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A KOLMOGOROV PREDATOR-PREY SYSTEM ON A TIME SCALE

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. We use a Leggett-Williams norm-type theorem for coincidences to study the existence of periodic solutions of a Kolmogorov predator-prey system on a time scale.

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1. Introduction

In this paper, we consider the existence of positive periodic solutions of the Kolmogorov predator-prey model

$$\begin{aligned} u^{\Delta}(t) &= u(t)f\big(t,u(t),v(t)\big) \\ v^{\Delta}(t) &= v(t)g\big(t,u(t),v(t)\big). \end{aligned} \tag{1.1}$$

Here u represents the population of the prey and v the population of the predator. We reformulate the problem as $Lx = Nx$ for appropriate operators L and N and employ a theorem due to O'Regan and Zima [25] to show the existence of a solution. Throughout we assume that the following conditions hold.

- There exists an $M > 0$ such that $f(t, M, 0) = 0$ and $f(t, u, v) < 0$ if $u > M$. (M is the carrying capacity of the prey.)
- The function $f(t, u, v)$ is decreasing in v. (Predation has a negative impact on the growth of the prey.)
- For some $J > 0$, $g(t, J, 0) = 0$. (There is a minimum prey population needed to support the predators.)
- The function $g(t, u, v)$ is decreasing in v and increasing in u.

We say that the functions f and g are Kolmogorov if they satisfy the above conditions.

The system (1.1) incorporates many different types of predator-prey models. For example, the system includes prey-dependent functional response models in which the functions f and g are given by $f(u, v) = r(1-u) - v\phi(u)$ and $g(u, v) = e\phi(u) - d$.

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In Holling type I - IV schemes, ϕ is given by $\phi(u) = 1$ $\phi(u) = au$, $\phi(u) = \frac{au}{b+u}$, $\phi(u) = \frac{au^\theta}{b+u^\theta}, \theta > 0$, respectively [19, 20].

One of the major issues with prey-dependent functional response models is the so-called *paradox of enrichment* [17, 28, 29] in which an enrichment of the environment that benefits the growth of either population leads to a destabilization of the system. Further enrichment causes the extinction of either the predator or prey. Some ecologist have argued that the prey-dependent functional response models themselves are flawed and that a ratio-dependent functional response, one in which the per capita predator growth rate is related to the ratio of the abundance of the prey to the predator, are more suitable; see for example the papers [3, 4, 6, 18]. In these models $f(u,v) = r(1-u) - \phi\left(\frac{u}{v}\right), g(u,v) = e\phi\left(\frac{u}{v}\right) - d$ where typical functional responses are given by $\phi\left(\frac{u}{v}\right) = a\frac{u}{v}$, and $\phi\left(\frac{u}{v}\right) = \frac{a\frac{u}{v}}{b+c\frac{u}{v}} = \frac{au}{bv+cu}$.

Ratio-dependent models have their detractors as well; see [1]. Many researchers instead consider predator-dependent functional response schemes. Here the functions f and g have the form $f(u,v) = r(1-u) - \phi(u,v)$ and $g(u,v) = e\phi(u,v) - d$. The Beddington-DeAngelis model is a particular predator-dependent functional response model in which $\phi(u,v) = \frac{cv}{\alpha+\beta u+\gamma v}$. This model, as well as the semi-ratio dependent model, has been studied extensively in both the continuum case and the time scale case; see for example [5, 7, 8, 12, 13, 14, 15, 16, 30] and references therein. The predator-dependent functional response model accounts for group hunting by the predators, density dependent and time-consuming social interactions among predators, aggressive interaction among predators searching for food, and limited number of sites where predators can capture prey, [2].

One last class of models that we mention are the Leslie-Gower models [22, 23, 24]. Here the carrying capacity of the predator is proportional to the size of the prey population. The function g is given by $g(u,v) = b(1 - \frac{v}{au})$.

Our goal is to study the more general model (1.1). In addition to relaxing some of the typical assumptions on f and g, see [11] for more about Kolmogorov systems, our model (1.1) relaxes the constraint on time. That is, time scale models include those of the discrete time case, the continuum time case, and all time scales in between.

In Section 2 we provide some details on periodic time scales. In Section 3 we give the concepts from coincidence degree theory and a theorem due to O'Regan and Zima [25] that we employ to show the existence of a positive periodic solution to (1.1). We state and prove our main result in Section 4.

2. **Time Scale Essentials**

In this section we provide a brief description of periodic time scales and periodic functions on time scales. At the end of the section we present some definitions that we will use in Sections 3 and 4.

A time scale $\mathbb{T}$ is a closed non-empty subset of $\mathbb{R}$ together with the forward and backward jump operators, σ and ρ. We assume the reader is familiar with the notation and basic results for dynamic equations on time scales. For a thorough review of time scales we direct the reader to the monographs [9] and [10].

Definition 2.1 ([21])**.** A time scale $\mathbb{T}$ is *periodic* if there exist a $p > 0$ such that if $t \in \mathbb{T}$ then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the *period* of the time scale.

Example 2.2. The following time scales are periodic.

1. $\mathbb{T} = \bigcup_{i=-\infty}^{\infty}[(2i-1)h, 2ih]$, $h > 0$ has period $p = 2h$.
2. $\mathbb{T} = h\mathbb{Z}$ has period $p = h$.
3. $\mathbb{T} = \mathbb{R}$.
4. $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z}, m \in \mathbb{N}_0\}$, where $0 < q < 1$, has period $p = 1$.

Remark: All periodic time scales are unbounded above and below.

Definition 2.3 ([21])**.** Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. The function $f\colon \mathbb{T} \to \mathbb{R}$ is periodic with period ω if there exists a natural number n such that $\omega = np$, $f(t \pm \omega) = f(t)$ for all $t \in \mathbb{T}$, and ω is the smallest number such that $f(t \pm \omega) = f(t)$.

If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $\omega > 0$ if ω is the smallest positive number such that $f(t \pm \omega) = f(t)$ for all $t \in \mathbb{T}$.

Remark: If $\mathbb{T}$ is a periodic time scale with period p, then $\sigma(t \pm np) = \sigma(t) \pm np$ for all natural numbers n. Consequently, the graininess function μ satisfies $\mu(t \pm np) = \sigma(t \pm np) - (t \pm np) = \sigma(t) - t = \mu(t)$ and is therefore a periodic function with period p.

By the notation $[a, b]$ we mean

$$[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$$

unless otherwise specified. The intervals $[a, b), (a, b]$, and (a, b) are defined similarly.

Let $\mathbb{T}$ be a periodic time scale. Given a periodic function f with period ω, define

$$\kappa = \min\left\{[0, +\infty) \cap \mathbb{T}\right\}, \quad I_\omega = [\kappa, \kappa + \omega] \cap \mathbb{T}, \quad \bar{f} = \frac{1}{\omega}\int_{I_\omega} f(s)\Delta s.$$

3. **Coincidence Degree Preliminaries**

Our goal in this section is to introduce the concepts from coincidence degree theory that we need in the sequel. We end this section by stating a Leggett-Williams norm-type theorem for coincidences due to O'Regan and Zima [25].

Let X and Z be normed spaces. A linear mapping $L : \text{dom } L \subset X \to Z$ is called a *Fredholm mapping* if the following two conditions hold:

(i) $\ker L$ has a finite dimension, and
(ii) $\text{Im } L$ is closed and has finite codimension.

If L is a Fredholm mapping, its (Fredholm) *index* is the integer, $\text{Ind } L$, given by $\text{Ind } L = \dim \ker L - \text{codim Im } L$.

Let $L : \text{ dom } L \subset X \to Z$ be a Fredholm map of index zero. Then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that

$$\text{Im } P = \ker L, \quad \ker Q = \text{Im } L, \quad X = \ker L \oplus \ker P, \quad Z = \text{Im } L \oplus \text{Im } Q$$

and the mapping

$$L|_{\text{dom } L \cap \ker P} : \text{dom } L \cap \ker P \to \text{Im } L$$

is invertible. The inverse of $L_P := L|_{\text{dom } L \cap \ker P}$ is denoted by

$$K_P : \text{Im } L \to \text{dom } L \cap \ker P.$$

Since $\dim \text{Im } Q = \text{codim Im } L$ there exists an isomorphism $J : \text{Im } Q \to \ker L$. Furthermore, the equation $Lx = \lambda Nx$ is equivalent to

$$x = (P + JQN)x + \lambda K_P(I - Q)Nx,$$

for all $\lambda \in (0, 1]$. Our Nagumo operator $N : X \to Z$ is defined in Section 4.

Let C be a cone in X. Then C induces a partial order on X by

$$x \preceq y \text{ if, and only if, } y - x \in C.$$

We will use the following property for cones in a Banach space (see [25, 26, 27]).

Lemma 3.1. *Let C be a cone in X. Then for every $s \in C \setminus \{\theta\}$ there exists a positive number $\eta(s)$ such that*

$$\|x + s\| \geq \eta(s)\|x\|$$

for all $x \in C$.

It follows that if $\eta(s) > 0$ is such that $\|x + s\| \geq \eta(s)\|x\|$ for all $x \in C$, then for all $\lambda > 0$,

$$\|x + \lambda s\| \geq \eta(s)\|x\|.$$

Let $\gamma : X \to C$ be a retraction, that is, γ is a continuous mapping such that $\gamma(x) = x$ for all $x \in C$. Let

$$\Psi = P + JQN + K_P(I - Q)N \tag{3.1}$$

and set

$$\Psi_\gamma = \Psi \circ \gamma. \tag{3.2}$$

We end this section with the following existence theorem, (see [25]).

Theorem 3.2. *Let $C \subset X$ be a cone and let Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Let L be a Fredholm operator of index zero. Assume that:*

(i) *$QN : X \to Y$ is continuous and bounded and $K_{P,Q} : X \to X$ is compact on every bounded subset of X;*
(ii) *$Lx \neq \lambda Nx$ for all $x \in C \cap \partial\Omega_2 \cap$ dom L and $\lambda \in (0, 1)$;*
(iii) *γ maps subset of $\overline{\Omega}_2$ into bounded subsets of C;*
(iv) $\deg_B \big([I - (P + JQN)\gamma]|_{\ker L}, \ker L \cap \Omega_2, \theta\big) \neq 0$;
(v) *there exists $s_0 \in C \setminus \{\theta\}$ such that $\|x\| \le \eta(s_0)\|\Psi x\|$ for all $x \in C(s_0) \cap \partial\Omega_1$, where $C(s_0) = \{x \in C : \lambda s_0 \preceq x$ for some $\lambda > 0\}$, and $\eta(s_0)$ is such that $\|x + s_0\| \ge \eta(s_0)\|x\|$ for every $x \in C$;*
(vi) *$(P + JQN)\gamma(\partial\Omega_2) \subset C$; and*
(vii) *$\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$.*

Then the equation $Lx = Nx$ has a solution in $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

4. **Main Result**

We begin this last section by defining the spaces in which we work. Let $\omega > 0$ and define

$$\mathcal{L}^\omega = \left\{(u, v) \in C(\mathbb{T}, \mathbb{R}^2) : u(t + \omega) = u(t), v(t + \omega) = v(t) \text{ for all } t \in \mathbb{T}\right\}.$$

The norm on $\mathcal{L}^\omega$ is given by $\|(u, v)\| = \max\left\{\max_{t\in I_\omega} |u(t)|, \max_{t\in I_\omega} |v(t)|\right\}$. Then, $\big(\mathcal{L}^\omega, \|\cdot\|\big)$ is a Banach space. Define the sets $\mathcal{L}^\omega_0$ and $\mathcal{L}^\omega_c$ by

$$\mathcal{L}^\omega_0 = \{(u, v) \in \mathcal{L}^\omega : \bar{u} = \bar{v} = 0\}$$

and

$$\mathcal{L}^\omega_c = \left\{(u, v) \in \mathcal{L}^\omega : \big(u(t), v(t)\big) \equiv (c_1, c_2) \in \mathbb{R}^2\right\}.$$

Lemma 4.1. $\mathcal{L}^\omega = \mathcal{L}^\omega_0 \oplus \mathcal{L}^\omega_c$.

Proof. Let $(u,v) \in \mathcal{L}^\omega$ and let $a = \bar{u}$ and $b = \bar{v}$. Note that $(a,b) \in \mathcal{L}_c^\omega$. Define

$$\mu(t) = u(t) - a, \quad \nu(t) = v(t) - b.$$

Then

$$\bar{\mu} = \frac{1}{\omega}\int_{I_\omega} \mu(t)\Delta t = \frac{1}{\omega}\int_{I_\omega} u(t)\Delta t - \frac{1}{\omega}\int_{I_\omega} a\Delta t = a - a\frac{1}{\omega}\omega = 0.$$

Likewise $\bar{\nu} = 0$. Hence there exist $(\mu,\nu) \in \mathcal{L}_0^\omega$ and $(a,b) \in \mathcal{L}_c^\omega$ such that $u = \mu + a$ and $v = \nu + b$.

Suppose there exist $(a_1,b_1),(a_2,b_2) \in \mathcal{L}_c^\omega$ and $(\mu_1,\nu_1),(\mu_2,\nu_2) \in \mathcal{L}_0^\omega$ such that for all $t \in \mathbb{T}$

$$\begin{aligned} u(t) &= \mu_1(t) + a_1 = \mu_2(t) + a_2 \text{ and} \\ v(t) &= \nu_1(t) + b_1 = \nu_2(t) + b_2. \end{aligned}$$

Then,

$$\bar{u} = \frac{1}{\omega}\int_{I_\omega} \big(\mu_1(t) + a_1\big)\Delta t = a_1$$

and

$$\bar{u} = \frac{1}{\omega}\int_{I_\omega} \big(\mu_2(t) + a_2\big)\Delta t = a_2.$$

Hence $a_1 = a_2$ and consequently $\mu_1(t) = \mu_2(t) + a_2 - a_1 = \mu_2(t)$ for all $t \in \mathbb{T}$. That is the representation of u is unique. Likewise the representation of v is unique and the proof is complete. $\square$

We are now ready to define our operators L and N. To this end let $X = Z = \mathcal{L}^\omega$. The operator $L : \text{dom } L \subset X \to Z$ is given by

$$L\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} u^\Delta(t) \\ v^\Delta(t) \end{bmatrix}, \quad t \in \mathbb{T},$$

and $N : X \to Z$ is given by

$$N\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} u(t)f(t,u(t),v(t)) \\ v(t)g(t,u(t),v(t)) \end{bmatrix}, \quad t \in \mathbb{T}.$$

Lemma 4.2. *The mapping L : dom $L \subset X \to Z$ is a Fredholm mapping of index zero.*

Proof. Let $g = (g_1,g_2) \in \text{Im } L \subset Z$. Then there exists $(u,v) \in \text{dom } L \subset X$ such that $u^\Delta = g_1, v^\Delta = g_2$. Since $(u,v) \in X = \mathcal{L}^\omega$ then

$$u(\kappa + \omega) - u(\kappa) = \int_{I_\omega} g_1(t)\Delta t = 0,$$

$$v(\kappa + \omega) - v(\kappa) = \int_{I_\omega} g_2(t)\Delta t = 0.$$

Hence $g = (g_1,g_2) \in \mathcal{L}_0^\omega$ and so $\text{Im } L \subseteq \mathcal{L}_0^\omega$.

Now let $g = (g_1, g_2) \in \mathcal{L}_0^\omega$ and define $u(t) = \int_\kappa^t g_1(t)\Delta t + u(\kappa)$ and $v(t) = \int_\kappa^t g_2(t)\Delta t + v(\kappa)$. Then $u^\Delta = g_1, v^\Delta = g_2$, $u(\kappa+\omega) = \int_{I_\omega} g_1(t)\Delta t + u(\kappa) = u(\kappa)$, and $v(\kappa+\omega) = \int_{I_\omega} g_2(t)\Delta t + v(\kappa) = v(\kappa)$. That is, there exists a $(u, v) \in \operatorname{dom} L$ such that

$$L \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}.$$

Consequently, $g = (g_1, g_2) \in \operatorname{Im} L$ and so $\mathcal{L}_0^\omega \subseteq \operatorname{Im} L$. As such we have $\mathcal{L}_0^\omega = \operatorname{Im} L$.

Let $(u, v) \in \ker L$. Then $u^\Delta = 0, v^\Delta = 0$ and so $u(t) = c_1, v(t) = c_1$ for all $t \in \mathbb{T}$. The converse also applies; If $u(t) = c_1, v(t) = c_2$ for all $t \in \mathbb{T}$ then $u^\Delta = 0 = v^\Delta$. Hence $\ker L = \mathcal{L}_c^\omega \cong \mathbb{R}^2$.

Since $\operatorname{Im} L$ is closed in Z and $\operatorname{codim} \operatorname{Im} L = \dim\ker L = 2$, then $\operatorname{Ind} L = \dim\ker L - \operatorname{codim} \operatorname{Im} L = 0$. The operator L is a Fredholm mapping of index 0 and the proof is complete. ☐

Define the functions $P : X \to X$ and $Q : Z \to Z$ by

$$P \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \overline{u} \\ \overline{v} \end{bmatrix}, \quad \text{and} \quad Q \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \overline{g}_1 \\ \overline{g}_2 \end{bmatrix}.$$

Note that

$$P^2 \begin{bmatrix} u \\ v \end{bmatrix} = P \begin{bmatrix} \overline{u} \\ \overline{v} \end{bmatrix} = \begin{bmatrix} \overline{u} \\ \overline{v} \end{bmatrix} = P \begin{bmatrix} u \\ v \end{bmatrix}$$

and that P is continuous. Hence P is a continuous projector and furthermore $\operatorname{Im} P = \ker L = \mathcal{L}_c^\omega$. We also can show that $Q : Z \to Z$ is a continuous projector such that $\ker Q = \mathcal{L}_0^\omega = \operatorname{Im} L$.

Define the function $G(t, s)$ by

$$G(t,s) = \frac{1}{\omega} \begin{cases} s+\omega, & \kappa \le s \le t \le \kappa+\omega \\ s, & \kappa \le t < s \le \kappa+\omega \end{cases}. \tag{4.1}$$

The inverse mapping of $L|_{\operatorname{dom} L \cap \ker P}$, $K_P : \operatorname{Im} L \subset Z \to \operatorname{dom} L \cap \ker P$, is given by

$$\big(K_P(g_1, g_2)\big)(t) = \begin{bmatrix} \int_{I_\omega} G(t,s) g_1(s)\Delta s \\ \int_{I_\omega} G(t,s) g_2(s)\Delta s \end{bmatrix}.$$

Assume that the following conditions hold.

(H1) For all u, v, the mappings $t \mapsto f(t, u, v)$ and $t \mapsto g(t, u, v)$ are measurable and satisfy $f(t+\omega, u, v) = f(t, u, v)$, $g(t+\omega, u, v) = g(t, u, v)$ for all $t \in \mathbb{T}$.

(H2) For all $t \in I_\omega$, the mappings $(u, v) \mapsto f(t, u, v)$ and $(u, v) \mapsto g(t, u, v)$ are continuous.

(H3) For all $\rho > 0$, there exists α_ρ with $\int_{I_\omega} \alpha_\rho(s)\Delta s < \infty$ such that for all $t \in I_\omega$ and for all (u, v) with $|u| < \rho$ and $|v| < \rho$, we have $|f(t, u, v)| \le \alpha_\rho(t)$ and $|g(t, u, v)| \le \alpha_\rho(t), t \in I_\omega$.

Lemma 4.3. *Under conditions* (H1)–(H3), $QN : X \to Z$ *is continuous and bounded and* $K_P(I-Q)N : X \to X$ *is compact on every bounded subset of* X.

Proof. Let $E \subset X$ be a nonempty bounded set and let $\rho > 0$ be such that $\|(u,v)\| \leq \rho$ for all $(u,v) \in E$. Let $\alpha_\rho(t)$ be such that $|f(t,u,v)| \leq \alpha_\rho(t)$ and $|g(t,u,v)| \leq \alpha_\rho(t)$. By (H3) we have $\frac{1}{\omega}\int_{I_\omega} |u(s)||f\big(s,u(s),v(s)\big)|\Delta s \leq \frac{\rho}{\omega}\int_{I_\omega} \alpha_\rho(s)\Delta s < \infty$ for all $t \in \mathbb{T}$. It follows easily that $QN(E)$ is uniformly bounded.

It is clear that the functions $QN(u,v)(t)$ are uniformly continuous on E. By the Arzelà-Ascoli Theorem $QN(E)$ is relatively compact. It also can be shown the $K_{PQ}N(E)$ is relatively compact and the proof is complete. ☐

Define the set

$$C = \{(u,v) \in X : u(t) \geq 0, v(t) \geq 0, t \in \mathbb{T}\}.$$

Then C is a cone in X. We employ the retraction $\gamma : X \to C$ given by

$$\big(\gamma(u,v)\big)(t) = \big(|u(t)|, |v(t)|\big), \tag{4.2}$$

and use the identity for our isomorphism, $J \equiv I : \operatorname{Im} Q \to \ker L$. Define

$$\tilde{G}(t,s) = \frac{1}{\omega}\begin{cases} (1+\omega+\kappa-\Gamma)-(t-s), & \kappa \leq s < t \leq \kappa+\omega, \\ (1+\kappa-\Gamma)+(s-t), & \kappa \leq t \leq s \leq \kappa+\omega, \end{cases} \tag{4.3}$$

where $\Gamma = \frac{1}{\omega}\int_{I_\omega} s\Delta s$.

In addition to (H1)–(H3), assume that f and g satisfy the following:

(H4) There exists an $R > M > 0$ such that $g(t,u,R) < 0$ for all $0 \leq u \leq R$ and $t \in \mathbb{T}$.

(H5) There exists a $\beta \in \big(0, \frac{1}{1+\kappa-\Gamma}\big]$ such that

$$f(t,u,v) > -\min\{\beta,1\} \text{ and } g(t,u,v) > -\min\{\beta,1\}$$

for all $(t,u,v) \in I_\omega \times [0,R] \times [0,R]$.

(H6) There exists an $r \in (0,R)$, a $t_0 \in I_\omega$, and continuous functions $h_1 : I_\omega \to [0,+\infty)$, $h_2 : (0,r] \times (0,r] \to [0,+\infty)$, $h_3 : I_\omega \to [0,+\infty)$, $h_4 : (0,r] \times (0,r] \to (-\infty,0]$, such that $h_2(u,v)$ is nonincreasing in u and v, $h_4(u,v)$ is nondecreasing in u and v, $f(s,u,v) \geq h_1(s)h_2(u,v)$ and $g(s,u,v) \leq h_3(s)h_4(u,v)$ for all $t \in I_\omega, u,v \in (0,r]$, and

$$h_2(r,r)\int_{I_\omega} \tilde{G}(t_0,s)h_1(s)\Delta s \geq 1, \tag{4.4}$$

and

$$h_4(r,r)\int_{I_\omega} \tilde{G}(t_0,s)h_3(s)\Delta s \leq -2. \tag{4.5}$$

Theorem 4.4. *Assume that f and g are Kolmogorov and satisfy conditions* (H1)–(H6)*. Then the predator-prey system*

$$\begin{aligned} u^{\Delta}(t) &= u(t)f\big(t, u(t), v(t)\big) \\ v^{\Delta}(t) &= v(t)g\big(t, u(t), v(t)\big) \end{aligned}$$

has a positive periodic solution $u^*(t), v^*(t), t \in \mathbb{T}$ such that $r \leq \|(u^*, v^*)\| \leq R$.

Proof. Lemma 4.3 ensures that condition (i) of Theorem 3.2 holds. By the Kolmogorov condition on f, we have $f(t, R, v) < 0$ for all $0 \leq v \leq R, t \in \mathbb{T}$. Define the sets

$$\begin{aligned} \Omega_1 = \Big\{ (u, v) : \frac{1}{2}\|(u, v)\| < |u(t)| < r \text{ and} \\ \frac{1}{2}\|(u, v)\| < |v(t)| < r \text{ for all } t \in I_\omega \Big\} \end{aligned}$$

and

$$\Omega_2 = \big\{ (u, v) : \|(u, v)\| < R \big\}.$$

It is easy to see that the sets Ω_1 and Ω_2 are open and bounded and that $\overline{\Omega}_1 \subset \Omega_2$. Moreover, $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$.

We first show that condition (ii) of Theorem 3.2 is satisfied. If not then there exist $(u_0, v_0) \in C \cap \partial\Omega_2 \cap \operatorname{dom} L$ and $\lambda_0 \in (0, 1)$ such that

$$L(u_0, v_0) = \lambda_0 N(u_0, v_0).$$

Suppose that $\|(u_0, v_0)\| = R = \max_{t \in I_\omega} |u(t)|$. Then there exists a $t_1 \in \mathbb{T}$ such that u^{Δ} has a generalized zero at t_1. That is, either $u^{\Delta}(t_1) = 0$ or $u^{\Delta}(t_1)u^{\Delta}(\sigma(t_1)) < 0$.

If $u^{\Delta}(t_1) = 0$, then $0 = u^{\Delta}(t_1) = u(t_1)f\big(t_1, u(t_1), v(t_1)\big)$. Hence

$$0 = f(t_1, R, v^*)$$

for some $v^* = v(t_1) \in [0, R]$, which is a contradiction. If $u^{\Delta}(t_1)u^{\Delta}(\sigma(t_1)) < 0$, then either $u^{\Delta}(t_1) > 0$ or $u^{\Delta}(\sigma(t_1)) > 0$. Suppose that $u^{\Delta}(t_1) > 0$. Then $f(t_1, R, v^*) < 0 < f(t_1, R, v^*)$, a clear contradiction. Likewise we have a contradiction if $u^{\Delta}(\sigma(t_1)) > 0$. Hence $\|(u_0, v_0)\| = R \neq \max_{t \in I_\omega} |u(t)|$. Using a similar argument we obtain $\|(u_0, v_0)\| = R \neq \max_{t \in I_\omega} |v(t)|$. Since we obtain a contradiction under all circumstances, condition (ii) holds.

The retraction is given by (4.2) trivially satisfies condition (iii) of Theorem 3.2.

We next show that condition (iv) holds. Define the function $\tilde{H}$ by

$$\tilde{H}\big((\xi_1, \xi_2), \lambda\big) = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} - \lambda \begin{bmatrix} |\xi_1| \\ |\xi_2| \end{bmatrix} - \lambda \begin{bmatrix} \int_{I_\omega} |\xi_1| f(s, |\xi_1|, |\xi_2|)\Delta s \\ \int_{I_\omega} |\xi_2| g(s, |\xi_1|, |\xi_2|)\Delta s \end{bmatrix}$$

for $\xi_1, \xi_2 \in \mathbb{R}$ and $\lambda \in [0,1]$. Let $(u,v) \in \ker L \cap \Omega_2$. Then $\big(u(t), v(t)\big) \equiv (c_1, c_2)$ for some $c_1, c_2 \in [0,R]$ and for all $t \in I_\omega$. Assume either $\tilde{H}\big((R,c_2),\lambda\big) = 0$ or $\tilde{H}\big((c_1,R),\lambda\big) = 0$. If $\tilde{H}\big((R,c_2),\lambda\big) = 0$, then

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R \\ c_2 \end{bmatrix} - \lambda \begin{bmatrix} R \\ c_2 \end{bmatrix} - \lambda \begin{bmatrix} \int_{I_\omega} R f(s,R,c_2)\Delta s \\ \int_{I_\omega} c_2 g(s,R,c_2)\Delta s \end{bmatrix}. \tag{4.6}$$

From the Kolmogorov conditions, condition (H4) and (4.6), we obtain the absurd statement

$$0 > \lambda R \int_{I_\omega} f(s,R,c_2)\Delta s = (1-\lambda)R > 0.$$

As such $\tilde{H}\big((R,c_2),\lambda\big) \neq 0$. Similarly we have $\tilde{H}\big((c_1,R),\lambda\big) \neq 0$. Thus,

$$\deg_B\left(\tilde{H}\big((c_1,c_2),1\big), \ker L \cap \Omega_2, \theta\right) = \deg_B\left(\tilde{H}\big((c_1,c_2),0\big), \ker L \cap \Omega_2, \theta\right).$$

Since

$$\deg_B\left(\tilde{H}\big((c_1,c_2),0\big), \ker L \cap \Omega_2, \theta\right) = \deg_B\left(I, \ker L \cap \Omega_2, \theta\right) = 1,$$

then $\deg_B\left([I - (P + JQN)\gamma]|_{\ker L}, \ker L \cap \Omega_2, \theta\right) \neq 0$ and condition (iv) of Theorem 3.2 holds.

Note that condition (v) of Theorem 3.2 reads that there exists a $(u_0,v_0) \in C \setminus \{\theta\}$ such that $\|(u,v)\| \leq \eta(u_0,v_0)\|\Psi(u,v)\|$ for all $(u,v) \in C(u_0,v_0) \cap \partial\Omega_1$, where

$$C(u_0,v_0) = \{(u,v) \in C : \lambda(u_0,v_0) \preceq (u,v) \text{ for some } \lambda > 0\}$$

and $\eta(u_0,v_0)$ is such that $\|(u,v) + (u_0,v_0)\| \geq \eta(u_0,v_0)\|(u,v)\|$ for all $(u,v) \in C$.

Let $u_0(t) = v_0(t) \equiv 1$ and let $\eta(u_0,v_0) = 1$. Then $(u_0,v_0) \in C \setminus \{\theta\}$ and

$$C(u_0,v_0) = \{(u,v) \in C : u(t) > 0, v(t) > 0, t \in I_\omega\}.$$

Since J is the identity map, then Ψ given in (3.1) becomes

$$\begin{aligned}
\Psi(u,v)(t) &= \begin{bmatrix} \Psi_1(u,v)(t) \\ \Psi_2(u,v)(t) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\omega}\int_{I_\omega} u(s)\Delta s + \int_{I_\omega} u(s)f(s,u(s),v(s))\Delta s \\ \quad + \int_{I_\omega} G(t,s)\Big(u(s)f(s,u(s),v(s)) \\ \quad - \int_{I_\omega} u(r)f(r,u(r),v(r))\Delta r\Big)\Delta s \\ \frac{1}{\omega}\int_{I_\omega} v(s)\Delta s + \int_{I_\omega} v(s)g(s,u(s),v(s))\Delta s \\ \quad + \int_{I_\omega} G(t,s)\Big(v(s)g(s,u(s),v(s)) \\ \quad - \int_{I_\omega} v(r)g(r,u(r),v(r))\Delta r\Big)\Delta s \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\omega}\int_{I_\omega} u(s)\Delta s + \int_{I_\omega} \tilde{G}(t,s)u(s)f(s,u(s),v(s))\Delta s \\ \frac{1}{\omega}\int_{I_\omega} v(s)\Delta s + \int_{I_\omega} \tilde{G}(t,s)v(s)g(s,u(s),v(s))\Delta s \end{bmatrix},
\end{aligned}$$

where $\tilde{G}(t,s)$ is define in (4.3).

Let $(u,v) \in C(u_0,v_0) \cap \partial\Omega_1$. Then $u(t) \geq 0, v(t) \geq 0, t \in I_\omega$, $0 < \|(u,v)\| < r$ and $u(t) \geq \frac{1}{2}\|(u,v)\|$ and $v(t) \geq \frac{1}{2}\|(u,v)\|$ for all $t \in I_\omega$. By (H6), we have

$$\begin{aligned}
\Psi_1(t_0) &= \frac{1}{\omega}\int_{I_\omega} u(s)\Delta s + \int_{I_\omega} \tilde{G}(t_0,s)u(s)f(s,u(s),v(s))\Delta s \\
&\geq \frac{1}{2\omega}\int_{I_\omega} \|(u,v)\|\Delta s + \frac{1}{2}\int_{I_\omega} \tilde{G}(t_0,s)\|(u,v)\|f(s,u(s),v(s))\Delta s \\
&\geq \|(u,v)\|\left[\frac{1}{2} + \frac{1}{2}\int_{I_\omega} \tilde{G}(t_0,s)h_1(s)h_2(u(s),v(s))\Delta s\right] \\
&\geq \|(u,v)\|\left[\frac{1}{2} + \frac{1}{2}h_2(r,r)\int_{I_\omega} \tilde{G}(t_0,s)h_1(s)\Delta s\right] \\
&\geq \|(u,v)\|.
\end{aligned}$$

In this case, $\max_{t\in I_\omega} |\Psi_1(t)| \geq |\Psi_1(t_0)| \geq \|(u,v)\|$.

Also, by (H6) we have

$$\begin{aligned}
\Psi_2(t_0) &= \frac{1}{\omega}\int_{I_\omega} v(s)\Delta s + \int_{I_\omega} \tilde{G}(t_0,s)v(s)g(s,u(s),v(s))\Delta s \\
&\leq r + r\int_{I_\omega} \tilde{G}(t_0,s)g(s,u(s),v(s))\Delta s \\
&\leq r + r\int_{I_\omega} \tilde{G}(t_0,s)h_3(s)h_4(u(s),v(s))\Delta s \\
&\leq r\left[1 + h_4(r,r)\int_{I_\omega} \tilde{G}(t_0,s)h_3(s)\Delta s\right] \\
&\leq -r.
\end{aligned}$$

So, $\max_{t\in I_\omega} |\Psi_2(t)| \geq |\Psi_2(t_0)| \geq r > \|(u,v)\|$. Thus,

$$\begin{aligned}
\|\Psi(u,v)\| &= \max\left\{\max_{t\in I_\omega}|\Psi_1(t)|, \max_{t\in I_\omega}|\Psi_2(t)|\right\} \\
&\geq \|(u,v)\| = \eta(u_0,v_0)\|(u,v)\|,
\end{aligned}$$

and condition (v) holds.

By (H5), if $(u,v) \in \partial\Omega_2$ then

$$\begin{aligned}
\frac{1}{\omega}\int |u(s)|\big(1 + f(s,|u(s)|,|v(s)|)\big)\Delta s &\geq 0, \quad \text{and} \\
\frac{1}{\omega}\int |v(s)|\big(1 + g(s,|u(s)|,|v(s)|)\big)\Delta s &\geq 0.
\end{aligned}$$

Hence $(P + JQN)\gamma(\partial\Omega_2) \subset C$ and condition (vi) is true.

Finally we show that condition (vii) of Theorem 3.2 holds. Let $(u,v) \in \overline{\Omega}_2 \setminus \Omega_1$. Then $\max_{t\in I_\omega} |u(t)| \leq R$ and $\max_{t\in I_\omega} |v(t)| \leq R$. By the definition of Ψ_γ, (3.2), we have

$$\begin{aligned}\Psi_\gamma(u,v)(t) &= \begin{bmatrix} \Psi_{\gamma 1}(u,v)(t) \\ \Psi_{\gamma 2}(u,v)(t) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\omega}\int_{I_\omega} |u(s)|\Delta s + \int_{I_\omega} \tilde{G}(t,s)|u(s)|f(s,|u(s)|,|v(s)|)\Delta s \\ \frac{1}{\omega}\int_{I_\omega} |v(s)|\Delta s + \int_{I_\omega} \tilde{G}(t,s)|v(s)|g(s,|u(s)|,|v(s)|)\Delta s \end{bmatrix}.\end{aligned}$$

Since $\beta \in \big(0, \frac{1}{1+\kappa-\Gamma}\big]$, then $1-\beta\omega\tilde{G}(t,s) \geq 0$ for all $t, s \in I_\omega$. By (H5),

$$\begin{aligned}\Psi_{\gamma 1}(u,v)(t) &= \frac{1}{\omega}\int_{I_\omega} |u(s)|\Delta s + \int_{I_\omega} \tilde{G}(t,s)|u(s)|f(s,|u(s)|,|v(s)|)\Delta s \\ &\geq \frac{1}{\omega}\int_{I_\omega} |u(s)|\big(1-\beta\omega\tilde{G}(t,s)\big)\Delta s \\ &\geq 0.\end{aligned}$$

Likewise, $\Psi_{\gamma 2}(u,v)(t) \geq 0$. Thus $\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$ and condition (vii) of Theorem 3.2 holds.

Since all the conditions of Theorem 3.2 hold then there exists a solution of (1.1) and the proof is complete. □

REFERENCES

[1] P. A. Abrams, The fallacies of "ratio-dependent" predation, *Ecology*, 75:1842–1850, 1994.

[2] P. A. Abrams and L. R. Ginzberg, The nature of predation: prey dependent, ratio dependent or neither?, *Trends Ecol. Evol.*, 15:337–341, 2000.

[3] R. Arditi and L. R. Ginzburg, Coupling in predator-prey dynamics:ratio-dependence, *J. Theor. Biol.*, 139:311–326, 1989.

[4] R. Arditi, L. R. Ginzberg, and H. R. Akcakaya, Variation in plankton densities among lakes: a case for ratio-dependent models, *Am. Nat.*, 138:1287–1296, 1991.

[5] J. R. Beddington, Mutual interference between parasites or predators and its effect on searching efficiency, *J. Animal Ecol.*, 44:331–340, 1975.

[6] E. Beretta and Y. Kuang, Global analyses in some delayed ratio-dependent predator-prey systems, *Nonlinear Anal.*, 32:381–408, 1998.

[7] M. Bohner, M. Fan, and J. Zhang, Existence of periodic solutions in predator-prey and competition dynamic systems, *Nonlinear Anal. Real World Appl.*, 7:1193–1204, 2006.

[8] M. Bohner, M. Fan, and J. Zhang, Periodicity of scalar dynamic equations and applications to population models, *J. Math. Anal. Appl.*, 330:1–9, 2007.

[9] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales, An introduction with Applications*, Birkhäuser, Boston, 2001.

[10] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[11] F. Brauer and C. Castillo-Chávex, *Mathematical Models in Population Biology and Epidemiology*, Springer-Verlag, New York, 2001.

[12] R. S. Cantrell and C. Cosner, On the dynamics of predator-prey models with the Beddington-DeAngelis functional response, *J. Math. Anal. Appl.*, 257:206–222, 2001.

[13] D. L. DeAngelis, R. A. Goldstein, and R. V. O'Neill, A model for trophic interaction, *Ecology*, 56:881–892, 1975.

[14] M. Fazly and M. Hesaaraki, Periodic solutions for predator-prey systems with Beddington-DeAngelis functional response on time scales, *Nonlinear Anal. Real World Appl.*, 9:1224–1235, 2008.

[15] M. Fazly and M. Hesaaraki, Periodic solutions for a semi-ratio-dependent predator-prey dynamical system with a class of functional responses on time scales, *Discrete Contin. Dyn. Syst. Ser. B*, 9:267–279, 2008.

[16] S. Gakkar and R. K Naji, Seasonally perturbed prey-predator system with predator-dependent functional response, *Chaos Solitons Fractals*, 18:1075–1083, 2003.

[17] N. G. Hairston, F. E. Smith L. B. Slobodkin, Community structure, population control and competition, *American Naturalist*, 94:421–425, 1960.

[18] I. Hanski, Theunctional response of predator: worries about scale, *TREE*, 6:141–142, 1991.

[19] C. S. Holling, The functional response of predator to prey density and its role in mimicry and population regulation, *Memories of the Entomological Society of Canada*, 45:1–60, 1965.

[20] C. Ji, D. Jiang, and N. Shi, Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation, *J. Math. Anal. Appl.*, 359:482–498, 2009.

[21] E. R. Kaufmann and Y. N. Raffoul, Periodic solutions for a neutral nonlinear dynamical equation on a time scale, *J. Math Anal. Appl.*, 319:315–325, 2006.

[22] P. H. Leslie, Some further notes on the use of matrices in certain population mathematics, *Biometrika*, 33:213–245, 1948.

[23] P. H. Leslie, A stochastic model for studying the properties of certain biological systems by numerical methods, *Biometrika*, 45:16–31, 1958.

[24] P. H. Leslie and J. C. Gower, The properties of a stochastic model for the predator-prey type of interaction between two species, *Biometrika*, 47:219–234, 1960.

[25] D. O'Regan and M. Zima, Leggett-Williams norm-type theorems for coincidences, *Arch. Math.*, 87:233–244, 2006.

[26] W. V. Petryshyn, On the solvability of $x \in Tx + \lambda Fx$ in quasinormal cones with T and F k-set contractive, *Nonlinear Anal.*, 5:585–591, 1981.

[27] W. V. Petryshyn, Existence of fixed points of positive k-set-contractive maps as consequences of suitable boundary conditions, *J. London Math. Soc.*, 38:503–512, 1988.

[28] M. L. Rosenzweig, Paradox of enrichment; destabilization of exploitative ecosystem in ecological time, *Science*, 171:385–387, 1971.

[29] S. Roy and J. Chattopadhyay, The stability of ecosystems: A brief overview of the paradox of enrichment, *J. Biosci*, 32:421–428, 2007.

[30] D. Xiaoquan, Periodicity in a delayed semi-ration-dependent predator-prey system, *Appl. Math. J. Chinese Univ. Ser. B*, 20:151–158, 2005.