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# GLOBAL ATTRACTIVITY IN A NONLINEAR DIFFERENCE EQUATION AND APPLICATIONS TO DISCRETE POPULATION MODELS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

**ABSTRACT.** Consider the following difference equation of order $k+1$

(0.1) $$x_{n+1} = (1-t_n)x_n + t_n f(x_{n-k}), \quad n = 0, 1, \dots$$

where $f : [0, B) \to [0, B)$, $B \le \infty$, is a continuous function, $\{t_n\}$ is a sequence in $[0, 1)$ and $k$ is a nonnegative integer. We establish some sufficient conditions for the global attractivity of positive solutions of this equation. By applying these results to some discrete population models, several new global attractivity criteria are obtained.

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## 1. Introduction

Consider the following difference equation of order $k+1$

(1.1) $$x_{n+1} = (1-t_n)x_n + t_n f(x_{n-k}), \quad n = 0, 1, \dots$$

where $f : [0, B) \to [0, B)$, $B \le \infty$, is a continuous function, $\{t_n\}$ is a sequence in $[0, 1)$ and $k$ is a nonnegative integer. When $k = 0$ and $f$ is a nonexpansive function defined on a finite closed interval, Eq. (1.1) reduces to the first order difference equation

(1.2) $$x_{n+1} = (1-t_n)x_n + t_n f(x_n), \quad n = 0, 1, \dots$$

which is often said to be a segmenting Mann iteration or to be of Kransnoselski-type. Various properties of (1.2) and extensions to more abstract spaces have been discussed by numerous authors, see for example, [1, 2, 15] and the references cited therein.

Eq. (1.1) can be viewed as a different kind of extension of Eq. (1.2). In analogy to delay differential equations, we may say that Eq. (1.1) is a first order difference equation with delay $k$. In general, the presence of the delay can cause more complicated dynamics. The global attractivity of solutions of Eq. (1.1) has been studied in [14] recently. Two explicit sufficient conditions for the global attractivity of solutions of

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Eq. (1.1) on a finite interval $[a, b]$ have been obtained for the cases that $\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty} t_n = \lambda > 0$, respectively. While for the case that $\{t_n\}$ does not necessarily have a limit, for instance $\{t_n\}$ is a periodic sequence, a global attractivity result has been obtained under the hypothesis that $f$ has a globally attracting fixed point $\bar{x} \in [a, b]$, that is, $\bar{x}$ is a global attractor of solutions of the first order difference equation $x_{n+1} = f(x_n)$ on $[a, b]$. However, this hypotheses is not a necessary condition for $\bar{x}$ to be a global attractor of solutions of Eq. (1.1). Our aim in this paper is to establish some sufficient conditions for the global attractivity of positive solutions of Eq. (1.1) without this hypothesis. Although the interval discussed here is $[0, B)$, as we will see, the results to be obtained may be applied to the global attractivity of solutions of Eq. (1.1) defined on a finite closed interval also.

Besides its theoretical interest, Eq. (1.1) has applications in mathematical biology also. For instance, the difference equation

$$x_{n+1} = (1-\mu)x_n + \mu x_{n-k}\left[1 + q\left(1 - \left(\frac{x_{n-k}}{K}\right)^z\right)\right]_+, \quad n = 0, 1, \ldots \tag{1.3}$$

where $k$ is a positive integer, $K, q, z \in (0, \infty), \mu \in (0, 1)$ and $[x]_+ = \max\{x, 0\}$, has been proposed by the International Whaling Commission as a model that describes the dynamics of baleen whales. The global stability of Eq. (1.3) has been studied by several authors, and a sufficient condition for the positive equilibrium $K$ to be globally asymptotically stable relative to the interval $(K - a, K + a) \subset (0, x^*)$ has been established, where $x^* = K\left(\frac{1+q}{q}\right)^{1/z}$ and $a$ is a positive constant with $a \leq \min\{K, x^* - K\}$, see [8] and the references cited therein. Later, a sufficient condition for the global asymptotic stability of the equilibrium $K$ of Eq. (1.3) relative to the whole interval $(0, x^*)$ has been obtained in [13].

Although simple difference equations sometimes are proper models in various applications, the need for more sophisticated models is evident due to the complexity of natural and laboratory systems. For example, the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. The assumptions of periodicity of the parameters are a way of incorporating the periodicity of the environment. Clearly, in the model (1.3), the parameter $\mu$ plays an important role. Hence, if we replace the constant parameter $\mu$ by a variable $t_n \in [0, 1)$, Eq. (1.3) becomes

$$x_{n+1} = (1-t_n)x_n + t_n x_{n-k}\left[1 + q\left(1 - \left(\frac{x_{n-k}}{K}\right)^z\right)\right]_+, \quad n = 0, 1, \ldots \tag{1.4}$$

which is in the form of Eq. (1.1).

By a solution of Eq. (1.1), we mean a sequence $\{x_n\}$ which is defined for $n \geq -k$ and which satisfies Eq. (1.1) for $n \geq 0$. With Eq. (1.1) we associate an initial condition of the form

$$x_{-k}, x_{-k+1}, \ldots, x_0 \in [0, B) \quad \text{with } x_0 > 0. \tag{1.5}$$

Then, by the method of steps, it follows IVP Eqs. (1.1) and (1.5) has a unique positive solution $\{x_n\}$ with $x_n \in (0, B)$, $n = 0, 1, \ldots$.

Clearly, when $t_n \not\equiv 0$, $\bar{x}$ is an equilibrium of Eq. (1.1) if and only if $\bar{x}$ is a fixed point of $f$. In Section 2, we will establish some sufficient conditions for a fixed point of $f$ to be a global attractor of all positive solutions of Eq. (1.1). Then in Section 3, we will apply our results obtained in Section 2 to Eq. (1.4) and several other difference equations derived from mathematical biology to establish some new global attractivity criteria for these equations.

## 2. Main Results

In this section, we establish some sufficient conditions for the global attractivity of positive solutions of Eq. (1.1). In the following discussion, we always assume that $f$ has a unique positive fixed point $\bar{x}$. In addition, we adopt the notation $\prod_{i=m}^{n}(1-t_i) = 1$ whenever $m > n$.

**Theorem 2.1.** *Assume that*

$$\sum_{n=0}^{\infty} t_n = \infty, \tag{2.1}$$

*and there is a positive integer $N_0 \geq k$ such that*

$$\sum_{j=n-k}^{n} t_j \prod_{i=j+1}^{n} (1 - t_i) \leq T_0, \quad n \geq N_0. \tag{2.2}$$

*where $T_0$ is a positive constant. Suppose also that $f$ satisfies*

$$(x - \bar{x})(f(x) - x) < 0 \quad \textit{for } x > 0 \textit{ and } x \neq \bar{x}, \tag{2.3}$$

*and that $f$ is L-Lipschitz with $LT_0 < 1$. Then every positive solution $\{x_n\}$ of Eq. (1.1) converges to $\bar{x}$ as $n \to \infty$.*

*Proof.* First, we show that

$$\limsup_{n\to\infty} x_n < B. \tag{2.4}$$

Otherwise, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i} > \max\{x_n : -k \leq n < n_i\}, \quad i = 1, 2, \ldots, \text{ and } \lim_{i\to\infty} x_{n_i} = B. \tag{2.5}$$

Then it follows from Eq. (1.1) that

$$t_{n_i-1}(f(x_{n_i-1-k}) - x_{n_i}) = (1 - t_{n_i-1})(x_{n_i} - x_{n_i-1}) > 0$$

which, in view of (2.5), yields

$$f(x_{n_i-1-k}) > x_{n_i} > x_{n_i-1-k}. \tag{2.6}$$

Then by noting (2.3) we see that $x_{n_i-1-k} < \bar{x}$. Hence, there is a subsequence $\{x_{n_{i_r}-1-k}\}$ of $\{x_{n_i-1-k}\}$ such that $\lim_{r\to\infty} x_{n_{i_r}-1-k} = b$, where $0 \le b \le \bar{x}$. From (2.6), we see that $f(x_{n_{i_r}-1-k}) > x_{n_{i_r}}$, which implies that $f(b) \ge B$. Clearly, this contradicts the hypothesis $f : [0, B) \to [0, B)$. Hence, (2.4) must hold and so $\{x_n\}$ is bounded no matter $B < \infty$ or $B = \infty$.

In the following, we show that $\{x_n\}$ tends to $\bar{x}$ as $n \to \infty$. First, we assume that $\{x_n\}$ is a nonoscillatory (about $\bar{x}$) solution. Suppose that $x_n - \bar{x}$ is eventually positive. The proof for the case that $x_n - \bar{x}$ is eventually negative is similar and will be omitted. Let $\limsup_{n\to\infty} x_n = R$. Then $\bar{x} \le R < B$. Clearly, it suffices to show that $R = \bar{x}$. First, we assume that $\{x_n\}$ is decreasing eventually. Then $\lim_{n\to\infty} x_n = R$. If $R > \bar{x}$, then by noting (2.1), it follows from Eq. (1.1) that

$$x_{n+1} - x_0 = \sum_{i=0}^{n} t_i[f(x_{i-k}) - x_i] \to -\infty \quad \text{as } n \to \infty,$$

which is a contradiction. Hence, $R = \bar{x}$.

Next, assume that $\{x_n\}$ is not eventually decreasing. Then, there is a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that

$$\lim_{m\to\infty} x_{n_m} = R \quad \text{and} \quad x_{n_m} > x_{n_m-1}, \quad m = 0, 1, \ldots.$$

Hence, it follows from Eq. (1.1) that

$$t_{n_m-1}[f(x_{n_m-1-k}) - x_{n_m}] = (1 - t_{n_m-1})(x_{n_m} - x_{n_m-1}) > 0, \quad m = 0, 1, \ldots.$$

and so

$$f(x_{n_m-1-k}) > x_{n_m}, \quad m = 0, 1, \ldots. \tag{2.7}$$

Since $x_{n_m-1-k} \ge \bar{x}$, $f(x_{n_m-1-k}) \le x_{n_m-1-k}$. Hence, $x_{n_m-1-k} > x_{n_m}$ and so $\lim_{m\to\infty} x_{n_m-1-k} = R$. Then by taking limit on both sides of (2.7), we see that $f(R) \ge R$ and so it follows that $R \le \bar{x}$. Hence, $R = \bar{x}$.

Finally, assume that $\{x_n\}$ is a solution of Eq. (1.1) and oscillates about $\bar{x}$. Let $y_n = x_n - \bar{x}$. Then $\{y_n\}$ satisfies

$$y_{n+1} = (1 - t_n)y_n + t_n(f(y_{n-k} + \bar{x}) - \bar{x}), \quad n = 0, 1, \ldots, \tag{2.8}$$

and $\{y_n\}$ oscillates about zero. Since $\{x_n\}$ is bounded, there is a positive constant $M$ such that $|y_n| = |x_n - \bar{x}| \le M$, $n = 0, 1, \ldots$. Then by noting $f(\bar{x}) = \bar{x}$ and the Lipschitz property of $f$, we see that

$$|f(y_{n-k} + \bar{x}) - \bar{x}| = |f(y_{n-k} + \bar{x}) - f(\bar{x})| \le L|y_{n-k}| \le LM, \quad n \ge k. \tag{2.9}$$

Let $y_l$ and $y_s$ be two consecutive members of the solution $\{y_n\}$ with $N_0 < l < s$ such that

$$y_l \le 0, y_{s+1} \le 0 \quad \text{and} \quad y_n > 0 \text{ for } l+1 \le n \le s. \tag{2.10}$$

Let

$$y_r = \max\{y_{l+1}, y_{l+2}, \dots, y_s\} \tag{2.11}$$

where $y_r$ is chosen as the first one to reach the maximum among $y_{l+1}, y_{l+2} \cdots y_s$. We claim that

$$r - (l+1) \le k. \tag{2.12}$$

Suppose, for the sake of contradiction, that $r-(l+1) > k$. Then, $y_r > y_{r-1-k} > 0$. By noting $y_{r-1-k} + \bar{x} > \bar{x}$ and (2.3), we see that $f(y_{r-1-k}+\bar{x}) < y_{r-1-k} + \bar{x}$ and so

$$\begin{aligned} f(y_{r-1-k}+\bar{x}) - y_r - \bar{x} &< y_{r-1-k} + \bar{x} - y_r - \bar{x} \\ &= y_{r-1-k} - y_r < 0. \end{aligned} \tag{2.13}$$

However, on the other hand, (2.8) yields

$$t_{r-1}(f(y_{r-1-k}+\bar{x}) - y_r - \bar{x}) = (1-t_{r-1})(y_r - y_{r-1}) > 0.$$

Then it follows that $f(y_{r-1-k}+\bar{x}) - y_r - \bar{x} > 0$ which contradicts (2.13). Hence, (2.12) holds.

Now, observe that (2.8) yields

$$\frac{y_{n+1}}{\prod_{i=0}^{n}(1-t_i)} - \frac{y_n}{\prod_{i=0}^{n-1}(1-t_i)} = \frac{t_n}{\prod_{i=0}^{n}(1-t_i)}.(f(y_{n-k}+\bar{x}) - \bar{x}) \tag{2.14}$$

Summing up from $l$ to $r-1$, we see that

$$\frac{y_r}{\prod_{i=0}^{r-1}(1-t_i)} - \frac{y_l}{\prod_{i=0}^{l-1}(1-t_i)} = \sum_{j=l}^{r-1} \frac{t_j}{\prod_{i=0}^{j}(1-t_i)}(f(y_{j-k}+\bar{x}) - \bar{x})$$

and so

$$y_r = \prod_{i=0}^{r-1}(1-t_i)\left\{\frac{y_l}{\prod_{i=0}^{l-1}(1-t_i)} + \sum_{j=l}^{r-1}\frac{t_j}{\prod_{i=0}^{j}(1-t_i)}(f(y_{j-k}+\bar{x}) - \bar{x})\right\}.$$

By noting $y_l \le 0$ and $f(\bar{x}) = \bar{x}$ we see that

$$y_r \le \prod_{i=0}^{r-1}(1-t_i)\left\{\sum_{j=l}^{r-1}\frac{t_j}{\prod_{i=0}^{j}(1-t_i)}|f(y_{j-k}+\bar{x}) - f(\bar{x})|\right\}$$

Then by combining (2.9), it follows that

$$y_r \le ML\prod_{i=0}^{r-1}(1-t_i)\sum_{j=l}^{r-1}\frac{t_j}{\prod_{i=0}^{j}(1-t_i)} = ML\sum_{j=l}^{r-1} t_j \prod_{i=j+1}^{r-1}(1-t_i).$$

Since (2.12) holds, that is, $r-1-k \le l$, and (2.2) holds, we find that

$$y_r \le ML \sum_{j=r-1-k}^{r-1} t_j \prod_{i=j+1}^{r-1} (1-t_i) \le MLT_0.$$

Hence, it follows that $y_n \le MLT_0$, $l \le n \le s$. Since $y_l$ and $y_s$ are two arbitrary members of the solution with property (2.10), we see that there is a positive integer $N_1' \ge N_0$ such that $y_n \le MLT_0$, $n \ge N_1'$. Then, by a similar argument, it can be shown that there is a positive integer $N_1'' \ge N_0$ such that $y_n \ge -MLT_0$, $n \ge N_1''$. Hence, there is a positive integer $N_1 \ge N_0$ such that

$$|y_n| \le MLT_0, \quad n \ge N_1. \tag{2.15}$$

Now, by noting the Lipschitz property of $f(x)$ and (2.15), we see that

$$|f(y_{n-k}+\bar{x}) - \bar{x}| = |f(y_{n-k}+\bar{x}) - f(\bar{x})| \le L|y_{n-k}| \le ML^2T_0, n \ge N_1 + k.$$

Let $y_l$ and $y_s$ be two consecutive members of the solution $\{y_n\}$ with $t_0 \le l < s$ such that (2.10) holds. Let $y_r$ be defined by (2.11). By a similar argument, we may show that (2.12) holds and

$$y_r \le ML^2T_0 \sum_{j=r-1-k}^{r-1} t_j \prod_{i=j+1}^{r-1} (1-t_i) \le M(LT_0)^2.$$

Then it follows that $y_n \le M(LT_0)^2$, $l \le n \le s$ and so again by noting $y_l$ and $y_s$ are two arbitrary members of the solution with property (2.10), there is a positive integer $N_2' \ge N_1 + k$ such tat $y_n \le M(LT_0)^2$, $n \ge N_2'$. Similarly, it can be shown that there is a positive integer $N_2'' \ge N_1 + K$ such that $y_n \ge -M(LT_0)^2$, $n \ge N_2''$. Hence, there is a positive integer $N_2 \ge N_1 + k$ such that $|y_n| \le M(LT_0)^2$, $n \ge N_2$. Finally, by induction, we find that for any positive integer $m$, there is a positive integer $N_m$ with $N_m \to \infty$ as $m \to \infty$ such that

$$|y_n| \le M(LT_0)^m, \quad n \ge N_m.$$

Then, by noting the hypotheses $LT_0 < 1$, we see that $y_n \to 0$ as $n \to \infty$, and so it follows that $x_n \to \bar{x}$ as $n \to \infty$. The proof is complete. □

**Remark 2.2.** The global attractivity of Eq. (1.1), where $f$ is defined on a finite interval $[a, b]$, has been studied in [14]. It has been shown that if (2.1) and (2.3) hold, $\lim_{n\to\infty} t_n = \lambda > 0$, and that $f$ is $L$-Lipschitz with

$$(1-(1-\lambda)^{k+1})L < 1, \tag{2.16}$$

then the equilibrium point $\bar{x}$ is a global attractor of all solutions of Eq. (1.1) on $[a, b]$. While for the case that $\{t_n\}$ does not necessarily have a limit, a sufficient condition for $\bar{x}$ to be a global attractor of solutions of Eq. (1.1) has been obtained under the hypothesis that $\bar{x}$ is a globally attracting fixed point of $f$ on $[a, b]$. From the proof of

Theorem 2.1, it is easy to see that by a slight modification of the proof, Theorem 2.1 holds also when Eq. (1.1) is defined on a finite interval $[a, b]$. This is a sufficient condition for the global attractivity of Eq. (1.1) without the hypothesis that $\bar{x}$ is a globally attracting fixed point of $f$. While for the case that $\{t_n\}$ has a limit $\lambda > 0$, the condition $LT_0 < 1$ assumed in Theorem 2.1 is equivalent to (2.16).

To establish next sufficient condition for the global attractivity of solutions of Eq. (1.1) with a different assumptions, we need the following result which is extracted from [9].

**Lemma 2.3.** *Consider the following difference equation*

$$x_{n+1} = g(x_n), \quad n = 0, 1, \ldots \tag{2.17}$$

*where $g : [0, \infty) \to [0, \infty)$ is a $S$-map, that is, $g$ is three times differentiable, $g'(x) < 0$ and $(Sg)(x) < 0$ for $x > 0$, where $(Sg)(x) = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2$ is the Schwarzian derivative of $g$. Assume that $\bar{x}$ is the unique fixed point of $g$ and $|g'(\bar{x})| \le 1$. Then $\bar{x}$ is a global attractor of all solutions of Eq.* (2.17)*.*

**Theorem 2.4.** *Assume that* (2.1) *and* (2.2) *assumed in Theorem* 2.1 *hold. Suppose also that $f$ is an $S$-map, and*

$$T_0|f'(\bar{x})| \le 1. \tag{2.18}$$

*Then every positive solution $\{x_n\}$ of Eq.* (1.1) *converges to $\bar{x}$ as $n \to \infty$.*

*Proof.* Since $f$ is decreasing, (2.3) assumed in Theorem 2.1 holds. From the proof of Theorem 2.1, we see that every nonoscillatory solution of Eq. (1.1) converges to $\bar{x}$. Hence, we only need to show that every oscillatory solution of Eq. (1.1) tends to $\bar{x}$ also as $n \to \infty$.

Assume that $\{x_n\}$ is a solution of Eq. (1.1) and oscillates about $\bar{x}$. Let $y_n = x_n - \bar{x}$. Then $y_n \ge -\bar{x}$ , $\{y_n\}$ oscillates about zero, and $\{y_n\}$ satisfies

$$y_{n+1} = (1 - t_n)y_n + t_n(f(y_{n-k} + \bar{x}) - \bar{x}). \tag{2.19}$$

We will say that $y_s$ is a local maximum of $\{y_n\}$ if

$$y_s \ge 0, y_s \ge y_{s-1} \quad \text{and} \quad y_s \ge y_{s+1}, \tag{2.20}$$

and $y_s$ is a local minimum of $\{y_n\}$ if

$$y_s \le 0, y_s \le y_{s-1} \quad \text{and} \quad y_s \le y_{s+1}. \tag{2.21}$$

Since $\{y_n\}$ oscillates, it has infinitely many local maximums and infinitely many local minimums. We claim that when $y_s$ is a local maximum or local minimum,

$$y_s \left( \frac{y_s}{\prod_{i=0}^{s-1}(1 - t_i)} - \frac{y_{s-1}}{\prod_{i=0}^{s-2}(1 - t_i)} \right) \ge 0. \tag{2.22}$$

In fact, if $\{y_s\}$ is a local maximum, then by noting (2.20), we see that

$$\begin{aligned}\frac{y_s}{\prod_{i=0}^{s-1}(1-t_i)} - \frac{y_{s-1}}{\prod_{i=0}^{s-2}(1-t_i)} &= \frac{1}{\prod_{i=0}^{s-2}(1-t_i)}\left(\frac{1}{1-t_{s-1}}y_s - y_{s-1}\right)\\ &\geq \frac{1}{\prod_{i=0}^{s-2}(1-t_i)}(y_s - y_{s-1}) \geq 0\end{aligned}$$

and so (2.22) holds; if $\{y_s\}$ is a local minimum, then by noting (2.21), we see that

$$\begin{aligned}\frac{y_s}{\prod_{i=0}^{s-1}(1-t_i)} - \frac{y_{s-1}}{\prod_{i=0}^{s-2}(1-t_i)} &= \frac{1}{\prod_{i=0}^{s-2}(1-t_i)}\left(\frac{1}{1-t_{s-1}}y_s - y_{s-1}\right)\\ &\leq \frac{1}{\prod_{i=0}^{s-2}(1-t_i)}(y_s - y_{s-1}) \leq 0\end{aligned}$$

and so (2.22) holds also. Next, observe that (2.19) yields

$$\frac{y_{n+1}}{\prod_{i=0}^{n}(1-t_i)} - \frac{y_n}{\prod_{i=0}^{n-1}(1-t_i)} = \frac{t_n}{\prod_{i=0}^{n}(1-t_i)}(f(y_{n-k}+\bar{x})-\bar{x}). \tag{2.23}$$

Then it follows from (2.22) and (2.23) that when $y_s$ is a local maximum or local minimum,

$$y_s\left(\frac{t_{s-1}}{\prod_{i=0}^{s-1}(1-t_i)}(f(y_{s-1-k}+\bar{x})-\bar{x})\right) \geq 0$$

and so

$$y_s(f(y_{s-1-k}+\bar{x})-\bar{x}) \geq 0.$$

Since $f(\bar{x})-\bar{x}=0$ and $f$ is decreasing, we see that $y_s y_{s-1-k} \leq 0$ which yields

$$\left(\frac{1}{\prod_{i=0}^{s-1}(1-t_i)}y_s\right)\left(\frac{1}{\prod_{i=0}^{s-2-k}(1-t_i)}y_{s-1-k}\right) \leq 0. \tag{2.24}$$

Now, suppose that $y_s$ is a local minimum of $\{y_n\}$. Then it follows from (2.23) and (2.24) that

$$\begin{aligned}\frac{1}{\prod_{i=0}^{s-1}(1-t_i)}y_s &\geq \frac{1}{\prod_{i=0}^{s-1}(1-t_i)}y_s - \frac{1}{\prod_{i=0}^{s-2-k}(1-t_i)}y_{s-1-k}\\ &= \sum_{j=s-1-k}^{s-1}\left(\frac{t_j}{\prod_{i=0}^{j}(1-t_i)}(f(y_{j-k}+\bar{x})-\bar{x})\right).\end{aligned}$$

Then, by noting $f(y_{j-1-k}+\bar{x}) \geq 0$, we see that

$$\frac{1}{\prod_{i=0}^{s-1}(1-t_i)}y_s \geq -\bar{x}\sum_{j=s-1-k}^{s-1}\frac{t_j}{\prod_{i=0}^{j}(1-t_i)}$$

and so

$$y_s \geq -\bar{x}\prod_{i=0}^{s-1}(1-t_i)\sum_{j=s-1-k}^{s-1}\frac{t_j}{\prod_{i=0}^{j}(1-t_i)} = -\bar{x}\sum_{j=s-1-k}^{s-1}t_j\prod_{i=j+1}^{s-1}(1-t_i) \geq -\bar{x}T_0.$$

Let $z_0 = -\bar{x}T_0$. Since $z_0$ is independent of the choice of $y_s$, it is easy to see that

$$y_n \geq z_0 \quad \text{for } n \geq N_1' \tag{2.25}$$

where $N_1' > N_0$ such that $y_{N_1'}$ is a local minimum of $\{y_n\}$.

Now, assume that $y_s$ with $s > N_1' + 2k$ is a local maximum of $\{y_n\}$. Then it follows from (2.23) and (2.24) that

$$\begin{aligned}\frac{1}{\prod_{i=0}^{s-1}(1-t_i)}y_s &\le \frac{1}{\prod_{i=0}^{s-1}(1-t_i)}y_s - \frac{1}{\prod_{i=0}^{s-2-k}(1-t_i)}y_{s-1-k}\\ &= \sum_{j=s-1-k}^{s-1}\left(\frac{t_j}{\prod_{i=0}^{j}(1-t_i)}(f(y_{j-k}+\bar{x})-\bar{x})\right).\end{aligned}$$

Clearly, $\sum_{j=n-k}^{n} t_j \prod_{i=j+1}^{n}(1-t_i) \le 1$ for any $t_n \in [0,1)$. Hence, we may have $T_0 \le 1$ in (2.2). Then it follows that $z_0 + \bar{x} = -\bar{x}T_0 + \bar{x} \ge 0$ and so $f(z_0 + \bar{x})$ is defined. Then by noting (2.25) and $f$ is decreasing, we see that

$$f(y_{i-k}+\bar{x}) \le f(z_0+\bar{x}) \quad \text{for } i \ge N_1'+k, \text{ and } f(z_0+\bar{x}) \ge \bar{x}.$$

Hence, it follows that

$$\frac{1}{\prod_{i=0}^{s-1}(1-t_i)}y_s \le \sum_{j=s-1-k}^{s-1}\frac{t_{j-1}}{\prod_{i=0}^{j-1}(1-t_i)}[f(z_0+\bar{x})-\bar{x}]$$

and so

$$\begin{aligned}y_s &\le \prod_{i=0}^{s-1}(1-t_i)\sum_{j=s-1-k}^{s-1}\frac{t_j}{\prod_{i=0}^{j}(1-t_i)}[f(z_0+\bar{x})-\bar{x}]\\ &= \sum_{j=s-1-k}^{s-1} t_j \prod_{i=j+1}^{s-1}(1-t_i)[f(z_0+\bar{x})-\bar{x}] \le T_0[f(z_0+\bar{x})-\bar{x}].\end{aligned}$$

Let $z_1 = T_0[f(z_0+\bar{x})-\bar{x}]$. Since $z_1$ is independent of the choice of $y_s$ as long as $s > N_1' + 2k$, it is easy to see that

$$y_n \le z_1 \quad \text{for } n > N_1$$

where $N_1 > N_1' + 2k$ such that $y_{N_1}$ is a local maximum of $\{y_n\}$. Then, by an easy induction, we see that for each $m \ge 0$, there is a positive integer $N_{m+1}$ such that

$$z_{2m} \le y_n \le z_{2m+1} \quad \text{for } n \ge N_{m+1}$$

where $\{z_m\}$ is defined by

$$\begin{cases} z_m = T_0[f(z_{m-1}+\bar{x})-\bar{x}], & m = 1,2,\dots \\ z_0 = -T_0\bar{x}. \end{cases} \tag{2.26}$$

Let $w_m = y_m + \bar{z}$, $m = 0, 1, \dots$. Then (2.26) reduces to

$$\begin{cases} w_m = T_0 f(w_{m-1}) + (1-T_0)\bar{x}, & m = 1,2,\dots \\ w_0 = (1-T_0)\bar{x} \end{cases}$$

From the above discussion, it is easy to see that to show $x_n \to \bar{x}$ as $n \to \infty$ it suffices to show that $w_m \to \bar{x}$ as $m \to \infty$. To this end, let $g(x) = T_0 f(x) + (1-T_0)\bar{x}$.

Clearly, $g$ is defined on $[0,\infty)$, $\bar{x}$ is a fixed point of $g$, $g'(x) = T_0 f'(x) < 0$ and $(Sg)(x) = T_0(Sf)(x) < 0$ for $x > 0$. Hence, all the conditions assumed in Lemma 1 are satisfied and so $w_m \to \bar{x}$ as $m \to \infty$. Then, it follows that $x_n \to \bar{x}$ as $n \to \infty$. The proof is complete. □

## 3. **Applications**

In this section, we apply the results obtained in the last section to some equations derived from mathematical biology.

First, consider the difference equation which has been discussed in Section 1

$$x_{n+1} = (1-t_n)x_n + t_n x_{n-k}\left[1 + q\left(1 - \left(\frac{x_{n-k}}{K}\right)^z\right)\right]_+, \quad n = 0,1,\ldots \tag{3.1}$$

where $k$ is a positive integer, $\{t_n\}$ is a sequence in $[0,1)$, $K, q, z \in (0,\infty)$ and $[x]_+ = \max\{0,x\}$, and the solutions $\{x_n\}$ of Eq. (3.1) satisfy the initial conditions of the form

$$x_k, x_{k+1}, \ldots, x_0 \in [0, x^*) \quad \text{with } x_0 > 0 \tag{3.2}$$

where $x^* = K\left(\frac{1+q}{q}\right)^{1/z}$. Let

$$f(x) = x[1 + q(1 - (x/K)^z)]_+$$

and assume that

$$q < \frac{(1+z)^{1+\frac{1}{z}}}{z} - 1. \tag{3.3}$$

It has been shown (see [8]) that $f : (0,x^*) \to (0,x^*)$. Then it is easy to see that the solutions $\{x_n\}$ of Eq. (3.1) with initial conditions of form (3.2) satisfy $x_n \in (0,x^*)$, $n = 0,1,\ldots$. Hence, with the initial conditions of form (3.2), Eq. (3.1) is equivalent to the equation

$$x_{n+1} = (1-t_n)x_n + t_n x_{n-k}\left[1 + q\left(1 - \left(\frac{x_{n-k}}{K}\right)^z\right)\right], \quad n = 0,1,\ldots \tag{3.4}$$

Clearly, $f$ has the unique fixed point $K \in (0,x^*)$ and

$$(x-K)(f(x)-x) < 0 \quad \text{for } x \in (0,x^*) \text{ and } x \neq K.$$

Observe that

$$f'(x) = 1 + q(1 - (1+z)(x/K)^z), \quad x \in (0,x^*).$$

Hence,

$$-z(1+q) \le f'(x) \le 1+q, \quad x \in (0,x^*).$$

and so the function $f$ is $L$-Lipschitz with $L = \max\{1+q, z(1+q)\}$. Then the following result comes from Theorem 2.1 immediately.

**Theorem 3.1.** *Assume that* (3.3) *holds,* $\sum_{n=0}^{\infty} t_n = \infty$*, and that there are a positive constant* $c < 1$ *and a positive integer* $N_0 > k$ *such that*

$$L \sum_{j=n-k}^{n} t_j \prod_{i=j+1}^{n} (1 - t_i) \le c, \quad n \ge N_0 \tag{3.5}$$

*where* $L = \max\{1 + q, z(1 + q)\}$*. Then the positive equilibrium* $K$ *is a global attractor of solutions of Eq.* (3.1) *relative to the interval* $(0, x^*)$*.*

Next, consider the difference equations

$$x_{n+1} = (1 - t_n)x_n + bt_n \frac{x_{n-k}}{1 + x_{n-k}^{\gamma}}, \quad n = 0, 1, \dots \tag{3.6}$$

and

$$x_{n+1} = (1 - t_n)x_n + pt_n x_{n-k} e^{-\sigma x_{n-k}}, \quad n = 0, 1, \dots \tag{3.7}$$

where $\{t_n\}$ is a sequence in $[0, 1)$, $b$, $p$, $\gamma$ and $\sigma$ are positive constants with $b > 1$ and $p > 1$, and $k$ is a nonnegative integer.

For the special case that $\{t_n\}$ is a positive constant $\lambda \in (0, 1)$, let $1 - \lambda \equiv \delta$, $\beta = (1 - \delta)b$, and $\rho = (1 - \delta)p$. Then Eqs. (3.6) and (3.7) can be written in the forms

$$x_{n+1} = \delta x_n + \beta \frac{x_{n-k}}{1 + x_{n-k}^{\gamma}}, \quad n = 0, 1, \dots \tag{3.8}$$

and

$$x_{n+1} = \delta x_n + \rho x_{n-k} e^{-\sigma x_{n-k}}, \quad n = 0, 1, \dots \tag{3.9}$$

respectively. Eq. (3.8) is a discrete analogue of a model of haematopoiesis [10], while Eq. (3.4) is a discrete version of a model used in describing the dynamics of Nicholson's blowflies [5]. The global atrractivity of positive solutions of Eqs. (3.8) and (3.9) have been studied by several authors, see for example, [6, 7, 8] and references cited therein. In addition, when $k = 0$, Eq. (3.8) was proposed by Milton and Belair [11] as a model for the bobwhite quail population of northern Wisconsin, and its local and global stability has been studied in [11].

First consider Eq. (3.6) and let $f(x) = \frac{bx}{1+x^{\gamma}}$. Clearly, $f$ has a unique positive fixed point $\bar{x} = (b - 1)^{\frac{1}{\gamma}}$ which is the unique positive equilibrium of Eq. (3.6), and

$$(x - \bar{x})(f(x) - x) < 0 \quad \text{for } x > 0 \text{ and } x \ne \bar{x}.$$

By observing

$$f'(x) = \frac{b(1 + (1 - \gamma)x^{\gamma})}{(1 + x^{\gamma})^2} \quad \text{and} \quad f''(x) = \frac{b\gamma x^{\gamma-1}((\gamma - 1)x^{\gamma} - (\gamma + 1))}{(1 + x^{\gamma})^3},$$

we see that when $\gamma \le 1$, $\sup_{x>0}\{|f'(x)|\} = f'(0) = b$; while for the case that $\gamma > 1$, $\sup_{x>0}\{|f'(x)|\} = b$ and

$$\inf_{x>0}\{f'(x)\} = f'\left(\left(\frac{\gamma+1}{\gamma-1}\right)^{1/\gamma}\right) = -\frac{b(\gamma-1)^2}{4\gamma}$$

and so it follows that

$$\sup_{x>0}\{|f'(x)|\} = b\max\left\{1, \frac{(\gamma-1)^2}{4\gamma}\right\}.$$

Hence, the function $f$ is $L$-Lipschitz with $L = b$ if $\gamma \le 1$ and $L = b\max\left\{1, \frac{(\gamma-1)^2}{4\gamma}\right\}$ if $\gamma > 1$. Therefore, by Theorem 2.1, we have the following result.

**Theorem 3.2.** *Assume that*

$$\sum_{n=0}^{\infty} t_n = \infty, \tag{3.10}$$

*and there a positive integer $N_0 \ge k$ such that*

$$\sum_{j=n-k}^{n} t_j \prod_{i=j+1}^{n} (1-t_i) \le T_0, \quad n \ge N_0. \tag{3.11}$$

*where $T_0$ is a positive constant. Then the positive equilibrium $\bar{x}$ of Eq.* (3.6) *is a global attractor of all positive solutions of the equation if either $\gamma \le 1$ and $bT_0 < 1$, or $\gamma > 1$ and $bT_0 \max\left\{1, \frac{(\gamma-1)^2}{4\gamma}\right\} < 1$.*

Next, consider Eq. (3.7) and let $f(x) = pxe^{-\sigma x}$. Clearly, $f$ has a unique positive fixed point $\bar{x} = \frac{1}{\sigma}\ln p$ which is the unique positive equilibrium of Eq. (3.7), and

$$(x-\bar{x})(f(x)-x) < 0 \quad \text{for } x > 0 \text{ and } x \ne \bar{x}.$$

Then by noting

$$f'(x) = p(1-\sigma x)e^{-\sigma x} \quad \text{and} \quad f''(x) = p\sigma e^{-\sigma x}(\sigma x - 2)$$

we see that $\sup_{x>0}\{|f'(x)|\} = f'(0) = p$. Hence, the function $f$ is $L$-Lipschitz with $L = p$ and so by Theorem 2.1 we have the following result.

**Theorem 3.3.** *Assume that* (3.10) *and* (3.11) *hold. Then the positive equilibrium $\bar{x}$ of Eq.* (3.7) *is a global attractor of all positive solutions of the equation if $pT_0 < 1$ holds.*

Now, let's consider the difference equations

$$x_{n+1} = (1-t_n)x_n + \frac{bt_n}{1+x_{n-k}^{\gamma}}, \quad n = 0, 1, \ldots \tag{3.12}$$

and

$$x_{n+1} = (1-t_n)x_n + pt_ne^{-\sigma x_{n-k}}, \quad n = 0, 1, \ldots \tag{3.13}$$

where $\{t_n\}$ is an sequence in $[0,1)$, $b$, $p$, $\gamma$ and $\sigma$ are positive constants, and $k$ is a nonnegative integer. For the special case that $\{t_n\}$ is a positive constant $\lambda \in (0,1)$, let $1-\lambda \equiv \delta$, $\beta = (1-\delta)b$, and $\rho = (1-\delta)p$. Then Eqs. (3.12) and (3.13) can be written in the forms

$$x_{n+1} = \delta x_n + \frac{\beta}{1+x_{n-k}^{\gamma}}, \quad n = 0, 1, \dots \tag{3.14}$$

and

$$x_{n+1} = \delta x_n + \rho e^{-\sigma x_{n-k}}, \quad n = 0, 1, \dots \tag{3.15}$$

respectively. Eq. (3.14) is a discrete analogue of a model that has been used to study blood cell production [10], while Eq. (3.15) is a discrete version of a model of the survival of red blood cells in an animal [16]. The global attractivity of positive solutions of Eqs. (3.14) and (3.15) has been studied by several authors, see for example, [3, 4, 6, 7, 8] and references cited therein.

Clearly, Eq. (3.12) has a unique positive equilibrium $\bar{x}$. It has been shown in [14] that $\bar{x}$ is a global attractor of all positive solutions of Eq. (3.12) if one of the following conditions holds:

(i) $\gamma \le 1$ and $\sum_{n=0}^{\infty} t_n = \infty$;

(ii) $\lim_{n\to\infty} t_n = 0$ and $\sum_{\infty} t_n = \infty$;

(iii) $\lim_{n\to\infty} t_n = \lambda > 0$ and $(1-(1-\lambda)^{k+1})b\gamma < 1$.

While for the case that $\gamma > 1$ and $\{t_n\}$ does not necessarily have a limit, it has been shown in [12] that if $\sum_{n=1}^{\infty} \frac{t_n}{1-t_n} = \infty$, and there is a positive constant $T_0$ such that $\sum_{j=n-k}^{n} \frac{t_j}{1-t_j} \le T_0$ for large $n$, and $b\gamma T_0 < 1$, then $\bar{x}$ is a global attractor of all positive solutions of Eq. (3.12).

However, by letting $f(x) = \frac{b}{1+x^{\gamma}}$ and by an easy calculation, we find that $f'(x) = -\frac{b\gamma x^{\gamma-1}}{(1+x^{\gamma})^2} < 0$ for $x > 0$ and when $\gamma > 1$,

$$(Sf)(x) = \frac{1}{2}(1-\gamma)(1+\gamma)x^{-2} < 0, \quad x > 0.$$

Hence, by Theorem 2.4, we have the following conclusion.

**Theorem 3.4.** *Assume that* (3.10) *and* (3.11) *hold, and*

$$\gamma > 1, \quad \frac{b\gamma \bar{x}^{\gamma-1}}{(1+\bar{x}^{\gamma})^2} T_0 < 1.$$

*Then $\bar{x}$ is a global attractor of all positive solutions of Eq.* (3.12).

It is easy to see that $\sum_{n=0}^{\infty} \frac{t_n}{1-t_n} = \infty$ if and only if $\sum_{n=0}^{\infty} t_n = \infty$, and that $\frac{b\gamma\bar{x}^{\gamma-1}}{(1+\bar{x}^{\gamma})^2} < b\gamma$. In addition, by noting $t_j \prod_{i=j+1}^{n}(1-t_i) \le \frac{t_j}{1-t_j}$, we see that

$$\sum_{j=n-k}^{n} t_j \prod_{i=j+1}^{n} (1-t_i) \le \sum_{j=n-k}^{n} \frac{t_j}{1-t_j}.$$

Hence, Theorem 3.4 is an improvement of the corresponding result obtained in [12].

**Example 3.5.** Consider the equation

$$x_{n+1} = (1 - t_n)x_n + \frac{2t_n}{1 + x_{n-k}^{\gamma}}, \quad n = 0, 1, \dots \tag{3.16}$$

which is in the form of (3.12) with $b = 2$. Clearly $\bar{x} = 1$ is the only positive equilibrium of Eq. (3.16). Hence, by Theorem 3.4, if (3.10) and (3.11) hold, $\gamma > 1$ and $\gamma T_0 < 2$, then every positive solution of Eq. (3.16) tends to 1 as $n \to \infty$. (By the corresponding result obtained in [14], as we mentioned above, when $\gamma \leq 1$ and (3.10) holds, every positive solution tends to 1 as $n \to \infty$.)

Finally, consider Eq. (3.13). It has a unique positive equilibrium $\bar{x}$. It has been shown in [14] that $\bar{x}$ is a global attractor of all positive solutions of Eq. (3.13) if one of the following conditions holds:

(i) $p\sigma \leq e$ and $\sum_{n=0}^{\infty} t_n = \infty$;

(ii) $\lim_{n\to\infty} t_n = 0$ and $\sum_{\infty} t_n = \infty$;

(iii) $\lim_{n\to\infty} t_n = \lambda > 0$ and $(1 - (1 - \lambda)^{k+1})p\sigma < 1$.

For the case that $\{t_n\}$ does not necessarily have a limit, it has been shown in [12] that if $\sum_{n=1}^{\infty} \frac{t_n}{1-t_n} = \infty$, and there is a positive constant $T_0$ such that $\sum_{j=n-k}^{n} \frac{t_j}{1-t_j} \leq T_0$ for large $n$, and $p\sigma T_0 < 1$, then $\bar{x}$ is a global attractor of all positive solutions of Eq. (3.13). Now, let $f(x) = pe^{-\sigma x}$ and observe that

$$f'(x) = -p\sigma e^{-\sigma x} < 0 \quad \text{and} \quad (Sf)(x) = -\frac{1}{2}\sigma^2 < 0.$$

Hence, by Theorem 2.4, we have the following result

**Theorem 3.6.** *Assume that* (3.10) *and* (3.11) *hold, and*

$$p\sigma e^{-\sigma\bar{x}} T_0 < 1.$$

*Then $\bar{x}$ is a global attractor of all positive solutions of Eq.* (3.13).

Again by noting that $\sum_{n=0}^{\infty} \frac{t_n}{1-t_n} = \infty$ is equivalent to $\sum_{n=0}^{\infty} t_n = \infty$, and $\sum_{j=n-k}^{n} t_j \prod_{i=j+1}^{n}(1 - t_i) \leq \sum_{j=n-k}^{n} \frac{t_j}{1-t_j}$ and by noting $p\sigma e^{-\sigma\bar{x}} < p\sigma$, we see that Theorem 7 is an improvement of the corresponding result obtained in [12].

**Example 3.7.** Consider the equation

$$x_{n+1} = (1 - t_n)x_n + t_n e^{\sigma(1-x_{n-k})}, \tag{3.17}$$

which is in the form of (3.13) with $p = e^{\sigma}$. Clearly, $\bar{x} = 1$ is the unique positive equilibrium of Eq. (3.17). Hence, by Theorem 3.6, if (3.10) and (3.11) hold and $\sigma T_0 < 1$, then every positive solution of Eq. (3.17) tends to 1 as $n \to \infty$.

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