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POSITIVE SOLUTION FOR MULTI-POINTS BVPS OF HIGHER ORDER FRACTIONAL DIFFERENTIAL EQUATIONS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. This paper is concerned with the existence of positive solutions for a multi-points boundary value problem of higher order fractional differential equations. Firstly, we give the Green's function and discuss its property; then the existence criteria of one and two positive solutions are established by means of fixed point theory. The results extend and generalize some related results in the literature.

AMS (MOS) Subject Classification. 39A10.

1. INTRODUCTION

Fractional differential equations have been of great interest recently which is due to its significant role in engineering, science, economy and other fields. During the last few decades, many monographs and papers on fractional calculus and fractional differential equations have appeared, see [1, 9, 13, 15, 16]. Recently, there have been a lot of works dealing with the existence and multiplicity of solutions (or positive solutions) of boundary value problems for nonlinear fractional differential equations by use of techniques of nonlinear analysis (fixed-point theorems, critical point theory, et. al.) [1, 2, 3, 5, 6, 7, 10, 17, 18, 19, 20]. However, few papers considered multi-point boundary value problems for higher-order fractional differential equations [5].

In [5], El-Shahed and Nieto considered the following nonlinear differential equation m-points boundary value problem

$$\begin{cases} {}_RD^{\alpha}_{0+}u(t) + f(t,u(t)) = 0, & t \in [0,1], \\ u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0, & u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i), \end{cases} \tag{P}$$

They transformed the differential equation into an integral equation and obtained some results for nontrivial solutions of problem (P) by using Leray-Schauder nonlinear alternative and Banach contraction mapping principle.

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Motivated by [5], in this paper, we further investigate existence of positive solutions of (P), by deriving the corresponding Green's function and analyzing its properties. We reduce (P) to the equivalent Fredholm integral equation of second order, then by applying fixed-point theorems in cones, the existence criteria of one and two positive solutions for the problem (P) are established. The results generalize and extend some corresponding results in the literature, for example, Eloe and Ahmad [4] ($\alpha \in \mathbb{N}$, $m = 3$), Ma [11] ($\alpha = 2$), and [12] ($\alpha = 2$, $m = 3$), Pang, Dong and Wei[14]($\alpha \in \mathbb{N}$).

The rest of the paper is organized as follows. In Section 2, we first present preliminaries and lemmas which are used to prove our main results and then discuss some properties of the Green's function. Section 3 develops existence criteria for positive solutions of the problem (P).

2. PRELIMINARIES

For convenience, we present some preliminaries which are needed later.

Definition 2.1 ([15]). The Riemann-Liouville fractional integral (derivative) of order $\alpha > 0$ of a function $u : (0, \infty) \to \mathbb{R}$ is defined by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad \left(D_{0+}^{\alpha} u(t) = \left(\frac{d}{dt} \right)^n I_{0+}^{n-\alpha} u(t) \right)$$

provided that the right side are pointwise defined on $(0, \infty)$, where $n = [\alpha] + 1$.

Lemma 2.2 ([15]). *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$, then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} f(t) = f(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

for some $c_i \in R$, $i = 1, 2, \ldots, n$, $n - 1 < \alpha \leq n$.

Lemma 2.3 ([8]). *Let P be a cone in a Banach space E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subseteq \Omega_2$. If $T : P \cap (\overline{\Omega_2} \backslash \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$, $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$, *or*
(ii) $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$, $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Next, we derive the corresponding Green's function for the problem (P) and obtain some properties of it.

Lemma 2.4. *Suppose $D = 1 - \sum_{i=1}^{m-2} a_i \eta_i^{\alpha-1} \neq 0$, for given $y \in C[0,1]$, then the solution of problem*

$$ {}_R D_{0+}^{\alpha} u(t) + y(t) = 0, \quad t \in (0,1), \tag{2.1}$$

$$u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i), \tag{2.2}$$

is given by

$$u(t) = \int_0^1 G(t,s)y(s)ds \tag{2.3}$$

where the Green's function $G(t,s)$ *is*

$$G(t,s) = G_1(t,s) + t^{\alpha-1}g(s). \tag{2.4}$$

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \le t \le s \le 1, \end{cases} \tag{2.5}$$

$$g(s) = \frac{1}{D\Gamma(\alpha)} \sum_{i=1}^{m-2} a_i [\eta_i^{\alpha-1}(1-s)^{\alpha-1} - (\eta_i - s)^{\alpha-1}\chi_{[0,\eta_i]}(s)]$$

and $\chi_{[0,\eta_i]}(s) = 1$ *for* $s \in [0,\eta_i]$, $\chi_{[0,\eta_i]}(s) = 0$ *otherwise.*

Proof. From Lemma 2.2 and (2.1), one gets

$$u(t) = -I_{0+}^{\alpha} y(t) + c_1 t^{\alpha-1} + \cdots + c_{n+1} t^{\alpha-n-1}.$$

In view of (2.2), we have $c_2 = 0, \ldots, c_{n+1} = 0$ and

$$c_1 = \frac{1}{D\Gamma(\alpha)} \Big[\int_0^1 (1-s)^{\alpha-1} y(s) - \sum_{i=1}^{m-2} a_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} y(s)ds \Big].$$

Therefore, the solution of (2.1), (2.2) is

$$\begin{aligned}
u(t) = &-\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds - \frac{t^{\alpha-1}}{D\Gamma(\alpha)} \sum_{i=1}^{m-2} a_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} y(s)ds \\
&+ \frac{t^{\alpha-1}}{D\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)ds \\
= &-\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds - \frac{t^{\alpha-1}}{D\Gamma(\alpha)} \sum_{i=1}^{m-2} a_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} y(s)ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-s)^{\alpha-1} y(s)ds + \frac{t^{\alpha-1}}{D\Gamma(\alpha)} \sum_{i=1}^{m-2} a_i \eta_i^{\alpha-1} \int_0^1 (1-s)^{\alpha-1} y(s)ds \\
= &\frac{1}{\Gamma(\alpha)} \int_0^t [t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}] y(s)ds + \frac{1}{\Gamma(\alpha)} \int_t^1 t^{\alpha-1}(1-s)^{\alpha-1} y(s)ds \\
&- \frac{t^{\alpha-1}}{D\Gamma(\alpha)} \sum_{i=1}^{m-2} a_i \Big\{ \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} y(s)ds + \eta_i^{\alpha-1} \int_0^1 (1-s)^{\alpha-1} y(s)ds \Big\} \\
= &\int_0^1 G_1(t,s)y(s)ds + \int_0^1 t^{\alpha-1} g(s) y(s)ds = \int_0^1 G(t,s)y(s)ds.
\end{aligned}$$

Conversely, let $u(t)$ be given by (2.3), it is easy to verify that $u(t)$ is the solution of (2.1), (2.2). □

Lemma 2.5. *Suppose $D > 0$, then the Green's function $G(t,s)$ given in* (2.4) *has the following properties:*

$$0 \le G(t,s) \le k(s) \quad \text{for } (t,s) \in [0,1] \times [0,1],$$

and there exists a constant $\gamma \in (0,1)$ such that

$$\min_{t\in[\eta_1,1]} G(t,s) \ge \gamma k(s), \quad s \in [0,1]$$

where $k(s) = \frac{\alpha-1}{\Gamma(\alpha)} s(1-s)^{\alpha-1} + g(s)$.

Proof. By (2.5), when $0 \le s \le t \le 1$, in view of the Lagrange mean value problem we get that

$$\begin{aligned} 0 &\le \Gamma(\alpha)G_1(t,s) = t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1} \\ &\le (\alpha-1)(t(1-s))^{\alpha-2}(t(1-s)-(t-s)) \le (\alpha-1)s(1-s)^{\alpha-1}, \end{aligned}$$

when $0 \le t \le s \le 1$, $0 \le \Gamma(\alpha)G_1(t,s) = t^{\alpha-1}(1-s)^{\alpha-1} \le (\alpha-1)s(1-s)^{\alpha-1}$. Therefore

$$0 \le G(t,s) = G_1(t,s) + t^{\alpha-1}g(s) \le \frac{\alpha-1}{\Gamma(\alpha)} s(1-s)^{\alpha-1} + g(s) = k(s).$$

Observe that $g(s), k(s)$ are nonnegative on $[0,1]$, $g(s) = 0, k(s) = 0$ if and only if $s = 0$ or 1, and

$$\begin{aligned} &\lim_{s\to 0^+} \frac{g(s)}{k(s)} = \lim_{s\to 0^+} \frac{\sum_{i=1}^{m-2} a_i[\eta_i^{\alpha-1}(1-s)^{\alpha-1} - (\eta_i - s)^{\alpha-1}]}{D(\alpha-1)s(1-s)^{\alpha-1} + \sum_{i=1}^{m-2} a_i[\eta_i^{\alpha-1}(1-s)^{\alpha-1} - (\eta_i - s)^{\alpha-1}]} \\ &= \lim_{s\to 0^+} \frac{\sum_{i=1}^{m-2} a_i[(\eta_i - s)^{\alpha-2} - \eta_i^{\alpha-1}(1-s)^{\alpha-2}]}{D(1-s)^{\alpha-1} - D(\alpha-1)s(1-s)^{\alpha-2} + \sum_{i=1}^{m-2} a_i[(\eta_i - s)^{\alpha-2} - \eta_i^{\alpha-1}(1-s)^{\alpha-2}]} \\ &= \frac{\sum_{i=1}^{m-2} a_i \eta_i^{\alpha-2}(1-\eta_i)}{D + \sum_{i=1}^{m-2} a_i \eta_i^{\alpha-2}(1-\eta_i)}. \end{aligned}$$

$$\begin{aligned} \lim_{s\to 1^-} \frac{g(s)}{k(s)} &= \lim_{s\to 1^-} \frac{\sum_{i=1}^{m-2} a_i \eta_i^{\alpha-1}(1-s)^{\alpha-1}}{D(\alpha-1)s(1-s)^{\alpha-1} + \sum_{i=1}^{m-2} a_i \eta_i^{\alpha-1}(1-s)^{\alpha-1}} \\ &= \frac{\sum_{i=1}^{m-2} a_i \eta_i^{\alpha-1}}{D(\alpha-1) + \sum_{i=1}^{m-2} a_i \eta_i^{\alpha-1}}. \end{aligned}$$

So by the continuity and positivity of $g(s)$ and $k(s)$ on $(0,1)$, we get that there is $\gamma \in (0,1)$ such that

$$\min_{t\in[\eta_1,1]} G(t,s) = \min_{t\in[\eta_1,1]} G_1(t,s) + t^{\alpha-1}g(s) \ge \eta_1^{\alpha-1}g(s) \ge \gamma k(s), \quad s \in [0,1].$$

□

Let the Banach space $E = C[0,1]$ be endowed with the maximum norm $\|u\| = \max_{0\le t\le 1}|u(t)|$ and choose the cone $K \subset E$ defined by

$$K = \left\{ u \in E : u(t) \ge 0,\ \ t \in [0,1].\ \ \min_{t\in[\eta_1,1]} u(t) \ge \gamma\|u\| \right\}. \tag{2.13}$$

We make the following assumptions:

(H) Suppose $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1, a_i \ge 0$ $(i = 1, 2, \dots, m-2)$, $D > 0$ and $n \ge 2,\ f \in C([0,1] \times [0,\infty), [0,\infty))$.

Define the operator $A : E \to E$ by

$$Au(t) = \int_0^1 G(t,s) f(s,u(s))ds. \tag{2.14}$$

Therefore, seeking positive solutions of the problem (P) turns into seeking fixed points of the operator A.

Lemma 2.6. *Suppose (H) is satisfied, then $A : K \to K$ is completely continuous.*

Proof. Notice from (2.14), Lemma 2.5 and (H) that for $u \in K$, we obtain

$$0 \le Au(t) \le \int_0^1 k(s) f(s,u(s))ds, \quad t \in [0,1].$$

and

$$\min_{t\in[\eta_1,1]} Au(t) \ge \int_0^1 \gamma k(s) f(s,u(s))ds \ge \gamma\|Au\|$$

Thus, $A(K) \subset K$. In view of Arzela-Ascoli theorem, it is routine to see that $A : K \to K$ is completely continuous. □

3. MAIN RESULTS

For convenience, throughout this section we assume (H) hold and introduce the following notations.

$$f_0 = \liminf_{u\to 0^+} \min_{t\in[0,1]} \frac{f(t,u)}{u}, \quad f^0 = \limsup_{u\to 0^+} \max_{t\in[0,1]} \frac{f(t,u)}{u},$$

$$f_\infty = \liminf_{u\to +\infty} \min_{t\in[0,1]} \frac{f(t,u)}{u}, \quad f^\infty = \limsup_{u\to +\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u},$$

$$M = \left(\int_0^1 k(s)ds \right)^{-1}, \quad N = \left(\gamma \int_{\eta_1}^1 G(1,s)ds \right)^{-1}.$$

Theorem 3.1. *Assume $f_0 > N$, $f^\infty < M$, then* (P) *has least one positive solution u.*

Proof. Since $f_0 > N$, we can find $r > 0$ so that $f(t,u) > Nu$ for $t \in [0,1]$, $0 \le u \le r$. Let $\Omega_r = \{u \in K \mid \|u\| < r\}$, then for $u \in \partial\Omega_r$, we have $\gamma r \le u(t) \le r$, $t \in [\eta_1, 1]$. So, by Lemma 2.5 and (2.14), one gets

$$\begin{aligned}(Au)(1) &\ge \int_{\eta_1}^1 G(1,s)f(s,u(s))ds \ge rN\int_{\eta_1}^1 G(1,s)u(s)ds \\ &\ge rN\gamma\int_{\eta_1}^1 G(1,s)ds = \|u\|.\end{aligned}$$

From which we know that

$$\|Au\| \ge \|u\|, \quad u \in \partial\Omega_r. \tag{3.1}$$

On the other hand, since $f^\infty < M$, there exists $H > 0$ such that

$$f(t,u) < Mu, \quad t \in [0,1], \quad u \ge H. \tag{3.2}$$

If $\max_{t\in[0,1]} f(t,u)$ is unbounded on $[0,\infty)$, then we choose $R > \max\{r, H\}$ so that

$$f(t,u) \le \max_{t\in[0,1]} f(t,R) \quad \text{ for } t \in [0,1] \quad u \in (0,R]. \tag{3.3}$$

For $u \in K$ with $\|u\| = R$, then it follows from Lemma 2.5, (3.3) and (3.2) that

$$\|Au\| \le \int_0^1 k(s)\max_{t\in[0,1]} f(t,R)ds \le RM\int_0^1 k(s)ds = R = \|u\|.$$

If $\max_{t\in[0,1]} f(t,u)$ is bounded on $[0,\infty]$, we have $f(t,u) \le L$ for $t \in [0,1]$, $u \ge 0$. In this case, we choose $R > r + \frac{L}{M}$. For $u \in K$ with $\|u\| = R$, from Lemma 2.5 we have $\|Au\| \le \int_0^1 k(s)Lds = \frac{L}{M} < R = \|u\|$.

Therefore, in either case we may put $\Omega_R = \{u \in K \mid \|u\| < R\}$ and we get

$$\|Au\| \le \|u\|, \quad u \in \partial\Omega_R. \tag{3.4}$$

Therefore, by (3.1), (3.4) and Lemma 2.3, we know that the operator A has at least one fixed point $u \in \Omega_R \setminus \overline{\Omega}_r$, which is a positive solution of (P). □

Theorem 3.2. *Assume that $f^0 < M$, $f_\infty > N$ hold, then the problem* (P) *has at least one positive solution u.*

Proof. By $f^0 < M$, there is $r > 0$ such that $f(t,u) < Mu$ for $t \in [0,1]$, $0 < u \le r$. Let $\Omega_r = \{u \in K \mid \|u\| < r\}$. For $u \in \partial\Omega_r$, then $0 \le u(t) \le r$ on $t \in [0,1]$. By Lemma 2.5, we know

$$(Au)(t) = \int_0^1 G(t,s)f(s,u(s))ds \le M\int_0^1 k(s)u(s)ds \le Mr\int_0^1 k(s)ds = r = \|u\|.$$

From which we can see that

$$\|Au\| \le \|u\|, \quad u \in \partial\Omega_r. \tag{3.5}$$

On the other hand, since $f_\infty > N$, there exists $H > 0$ such that $f(t,u) > Nu$ for $t \in [0,1]$, $u > H$. Let $R = \max\{\frac{H}{\gamma}, 2r\}$, $\Omega_R = \{u \in K \mid \|u\| < R\}$. For $u \in \partial\Omega_R$, we have $u(t) \geq \gamma\|u\| = \gamma R \geq H$ on $t \in [\eta_1, 1]$. By Lemma 2.5, this implies

$$\begin{aligned}\|Au\| \geq |(Au)(1)| &\geq \int_{\eta_1}^{1} G(1,s)f(s,u(s))ds \\ &\geq N\int_{\eta_1}^{1} G(1,s)u(s)ds \geq \gamma N\int_{\eta_1}^{1} G(1,s)ds\|u\| = \|u\|.\end{aligned}$$

So

$$\|Au\| \geq \|u\| \quad \text{for } u \in \partial\Omega_R. \tag{3.6}$$

Thus, from (3.5), (3.6) and Lemma 2.3, we know that the operator A has at least one fixed point $u \in \Omega_R \setminus \overline{\Omega}_r$, which is a positive solution of (P). □

Theorem 3.3. *Under the assumptions $f_0 > N$, $f_\infty > N$, there exists a $p > 0$ such that $f(t,u) < Mp$, $(t,u) \in [0,1] \times [0,p]$, then* (P) *has at least two positive solutions u_1 and u_2 satisfying* $0 < \|u_1\| < p < \|u_2\|$.

Proof. Since $f_0 > N$, we can find $0 < r_0 < p$ so that $f(t,u) > Nu$, $t \in [0,1]$, $0 < u \leq r_0$. Let $\Omega_{r_0} = \{u \in K \mid \|u\| < r_0\}$, similar to the proof of Theorem 3.1, one gets

$$\|Au\| \geq \|u\| \quad \text{for } u \in \partial\Omega_{r_0}. \tag{3.7}$$

On the other hand, since $f_\infty > N$, there exist a $H > 0$ such that $f(t,u) > Nu$ for $t \in [0,1]$, $u \geq H$. Choose $R > R_0 := \max\{\frac{H}{\gamma}, p+1\}$, let $\Omega_R = \{u \in K \mid \|u\| < R\}$, similar to the proof of Theorem 3.2, we have

$$\|Au\| \geq \|u\|, \quad u \in \partial\Omega_R. \tag{3.8}$$

Let $\Omega_p = \{u \in K \mid \|u\| < p\}$. For $u \in \partial\Omega_p$, we know $f(t,u(t)) < Mp$ for $(t,u) \in [0,1] \times [0,p]$. By Lemma 2.5, so $\|Au\| \leq \int_0^1 G(t,s)f(s,u(s))ds \leq \int_0^1 k(s)Mp\,ds = p = \|u\|$. Thus

$$\|Au\| \leq \|u\| \quad \text{for } u \in \partial\Omega_p. \tag{3.9}$$

Therefore, by (3.7), (3.8), (3.9) and Lemma 2.3, we know that operator A has at least two fixed points u_1 and u_2 satisfying $0 < \|u_1\| < p < \|u_2\|$, which are two positive solutions of (P). □

Theorem 3.4. *Under the assumptions that $f^0 < M$, $f^\infty < M$, and there exists a $p > 0$ such that $f(t,u) > Nu$, $(t,u) \in [\eta_1, 1] \times [\gamma p, p]$, then* (P) *has at least two positive solutions u_1 and u_2 satisfying* $0 < \|u_1\| < p < \|u_2\|$.

Proof. Since $f^0 < M$, there exists $0 < r_0 < p$ such that $f(t,u) < Mu$, $t \in [0,1]$, $0 < u \le r_0$. Let $\Omega_{r_0} = \{u \in K \mid \|u\| < r_0\}$, similar to the proof of Theorem 3.2, one has

$$\|Au\| \le \|u\|, \quad u \in \partial\Omega_{r_0}. \tag{3.10}$$

On the other hand, let $R = \max\{r_0 + \frac{L}{M}, H, p+1\}$, $\Omega_R = \{u \in K \mid \|u\| < R\}$, similar to the proof of Theorem 3.1, we get

$$\|Au\| \le \|u\| \quad \text{for } u \in \partial\Omega_R. \tag{3.11}$$

Let $\Omega_p = \{u \in K \mid \|u\| < p\}$, for $u \in \partial\Omega_p$, we obtain $\gamma p \le u(t) \le p$, $t \in [\eta_1, 1]$. Thus $f(t,u) > Nu$, $(t,u) \in [\eta_1, 1] \times [\gamma p, p]$. In view of Lemma 2.5, then

$$Au(1) \ge \int_{\eta_1}^{1} G(1,s) f(s,u(s)) ds > N\gamma p \int_{\eta_1}^{1} G(1,s) ds = p = \|u\|.$$

So

$$\|Au\| \ge \|u\| \quad \text{for } u \in \partial\Omega_p. \tag{3.12}$$

Therefore, by (3.10), (3.11), (3.12) and Lemma 2.3, we know that operator A has at least two fixed points u_1 and u_2 satisfying $0 < \|u_1\| < p < \|u_2\|$, which are two positive solutions of (P). □

Remark 3.5. In this paper, if $\alpha = 2$, (P) is reduced to 2^{nd}-order BVPs. Through the verification, we can verify that the properties of the Green's function are still satisfied, so these results still hold. The results of this paper generalized the main results in [4] (α is an integer, m=3), [11] ($\alpha = 2$), [12] ($\alpha = 2$, $m = 3$) and [14] (α is an integer).

Remark 3.6. In this paper, we get some existence results for positive solutions to (P), when the nonlinearity term is nonnegative. Future research will consider the existence of positive solution to (P) with sign changing nonlinearity.

Example 3.7. Consider the following problem

$$\begin{cases} {}_R D_{0^+}^{2.5} u(t) + u^p(t) = 0, & t \in [0,1], \\ u(0) = u'(0) = 0, & u(1) = 2u(1/16) + 4u(1/4). \end{cases}$$

Clearly, $D = \frac{11}{16} > 0$, $f(u) = u^p$ with $p > 0$ and $p \ne 1$, then $f : [0,\infty) \to [0,\infty)$ is continuous. When $0 < p < 1$, we have $f_0 = \infty$, $f^\infty = 0$; when $p > 1$, we have $f^0 = 0$, $f_\infty = \infty$, so, by Theorem 3.1 and 3.2, the problem has at least one positive solution.

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