal{I}_2 = \int_0^\infty |G(t/A) - F(t/A)|\, |\hat{g_A}(t)|^2 t^{2\nu} dt.$$

Since $g_A = \sum_{j=1}^k c_j g_{j,A}$ and the Hankel transform is a linear operation, we can write

$$\mathcal{I}_1 = \sum_{j,\ell=1}^k c_j \overline{c_\ell} \int_0^\infty (1 - F_S(t/A))^n \hat{g}_{j,A}(t) \overline{\hat{g}_{\ell,A}(t)} t^{2\nu} dt.$$

Because $g_{j,A} = (1 + \frac{\tau}{A^2})^n h_j$ and Lemma 2.13 implies $\widehat{\tau^k h_j} = t^{2k}\hat{h_j}$, we have $\hat{g}_{j,A} = (1 + (t/A)^2)^n \hat{h_j}$. Also, since $(1 - F_S(t/A)) = \frac{(t/A)^2}{1+(t/A)^2}$, we have

$$\mathcal{I}_1 = \sum_{j,\ell=1}^k c_j \overline{c_\ell} \sum_{m=0}^n \binom{n}{m} \left\langle \left(\frac{t^2}{A^2}\right)^n \hat{h_j}, \left(\frac{t^2}{A^2}\right)^m \hat{h_\ell} \right\rangle.$$

Then by Theorem 2.5, Lemma 2.13, and the fact that each h_j vanishes for $t > 1$, we can rewrite these inner products as

$$\mathcal{I}_1 = \sum_{j,\ell=1}^k c_j \overline{c_\ell} \sum_{m=0}^n \binom{n}{m} \left(\left(\frac{\tau}{A^2}\right)^n h_j, \left(\frac{\tau}{A^2}\right)^m h_\ell \right).$$

Multiplying through by A^{2n} and rearranging we have

$$A^{2n}\mathcal{I}_1 = \sum_{j,\ell=1}^k c_j \overline{c_\ell} \left[(\tau^n h_j, h_\ell) + \sum_{m=1}^n \frac{1}{A^{2m}} \binom{n}{m} (\tau^n h_j, \tau^m h_\ell) \right],$$

and noting that the h_j functions have no dependence on A, we conclude that

$$\lim_{A\to\infty} A^{2n}\mathcal{I}_1 = \sum_{j,\ell=1}^k c_j \overline{c_\ell}\, (\tau^n h_j, h_\ell).$$

Then since $\mathcal{I}_1$ is a nonnegative quantity, using the triangle inequality and letting $A \to \infty$ we have

$$\limsup_{A\to\infty} A^{2n}\mathcal{I}_1 \le \sum_{j,\ell=1}^k |c_j \overline{c_\ell}| \left| (L_n h_j, h_\ell) - (\tilde{L_n} f_j, f_\ell) \right| + \sum_{j,\ell=1}^k |c_j \overline{c_\ell}|\, (\tilde{L_n} f_j, f_\ell)$$

$$< \sum_{j,\ell=1}^{k} \epsilon + \sum_{j=1}^{k} |c_j|^2 \Lambda_j \le \epsilon k^2 + \Lambda_k.$$

We now consider the integral $\mathcal{I}_2$. From Lemma 3.1, we can write

$$\begin{aligned}\mathcal{I}_2 &= A \int_\delta^\infty |G(t) - F(t)|\, |\hat{g_A}(At)|^2 (At)^{2\nu} dt \\ &\le KA \int_\delta^\infty |\hat{g_A}(At)|^2 (At)^{2\nu} dt\end{aligned}$$

for some constant K. As shown above (from Lemma 2.13),

$$\hat{g_A}(At) = \sum_{j=1}^{k} c_j (1 + (t/A)^2)^n \hat{h_j}(At) = \sum_{j=1}^{k} \sum_{m=0}^{n} \frac{c_j}{A^{2m}} \binom{n}{m} \widehat{\tau^m h_j}(At).$$

We define $H(t) = \sum_{j=1}^{k} \sum_{m=0}^{n} \binom{n}{m} \left| \widehat{\tau^m h_j}(t) \right|$. Since $\tau^m h_j \in \mathcal{G}$ for all values of m and j, from Lemma 2.13, $\widehat{\tau^m h_j}$ is rapidly decreasing for all m, j, and so $H(t)$ is a rapidly decreasing function which is independent of A. Thus we have, for A sufficiently large,

$$|\hat{g_A}(At)| \le H(At) \le (At)^{-(1+n+\nu)}$$

for $t \ge \delta$. Then

$$\mathcal{I}_2 \le KA^{-2n-1} \int_\delta^\infty t^{-2n-2} dt$$

which implies

$$\lim_{A \to \infty} A^{2n} \mathcal{I}_2 = 0.$$

Given these properties of $\mathcal{I}_1$ and $\mathcal{I}_2$, we now have

$$\limsup_{A \to \infty} A^{2n} \left((MI - T_A) g_A, g_A \right) \le (\sigma^2 + \epsilon)(\epsilon k^2 + \Lambda_k).$$

We will now consider the denominator (g_A, g_A). From Theorem 2.5 and Lemma 2.13,

$$\begin{aligned}(g_A, g_A) &= \sum_{j,\ell=1}^{k} c_j \overline{c_\ell} \langle g_{j,A}, g_{\ell,A} \rangle = \sum_{j,\ell=1}^{k} c_j \overline{c_\ell} \langle \hat{g}_{j,A}, \hat{g}_{\ell,A} \rangle \\ &= \sum_{j,\ell=1}^{k} c_j \overline{c_\ell} \left\langle \sum_{m=0}^{n} \frac{1}{A^{2m}} \binom{n}{m} \widehat{\tau^m h_j}, \sum_{p=0}^{n} \frac{1}{A^{2p}} \binom{n}{p} \widehat{\tau^p h_\ell} \right\rangle \\ &= \sum_{j,\ell=1}^{k} c_j \overline{c_\ell} \left\langle \sum_{m=0}^{n} \frac{1}{A^{2m}} \binom{n}{m} \tau^m h_j, \sum_{p=0}^{n} \frac{1}{A^{2p}} \binom{n}{p} \tau^p h_\ell \right\rangle\end{aligned}$$

Since the inner products $(\tau^m h_j, \tau^p h_\ell)$ are independent of A, we have

$$\begin{aligned}\lim_{A \to \infty} (g_A, g_A) &= \sum_{j,\ell=1}^{k} c_j \overline{c_\ell}\, (h_j, h_\ell) \\ &= \sum_{j,\ell=1}^{k} c_j \overline{c_\ell}\, (f_j, f_\ell) + \sum_{j,\ell=1}^{k} c_j \overline{c_\ell}\, \left[(h_j, h_\ell) - (f_j, f_\ell) \right]\end{aligned}$$

$$= 1 + \sum_{j,\ell=1}^{k} c_j \overline{c_\ell}\ [(h_j, h_\ell) - (f_j, f_\ell)]$$
$$\geq 1 - \epsilon k^2.$$

Therefore we have

$$\limsup_{A\to\infty} A^{2n}\mu_{k,A} \leq \frac{(\sigma^2+\epsilon)(\epsilon k^2 + \Lambda_k)}{1-\epsilon k^2}.$$

Since the left side has no dependence on ϵ, which was chosen arbitrarily, we have

$$\limsup_{A\to\infty} A^{2n}\mu_{k,A} \leq \sigma^2\Lambda_k.$$

Thus for sufficiently large A, $\mu_{k,A} < M$, and by Theorem 2.9, we have

$$\lambda_{k,A} = \sup_{\mathcal{M}_k}\left[\inf_{g\in\mathcal{M}_k}\frac{(T_A g, g)}{(g,g)}\right] \geq \min_{g\in\mathcal{L}_k}\frac{(T_A g, g)}{(g,g)} = M - \mu_{k,A} > 0.$$

Thus the operator T_A will have at least k positive eigenvalues with

$$M \geq \lambda_{1,A} \geq \lambda_{2,A} \geq \cdots \geq \lambda_{k,A} > 0.$$

We also see that $M - \lambda_{k,A} \leq \mu_{k,A}$ so that we have

$$\limsup_{A\to\infty} A^{2n}(M-\lambda_{k,A}) \leq \sigma^2\Lambda_k.$$

□

4. **Second Main Theorem**

The following integral can be found in [9, pg. 336] and is a consequence of a more general result proved in [13, pg. 429].

Lemma 4.1.

$$\int_0^\infty \frac{tJ_\nu(at)J_\nu(bt)}{z^2+t^2}dt = I_\nu(bz)K_\nu(az), \quad R(z)>0, \quad a\geq b>0, \quad R(\nu)>-1.$$

Thus, we are able to evaluate the corresponding ρ_S integral:

$$\rho_S(u,v) = (uv)^{1/2-\nu}\int_0^\infty \frac{tJ_{\nu-1/2}(ut)J_{\nu-1/2}(vt)}{1+t^2}\,dt$$

and from Lemma 4.1, letting $z=1$, we have

$$\rho_S(u,v) = (uv)^{1/2-\nu} I_{\nu-1/2}(v)K_{\nu-1/2}(u), \quad \text{for } u \geq v > 0. \tag{4.1}$$

Armed with this formula, we obtain the following two results which are crucial for our second main theorem. As mentioned in the introduction, our failure to properly generalize these results is the reason we have not succeeded with our full goal.

Lemma 4.2. *Let $f \in C_1^\infty(0,1)$ and $g = S_A f$. Then there exists $\epsilon > 0$ such that*

$$g'(x) = -A\frac{K_{\nu+1/2}(Ax)}{K_{\nu-1/2}(Ax)}g(x) \quad \text{for } x > 1-\epsilon.$$

Proof. From Equation (2.3),

$$g(x) = A^{2\nu+1}\int_0^1 f(y)\rho_S(Ax, Ay)y^{2\nu}dy.$$

Since $f \in C_1^\infty(0,1)$ there exists $\epsilon > 0$ such that $f(x) = 0$ for $x > 1-\epsilon$. Thus

$$g(x) = A^{2\nu+1}\int_0^{1-\epsilon} f(y)\rho_S(Ax, Ay)y^{2\nu}dy$$

and for $x > 1-\epsilon$, since $y < x$ in the integrand, we can substitute for $\rho_S(Ax, Ay)$ using Equation (4.1), yielding

$$\begin{aligned} g(x) &= A^{2\nu+1}\int_0^{1-\epsilon} f(y)(Ay)^{1/2-\nu}I_{\nu-1/2}(Ay)(Ax)^{1/2-\nu}K_{\nu-1/2}(Ax)y^{2\nu}dy \\ &= \left[A^{2\nu+1}\int_0^{1-\epsilon} f(y)(Ay)^{1/2-\nu}I_{\nu-1/2}(Ay)y^{2\nu}\,dy\right](Ax)^{1/2-\nu}K_{\nu-1/2}(Ax). \end{aligned}$$

Differentiating using Lemma 2.1,

$$g'(x) = \left[A^{2\nu+1}\int_0^{1-\epsilon} f(y)(Ay)^{1/2-\nu}I_{\nu-1/2}(Ay)y^{2\nu}\,dy\right]\left(-A(Ax)^{1/2-\nu}K_{\nu+1/2}(Ax)\right)$$

and so

$$\frac{g'(x)}{g(x)} = -A\frac{K_{\nu+1/2}(Ax)}{K_{\nu-1/2}(Ax)}.$$

Thus, for $x > 1-\epsilon$ we have the identity

$$g'(x) = -A\frac{K_{\nu+1/2}(Ax)}{K_{\nu-1/2}(Ax)}g(x).$$

□

Lemma 4.3. *Let $f \in C_1^\infty(0,1) \cap \mathcal{G}$ and $g = S_A f$. Then for $0 \le x_1 \le x_2 \le 1$,*

$$|g(x_2) - g(x_1)|^2 + \left(\int_{x_1}^{x_2}\frac{1}{x^{2\nu}}\,dx\right)\left(A\frac{K_{\nu+1/2}(A)}{K_{\nu-1/2}(A)}|g(1)|^2\right) \le (\tau g, g)\left(\int_{x_1}^{x_2}\frac{1}{x^{2\nu}}\,dx\right).$$

Proof. Let $0 \le x_1 \le x_2 \le 1$, then using the Schwarz inequality and integration by parts, we obtain

$$\begin{aligned} |g(x_2) - g(x_1)|^2 &= \left|\int_{x_1}^{x_2}\frac{1}{x^\nu}\,g'(x)x^\nu dx\right|^2 \le \left(\int_{x_1}^{x_2}\frac{1}{x^{2\nu}}dx\right)\left(\int_0^1 g'(x)\overline{g'(x)}x^{2\nu}dx\right) \\ &\le \left(\int_{x_1}^{x_2}\frac{1}{x^{2\nu}}dx\right)\left(g'(1)\overline{g(1)} + (\tau g, g)\right). \end{aligned}$$

Here the boundary term at 0 vanishes since g and g' are both bounded from Lemma 2.17. Substituting the identity from Lemma 4.2 for $x = 1$, the result follows. □

It is important to realize, in these last two results, that (see [8, p. 119, (5.10.24)], [13, p. 172 (4)]) $K_\alpha(x) > 0$, for $x > 0$.

Lemma 4.4. *Let $f \in D(L_1)$ and $g = S_A f$. Then $(I + \frac{\tau}{A^2})g = f$ and $(I - S_A)f = \frac{\tau}{A^2}g$.*

Proof. From Equation (2.4) we have $g(x) = \int_0^\infty \frac{1}{1+\frac{t^2}{A^2}} \hat{f}(t)J(xt)t^{2\nu}dt$. Since $J(xt)$ and $J'(xt)$ are bounded from Lemma 2.10 and $\hat{f}(t)$ is rapidly decreasing, we may differentiate twice and use Lemma 2.3 to get

$$\frac{\tau_x}{A^2}g(x) = \int_0^\infty \frac{1}{1+\frac{t^2}{A^2}} \hat{f}(t)\frac{\tau_x}{A^2}J(xt)t^{2\nu}dt = \int_0^\infty \frac{\frac{t^2}{A^2}}{1+\frac{t^2}{A^2}} \hat{f}(t)J(xt)t^{2\nu}dt.$$

Therefore

$$(I + \frac{\tau}{A^2})g(x) = \int_0^\infty \hat{f}(t)J(xt)t^{2\nu}dt = f(x).$$

and also $\frac{\tau}{A^2}g = f - g = (I - S_A)f$. □

Theorem 4.5. *For each integer $k \geq 1$ and A sufficiently large, the integral operator T_A, with $F(t)$ satisfying conditions* H1–H3*, will have k positive eigenvalues satisfying $M \geq \lambda_{1,A} \geq \lambda_{2,A} \geq \cdots \geq \lambda_{k,A} > 0$ and*

$$\lim_{A\to\infty} A^2(M - \lambda_{k,A}) = \sigma^2 {z_k}^2$$

where z_k is the k^{th} positive zero of the Bessel function $J_{\nu-1/2}$.

Proof. Note that in the case $n = 1$,

$$D(L_1) = \{v \in C^\infty(0,1) : v = 0 \text{ near } x = 1 \text{ and } v' = 0 \text{ near } x = 0\}.$$

Since H2 is stronger than H5, from Theorem 3.4 with $n = 1$ and Theorem 2.8, we know

$$\limsup_{A\to\infty} A^2(M - \lambda_{j,A}) \leq \sigma^2 z_j^2 \quad \text{ for } j = 1, 2, \ldots, k.$$

Define $\alpha_k = \liminf_{A\to\infty} A^2(M - \lambda_{k,A})$. We will show that $\alpha_k = \sigma^2 z_k^2$, which will complete the proof. Since $\alpha_k = \liminf_{A\to\infty} A^2(M - \lambda_{k,A})$, there exists a sequence $\mathcal{S}$ of real numbers tending monotonically to infinity such that

$$\alpha_k = \lim_{A\to\infty} A^2(M - \lambda_{k,A}), \quad (A \in \mathcal{S})$$

exists. Since the sequences $\{A^2(M - \lambda_{j,A}) : A \in \mathcal{S}\}$, for $1 \leq j < k$, are each bounded from Theorem 3.4, by the Bolzano-Weierstrass Theorem we may assume without loss of generality that

$$\alpha_j = \lim_{A\to\infty} A^2(M - \lambda_{j,A}), \quad (A \in \mathcal{S}) \text{ for } \; j = 1, 2, \ldots, k$$

exists. Henceforth, we restrict attention to values of $A \in S$. Let $f_{j,A}$ be a normalized eigenfunction of T_A, corresponding to the eigenvalue $\lambda_{j,A}$, for $j = 1, 2, \ldots, k$. Since $D(L_1)$ is dense, we may choose $g_{j,A} \in D(L_1)$ such that $\|g_{j,A} - f_{j,A}\| < \min\left\{\frac{1}{A^3\|MI - T_A\|}, \frac{1}{A}\right\}$

and $\|g_{j,A}\| = 1$. Let $u_{j,A} = S_A g_{j,A}$. Then from Lemma 4.4, $(I + \frac{\tau}{A^2})u_{j,A} = g_{j,A}$ and $(I - S_A)g_{j,A} = \frac{\tau}{A^2}u_{j,A}$.

We will now establish a useful inequality, which will reveal the role played by H2. We have

$$\begin{aligned}\|g_{j,A} - u_{j,a}\|^2 + \left(\frac{\tau}{A^2}u_{j,A}, u_{j,A}\right) &= ((I - S_A)g_{j,A}, g_{j,A}) = \langle (I - S_A)g_{j,A}, g_{j,A}\rangle \\ &= \langle (1 - F_S(t/A))\hat{g}_{j,A}, \hat{g}_{j,A}\rangle \le \frac{1}{q^2}\langle (M - F(t/A))\hat{g}_{j,A}, \hat{g}_{j,A}\rangle \\ &= \frac{1}{q^2}\langle (MI - T_A)g_{j,A}, g_{j,A}\rangle = \frac{1}{q^2}((MI - T_A)g_{j,A}, g_{j,A}) \\ &= \frac{1}{q^2}[((MI - T_A)f_{j,A}, g_{j,A}) + ((MI - T_A)(g_{j,A} - f_{j,A}), g_{j,A})] \\ &\le \frac{1}{q^2}[(M - \lambda_{j,A}) + \|MI - T_A\|\,\|g_{j,A} - f_{j,A}\|] \\ &\le \frac{1}{q^2}\left[(M - \lambda_{j,A}) + \frac{1}{A^3}\right].\end{aligned}$$

From Lemma 4.3, $(\tau u_{j,A}, u_{j,A})$ is a nonnegative quantity, so we apply Theorem 3.4 to obtain

$$\lim_{A\to\infty} \|g_{j,A} - u_{j,a}\| = 0 \tag{4.2}$$

and $(\tau u_{j,A}, u_{j,A})$ is bounded, by M_1 say. We now wish to show that for $1 \le j \le k$, $\{u_{j,A}(x) : A \in \mathcal{S}\}$ is bounded at $x = 1$ and equicontinuous on compact subsets of $(0, 1]$. We first show equicontinuity on compact subsets of $(0, 1]$. For $0 < \epsilon \le x_1 \le x_2 \le 1$,

$$|u_{j,A}(x_2) - u_{j,A}(x_1)|^2 \le (\tau u_{j,A}, u_{j,A})\left(\int_{x_1}^{x_2} \frac{1}{x^{2\nu}}\,dx\right) \le M_1\epsilon^{-2\nu}(x_2 - x_1)$$

which implies equicontinuity on $[\epsilon, 1]$. To prove that $\{u_{j,A} : A \in \mathcal{S}\}$ is bounded at $x = 1$, we apply Lemma 4.3 to get

$$A\frac{K_{\nu+1/2}(A)}{K_{\nu-1/2}(A)}|u_{j,A}(1)|^2 \le (\tau u_{j,A}, u_{j,A}) \le M_1.$$

Then from Lemma 2.2

$$\limsup_{A\to\infty} |u_{j,A}(1)|^2 \le \lim_{A\to\infty} \frac{M_1}{A}\frac{K_{\nu-1/2}(A)}{K_{\nu+1/2}(A)} = 0$$

and thus $\{u_{j,A} : A \in \mathcal{S}\}$ is bounded at $x = 1$, and in fact $\lim_{A\to\infty} u_{j,A}(1) = 0, (A \in \mathcal{S})$. Thus, using Ascoli's theorem (boundedness at one point and equicontinuity on an interval implies uniform boundedness on that interval), there exists a subsequence $\mathcal{S}_\epsilon$ of $\mathcal{S}$ such that $\{u_{j,A} : A \in \mathcal{S}_\epsilon\}$ converges uniformly on $[\epsilon, 1]$. Since $\epsilon > 0$ can be made arbitrarily small, we can use a diagonalization argument to find a subsequence $\mathcal{S}'$ of $\mathcal{S}$ such that $\{u_{j,A} : A \in \mathcal{S}_j\}$ converges uniformly on each compact subset of $(0, 1]$.

Since we can use this argument for each j, we first find a subsequence $\mathcal{S}_1$ such that $\{u_{1,A} : A \in \mathcal{S}_1\}$ converges uniformly on each compact subset of $(0, 1]$, then we find a subsequence $\mathcal{S}_2$ of $\mathcal{S}_1$ such that $\{u_{2,A} : A \in \mathcal{S}_2\}$ converges uniformly on each compact subset of $(0,1]$, and continue this process until we find a subsequence $\mathcal{S}_k$ such that $\{u_{k,A} : A \in \mathcal{S}_k\}$ converges uniformly on each compact subset of $(0,1]$ for all $j = 1, 2, \dots, k$ simultaneously. We will assume without loss of generality that $\mathcal{S}$ is this deep subsequence. Also, we define for $0 < x \leq 1$,

$$u_j(x) = \lim_{A\to\infty} u_{j,A}(x) \quad (A \in \mathcal{S}) \text{ for } j = 1, 2, \dots, k$$

and we note that from the pointwise convergence at $x = 1$ given above,

$$u_j(1) = 0 \qquad \text{for } j = 1, 2, \dots, k.$$

The goal now is to show that $u_j \in D({L_1}^*)$ and that ${L_1}^* u_j = \frac{\alpha_j}{\sigma^2} u_j$ for each j. We begin by investigating the equality

$$\big(A^2(MI - T_A)f_{j,A}, v\big) = \big(A^2(M - \lambda_{j,A})f_{j,A}, v\big) \tag{4.3}$$

for a fixed $v \in D(L_1)$ and evaluating the limit of each side as $A \to \infty$ with $A \in \mathcal{S}$. We first show that the right side tends to $(\alpha_j u_j, v)$ and then argue that the left side tends to $\sigma^2(u_j, L_1 v)$ for each j. This will give the equality, for arbitrary $v \in D(L_1)$,

$$(u_j, L_1 v) = \Big(\frac{\alpha_j}{\sigma^2} u_j, v\Big)$$

which implies that $u_j \in D({L_1}^*)$ and that ${L_1}^* u_j = \frac{\alpha_j}{\sigma^2} u_j$. Further, since $u_j(1) = 0$, from Theorem 2.7, $u_j \in D(\tilde{L}_1)$ and so for each j, $(\frac{\alpha_j}{\sigma^2}, u_j)$ is an eigenpair of the Friedrichs extension after we show that u_j is a nontrivial function.

We write

$$\big(A^2(M - \lambda_{j,A})f_{j,A}, v\big) = A^2(M - \lambda_{j,A})(f_{j,A} - u_{j,A}, v) + A^2(M - \lambda_{j,A})(u_{j,A}, v)$$

From Theorem 3.4, $A^2(M - \lambda_{j,A})$ is bounded, so we decompose

$$(f_{j,A} - u_{j,A}, v) = (f_{j,A} - g_{j,A}, v) + (g_{j,A} - u_{j,A}, v)$$

and note that our choice of $g_{j,A}$ and (4.2) force each term to tend to 0. Thus

$$\lim_{A\to\infty} \big(A^2(M - \lambda_{j,A})f_{j,A}, v\big) = \lim_{A\to\infty} A^2(M - \lambda_{j,A})(u_{j,A}, v) = \alpha_j \lim_{A\to\infty} (u_{j,A}, v), \quad (A \in \mathcal{S}).$$

In order to apply the Lebesgue Dominated Convergence Theorem, we need an integrable function that bounds $|u_{j,A}(x)v(x)|x^{2\nu}$. From Lemma 4.3, letting $x_2 = 1$ and $x_1 = x$, we have

$$|u_{j,A}(1) - u_{j,A}(x)| \leq \sqrt{M_1}\, \frac{1}{x^\nu}$$

and since sequence $\{u_{j,A}(1)\}$ is bounded, by M_2 say, we have

$$|u_{j,A}(x)| \leq \sqrt{M_1} \frac{1}{x^\nu} + M_2 \tag{4.4}$$

so

$$|u_{j,A}(x)v(x)|x^{2\nu} \leq (\sqrt{M_1} x^\nu + M_2 x^{2\nu})|v(x)|.$$

Thus $\lim_{A\to\infty}(u_{j,A}, v) = (u_j, v)$, and we are done with the right side of (4.3).

The left side of (4.3) is more delicate. First we decompose

$$\big(A^2(MI-T_A)f_{j,A},v\big)=\big(A^2(MI-T_A)(f_{j,A}-g_{j,A}),v\big)+\big(A^2(MI-T_A)g_{j,A},v\big)$$

and the Schwarz inequality with our choice of $g_{j,A}$ shows that the first term tends to 0, so we turn to the second term and use Theorem 2.5 to obtain

$$\big(A^2(MI-T_A)g_{j,A},v\big)=\langle A^2(MI-T_A)g_{j,A},v\rangle=\langle A^2(M-F(t/A))\hat{g}_{j,A},\hat{v}\rangle.$$

We use the same decomposition of $M-F(t/A)$, with $n=1$, as in the proof of Lemma 3.1. For a given $\epsilon>0$ and corresponding $\delta>0$, we have

$$\big\langle A^2(M-F(t/A))\hat{g}_{j,A},\hat{v}\big\rangle-\big\langle A^2\sigma^2(1-F_S(t/A))\hat{g}_{j,A},\hat{v}\big\rangle=Q_1(A)+Q_2(A) \tag{4.5}$$

where

$$Q_1(A)=\big\langle A^2\left(M-G(t/A)-\sigma^2(1-F_S(t/A))\right)\hat{g}_{j,A},\hat{v}\big\rangle$$
$$\text{and } Q_2(A)=\big\langle A^2\left(G(t/A)-F(t/A)\right)\hat{g}_{j,A},\hat{v}\big\rangle.$$

Note that the left side of (4.5) has no dependence on ϵ; the terms on the right depend on G and thus on ϵ. We shall show that $Q_2(A)$ tends to 0 while $Q_1(A)$ is bounded by a constant multiple of ϵ. Since $G(u)=F(u)$ for $u<\delta$,

$$Q_2(A)=A^2\int_{\delta A}^{\infty}\left(G(t/A)-F(t/A)\right)\hat{g}_{j,A}(t)\overline{\hat{v}(t)}t^{2\nu}dt.$$

From Theorem 2.5, since $g_{j,A}\in D(L_1)$ we have

$$\int_0^\infty|\hat{g}_{j,A}(t)|^2t^{2\nu}dt=\int_0^1|g_{j,A}(t)|^2t^{2\nu}dt=\|g_{j,A}\|^2=1. \tag{4.6}$$

Using the Schwarz inequality and (4.6), we get

$$|Q_2(A)|^2\le A^4\int_{\delta A}^{\infty}|(G(t/A)-F(t/A))\hat{v}(t)|^2t^{2\nu}dt.$$

From Lemma 3.1, $|G-F|$ is bounded and a change of variables gives

$$|Q_2(A)|^2\le M_3A^5\int_\delta^\infty|\hat{v}(At)|^2(At)^{2\nu}dt$$

where M_3 is the bound on $|G-F|$. Since $\hat{v}$ is rapidly decreasing from Lemma 2.13, for A sufficiently large,

$$|Q_2(A)|^2\le M_3A^5\int_\delta^\infty(At)^{-6}\,dt\le\frac{M_3}{A}\int_\delta^\infty t^{-6}dt$$

which implies

$$\lim_{A\to\infty}|Q_2(A)|=\lim_{A\to\infty}\left|\big\langle A^2\left(G(t/A)-F(t/A)\right)\hat{g}_{j,A},\hat{v}\big\rangle\right|=0.$$

From the proof of Lemma 3.1, we have

$$|Q_1(A)|<\epsilon\int_0^\infty A^2(1-F_S(t/A))|\hat{g}_{j,A}(t)||\hat{v}(t)|t^{2\nu}dt$$
$$\le\epsilon\int_0^\infty t^2|\hat{g}_{j,A}(t)||\hat{v}(t)|t^{2\nu}dt$$

Then the Schwarz inequality and (4.6) imply

$$|Q_1(A)|^2 < \epsilon^2 \int_0^\infty t^4 |\hat{v}(t)|^2 t^{2\nu}\, dt$$

and since $\hat{v}(t)$ is a fixed rapidly decreasing function, $\int_0^\infty t^4|\hat{v}(t)|^2 t^{2\nu}\, dt$ converges to a constant M_4 independent of A and we have

$$|Q_1(A)| = \left|\left\langle A^2 \left(M - G(t/A) - \sigma^2(1 - F_S(t/A))\right) \hat{g}_{j,A}, \hat{v}\right\rangle\right| < \epsilon\sqrt{M_4}.$$

Thus from (4.5), we now have

$$\limsup_{A\to\infty} \left|\left\langle A^2(M - F(t/A))\hat{g}_{j,A}, \hat{v}\right\rangle - \left\langle A^2\sigma^2(1 - F_S(t/A))\hat{g}_{j,A}, \hat{v}\right\rangle\right| < \sqrt{\epsilon M_4},$$

which is true for arbitrary $\epsilon > 0$ and therefore the left side of (4.3) has the same limit as

$$\lim_{A\to\infty} \left\langle A^2(M - F(t/A))\hat{g}_{j,A}, \hat{v}\right\rangle = \lim_{A\to\infty} \left\langle A^2\sigma^2(1 - F_S(t/A))\hat{g}_{j,A}, \hat{v}\right\rangle.$$

Using Theorem 2.5 and the fact that $v(t) = 0$ for $t \geq 1$,

$$\left\langle A^2\sigma^2(1 - F_S(t/A))\hat{g}_{j,A}, \hat{v}\right\rangle = \sigma^2 \left(A^2(I - S_A)g_{j,A}, v\right) = \sigma^2 \left(\tau u_{j,A}, v\right).$$

Further, integrating by parts,

$$(\tau u_{j,A}, v) = -\, x^{2\nu} u'_{j,A}(x)\overline{v(x)}\Big|_0^1 + x^{2\nu} u_{j,A}(x)\overline{v'(x)}\Big|_0^1 + (u_{j,A}, \tau v).$$

Since $v \in D(L_1)$, both v and v' vanish in a neighborhood of $x = 1$ and are bounded near $x = 0$. Also, from Lemma 2.17, $u_{j,A}$ and $u'_{j,A}$ are both bounded on $[0, 1]$, so all the boundary terms vanish. So we have finally

$$\lim_{A\to\infty} \left\langle A^2(M - F(t/A))\hat{g}_{j,A}, \hat{v}\right\rangle = \sigma^2 \lim_{A\to\infty} (u_{j,A}, \tau v) = \sigma^2(u_j, L_1 v).$$

where we have used (4.4) to invoke the Lebesgue Dominated Convergence Theorem.

We will next show that the u_j are orthonormal eigenfunctions of $\tilde{L}_1$. Using (4.4) as before, the Lebesgue dominated convergence theorem shows that

$$\lim_{A\to\infty} (u_{j,A}, u_{i,A}) = (u_j, u_k).$$

On the other hand, for $1 \leq i, j \leq n$, the triangle inequality gives

$$\begin{aligned} |(u_{j,A}, u_{i,A}) - (f_{j,A}, f_{i,A})| &\leq \|u_{j,A} - f_{j,A}\|\, \|u_{i,A}\| + \|f_{j,A}\|\, \|u_{i,A} - f_{i,A}\| \\ &\leq \|u_{j,A} - f_{j,A}\| \left(\|u_{i,A} - f_{i,A}\| + 1\right) + \|u_{i,A} - f_{i,A}\| \end{aligned}$$

From our choice of $g_{j,A}$, we have

$$\|u_{j,A} - f_{j,A}\| \leq \|u_{j,A} - g_{j,A}\| + \|g_{j,A} - f_{j,A}\| \leq \|u_{j,A} - g_{j,A}\| + \frac{1}{A}.$$

It then follows from (4.2) that and we can conclude

$$\lim_{A\to\infty} |(u_{j,A}, u_{i,A}) - (f_{j,A}, f_{i,A})| = 0$$

and so

$$\lim_{A\to\infty}(u_{j,A},u_{i,A})=\lim_{A\to\infty}(f_{j,A},f_{i,A})=\begin{cases}0 & \text{if } j\neq k\\ 1 & \text{if } j=k\end{cases}.$$

Thus the u_j are orthonormal eigenfunctions of $\tilde{L}_1$.

Finally, since the $\{\lambda_{j,A}\}$ are nondecreasing, it follows that the $\{\alpha_j\}$ are nonincreasing. From Theorem 3.4, $\alpha_1 \leq \sigma^2 z_1^2$ and thus $\alpha_1 = \sigma^2 z_1^2$ since α_1/σ^2 is an eigenvalue of $\tilde{L}_1$. Using Theorem 3.4, $\alpha_j \leq \sigma^2 z_j^2$. Since the eigenvalues of $\tilde{L}_1$ are simple from Theorem 2.8, an easy induction argument concludes the proof. □

5. **Lingering Difficulties**

The careful reader will have noted that our results have depended on two formulas for T_A, given first by (1.1) via the intermediate formula for $\rho(x,y)$, and then by the formula (2.2) directly in terms of the function $F(t)$. Morevover, the inequality result of the first main theorem (Theorem 3.4), holding for all $n\geq 1$, depends only on the second of these formulas, which reveals the role played by the differential operator τ in the inversion result Lemma 2.16 (for the special case F_S). On the other hand, the second main theorem (Theorem 4.5), the equality result holding only for $n=1$, depends critically on the first of these formulas and the fact that we know the integral given in Lemma 4.1. This result is useful in two ways: it gives us knowledge of the boundary behavior of $g=S_Af$ in Lemma 4.2, which then gives Lemma 4.3, the key which allows the use of Ascoli's theorem. Our failure to find a suitable replacement for this integral in the case $n>1$ is the barrier to our effort to generalize the equality result to the case $n>1$.

Finally, we offer a brief discussion of difficulties in the Jacobi case. It is straightfoward to extend Lemma 2.16 to this case, where F_S is unchanged, but now the operator $I+\tau/A^2$ of that lemma is replaced by $I+\tau_A/A^2$ and

$$\tau_A u=(p(Ax)^{-1}((p(Ax)u')', 0<x\leq 1.$$

The added difficulty now is that τ_A depends in a complicated fashion on A. It is then necessary to know the behavior of $p(Ax)$ as $A\to\infty$, where, recall, $p(x)=\phi_0^2(x)\Delta(x)$. Using the formula

$${}_2F_1(a,b;c;z)=(1-z)^{-a}{}_2F_1(a,c-b;c;z/(z-1))$$

and the known asymptotic behavior of ${}_2F_1(z)$, it is not hard to show that $p(Ax)$ converges uniformly on compact subsets of $(0,1]$ to cx^2, where c is a constant (dependent on α and β via the gamma function and also on ρ). Thus the limiting form of the operator τ_A has the form $\tau_\infty u=x^{-2}(x^2u')'$. There are added technical difficulties, but after some effort, the inequality result of our first main theorem can be extended to the Jacobi case.

Unfortunately, we have found it more difficult to extend the equality result in our second main theorem. A search of the literature has not revealed an integral which corresponds in the Jacobi case to the one in Lemma 4.1 in the Hankel case. We have exerted some effort, without success, to the evaluation of the required integral. In the absence of such knowledge, or an entirely different approach which dispenses with such knowledge, it does not seem possible to implement the methods used here to extend the second main theorem to the Jacobi case. One cannot fail to notice, however, that the differential operator lurking in the background here is the same as the one in the Hankel case, with $\nu = 1$.

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