Dynamic Systems and Applications 23 (2014) 601-618

AN EXISTENCE RESULT FOR SYSTEMS OF SECOND-ORDER BOUNDARY VALUE PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. In this note we consider the system

$$\begin{aligned} x''(t) &= -\lambda_1 f(t, x(t), y(t)), \quad t \in (0,1) \\ y''(t) &= -\lambda_2 g(t, x(t), y(t)), \quad t \in (0,1) \\ x(0) &= \varphi(x),\ y(0) = \psi(y) \\ x(1) &= 0 = y(1), \end{aligned}$$

and demonstrate that under suitable conditions on the functions f, $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to [0,+\infty)$ and the functionals φ, $\psi : \mathcal{C}([0,1]) \to \mathbb{R}$ this problem admits at least one positive solution. Since the functionals φ and ψ can be nonlinear, the boundary condition at $t = 0$ can be quite general. Our approach is based on supposing that each of $x \mapsto \varphi(x)$ and $y \mapsto \psi(y)$ behaves, in some sense, like a linear functional as $\|(x,y)\| \to +\infty$, and to this end we utilize the concept of the Fréchet derivative at $+\infty$ in our existence proof. We conclude by providing an explicit example of and discussion regarding our existence theorem.

AMS (MOS) Subject Classification. Primary: 34B09, 34B10, 34B15, 34B18, 47H07. Secondary: 47G10, 47H10.

1. Introduction

In this note we consider the boundary value problem

$$\begin{aligned} x''(t) &= -\lambda_1 f(t, x(t), y(t)), \quad t \in (0,1) \\ y''(t) &= -\lambda_2 g(t, x(t), y(t)), \quad t \in (0,1) \\ x(0) &= \varphi(x),\ y(0) = \psi(y) \\ x(1) &= 0 = y(1). \end{aligned} \tag{1.1}$$

Both the functions f, $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to [0,+\infty)$ and the functionals φ, ψ : $\mathcal{C}([0,1]) \to [0,+\infty)$ will always be assumed to be continuous; furthermore, λ_1, $\lambda_2 > 0$ are parameters. As will be clarified momentarily the functionals φ and ψ do not

Received September 2013

need to be linear. Rather, our primary assumption is that, in some sense, they are asymptotically related to a linear functional possessing suitable structure.

More precisely, we assume that there are *linear* functionals $L_1,\, L_2 : \mathcal{C}([0,1]) \to \mathbb{R}$ with the property that for each $\varepsilon > 0$ there is $M_\varepsilon > 0$ such that

$$|\varphi(x) - L_1(x)| \leq \varepsilon \|(x,y)\| \text{ and } |\psi(y) - L_2(y)| \leq \varepsilon \|(x,y)\| \tag{1.2}$$

whenever $(x,y) \in \mathcal{C}([0,1]) \times \mathcal{C}([0,1])$ satisfies $\|(x,y)\| \geq M_\varepsilon$; we provide in Example 3.3 an explicit demonstration of this condition. As will be stated later, throughout this work for functions $(x,y) \in \mathcal{C}([0,1]) \times \mathcal{C}([0,1])$ we define $\|x\| := \max_{t\in[0,1]} |x(t)|$ and $\|(x,y)\| := \|x\| + \|y\|$. A second important feature of our structure assumptions is that there exist linear functionals $\widetilde{L}_1,\, \widetilde{L}_2 : \mathcal{C}([0,1]) \to \mathbb{R}$ such that $\varphi(x) \geq \widetilde{L}_1(x) \geq 0$ and $\psi(y) \geq \widetilde{L}_2(y) \geq 0$ for each vector (x,y) in a suitable cone $\mathcal{K} \subseteq \mathcal{C}([0,1]) \times \mathcal{C}([0,1])$. In particular, this allows for $\varphi(x)$ and $\psi(y)$ to be nonpositive for some elements of $\mathcal{C}([0,1]) \times \mathcal{C}([0,1])$. Specifically, this latter point is achieved by ensuring that $\widetilde{L}_i$ remains nonnegative, for each i, on the same cone $\mathcal{K}$. By means of some ideas of Infante and Webb [27] we are able to control the sign of each of the functionals $\widetilde{L}_1$ and $\widetilde{L}_2$. In particular, this allows us to control the sign of the *nonlinear* functionals φ and ψ, whose signs we ordinarily could not control.

Let us next mention briefly certain of the relevant recent work in nonlocal BVPs (boundary value problems) and how these works are related to this present note. First of all, the study of nonlocal BVPs in general has seen much recent work. This has occurred in the context of the p-Laplacian problem [4, 5], discrete and continuous fractional calculus [2, 3, 7, 11, 15], differential inclusions [14], and integer-order differential equations [6, 8, 9, 10, 12, 13, 17, 18, 19, 20, 21, 22, 24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35].

More specifically, Infante and Webb [27, 28, 29, 30, 31] have produced some very general results in case the nonlocalities are linear functionals realized as Stieltjes integrals – say, the problem

$$\begin{aligned} -y''(t) &= f(t, y(t)), \quad t \in (0,1) \\ y(0) &= \int_{[0,1]} y(s)\,d\alpha(s) \\ y(1) &= \int_{[0,1]} y(s)\,d\beta(s), \end{aligned} \tag{1.3}$$

where $\alpha,\, \beta : [0,1] \to \mathbb{R}$ are of bounded variation on $[0,1]$ and not necessarily monotone. This leads to the nontrivial and mathematically interesting question of whether problem (1.3) may admit a nontrivial *positive* solution in light of the fact that the measures associated to the Stieltjes integrals in (1.3) are *signed*. One of they key advances produced by Infante and Webb was to incorporate the nonnegativity conditions

$$\int_{[0,1]} y(s)\ d\alpha(s) \geq 0 \text{ and } \int_{[0,1]} y(s)\ d\beta(s) \geq 0$$

into a suitable, new cone, thereby obtaining sufficient control over the sign of the fixed points of an appropriate integral operator. This novel insight has proved to be very useful in studying nonlocal BVPs with linear boundary conditions. Furthermore, in addition to the aforementioned works of Infante and Webb, some papers by Yang [34, 35] as well as Graef and Webb [17] have also addressed nonlocal BVPs of the type embodied by (1.3).

On the other hand, in the case of *nonlinear*, nonlocal boundary conditions a number of recent papers by Infante [19], Infante and Pietramala [20, 22, 23], Goodrich [8, 9, 10, 12, 13, 16], and Yang [32, 33] have all addressed this class of problem. The archetypical problem in this setting is essentially problem (1.1), of which the simpler scalar version is

$$\begin{aligned} -y''(t) &= f(t, y(t)), \quad t \in (0,1) \\ y(0) &= \varphi(y) \\ y(1) &= \psi(y), \end{aligned} \tag{1.4}$$

where φ, $\psi : \mathcal{C}([0,1]) \to \mathbb{R}$ are (possibly) nonlinear functions. Many recent studies such as [8, 9, 10, 13, 19, 20, 22, 23, 32, 33] assume a rather restricted form of (1.4) – namely, that the nonlinear functionals may be realized in the form $H \circ L$, where $H : \mathbb{R} \to \mathbb{R}$ is a continuous function, equipped with some growth or structural properties, and $L : \mathcal{C}([0,1]) \to \mathbb{R}$ is a *linear* functional. By requiring this special form, one may utilize the growth and structural hypotheses imposed on H together with the linearity of L to deduce existence of a nontrivial positive solution to problem (1.4).

In case either φ or ψ does not possess this special form, then existence of a nontrivial positive solution becomes more difficult to demonstrate. Indeed, the goal of this note is to address this difficulty in the context of problem (1.1). It is worth noting that we considered in [12] a problem similar to (1.1) – namely, the boundary value problem

$$\begin{aligned} -y''(t) &= \lambda f(t, y(t)), \quad t \in (0,1) \\ y(0) &= \varphi(y) \\ y(1) &= 0, \end{aligned} \tag{1.5}$$

for $\lambda > 0$ a real-valued parameter; in [12] the special case $f(t, y) := a(t)g(y)$ was especially treated. Consequently, one difference is that in [12] we did not consider systems of second-order equations. Thus, the results of this work extend certain of the ideas of [12] to the systems case. Another difference is that although we studied similar boundary conditions, as (1.5) demonstrates, we utilized the slightly weaker condition: there exists $\rho \in [0, 1)$ such that

$$\limsup_{\|y\|\to+\infty} \frac{|\varphi(y)|}{\|y\|} < \rho, \tag{1.6}$$

for all y in a suitable cone. On the other hand, in [12] we only obtained results in case either $\lambda = 1$ or λ was large; in particular, the case of small λ was entirely excluded. By contrast, the result we present here holds for each $(\lambda_1, \lambda_2) \in (0, +\infty) \times (0, +\infty)$. In addition, in [12] we had to make more substantial assumptions about the function f than we do here on the corresponding functions f and g appearing in (1.1). In summary, then, we show in this work that by slightly strengthening the asymptotic condition on φ (and ψ) we can correspondingly weaken the other structural conditions, whilst simultaneously obtaining a result that imposes no restriction on λ_1, λ_2 – other than a positivity restriction.

We accomplish this generalization of [12] by utilizing a different fixed point theorem. In particular, in [12] we utilized the well known Krasnosel'skiĭ's fixed point theorem. While a very common and successful approach to deducing existence of solution to ordinary differential equations equipped with a variety of boundary conditions, this approach does have its limitations. In this note we demonstrate that by instead using a fixed point theorem related to asymptotically linear operators along a cone we can, in fact, achieve a somewhat different and, in certain settings, rather more general result.

2. Preliminaries

Letting $\mathcal{C}([0, 1])$ be equipped with the usual maximum norm

$$\|x\| := \max_{t\in[0,1]} |x(t)|,$$

define the operators T_1, $T_2 : \mathcal{C}([0, 1]) \times \mathcal{C}([0, 1]) \to \mathcal{C}([0, 1])$ by

$$(T_1(x, y))(t) := (1 - t)\varphi(x) + \lambda_1 \int_0^1 G(t, s)f(s, x(s), y(s))\, ds \tag{2.1}$$

and

$$(T_2(x, y))(t) := (1 - t)\psi(y) + \lambda_2 \int_0^1 G(t, s)g(s, x(s), y(s))\, ds. \tag{2.2}$$

Now consider the normed space $\mathcal{X} := \mathcal{C}([0, 1]) \times \mathcal{C}([0, 1])$ when equipped with the norm $\|(x, y)\| := \|x\| + \|y\|$. Then it is well known that $\mathcal{X}$ is a Banach space – see [1]. Define the operator $T : \mathcal{X} \to \mathcal{X}$ by

$$(T(x, y))(t) := ((T_1(x, y))(t), (T_2(x, y))(t)). \tag{2.3}$$

Then we see that T may be studied as a means of deducing the existence of positive solutions to problem (1.1). In particular, a fixed point of T is a solution of the system (1.1).

Let us also remark that in the case of (2.1)–(2.2) the function $G : [0,1]\times[0,1] \to \mathbb{R}$ is the Green's function associated to the conjugate problem – namely (see [25]),

$$G(t,s) := \begin{cases} t(1-s), & 0 \le t \le s \le 1 \\ s(1-t), & 0 \le s \le t \le 1 \end{cases}. \tag{2.4}$$

We assume here and throughout that $[a,b]$ is a given fixed subinterval of $(0,1)$ with $0 < a < b < 1$. Then there exists a constant $\gamma = \gamma(a,b) := \min_{t\in[a,b]}\{t, 1-t\}$ such that

$$\min_{t\in[a,b]} G(t,s) \ge \gamma \max_{t\in[0,1]} G(t,s) = \gamma G(s,s), \tag{2.5}$$

for each $s \in [0,1]$; observe that γ also satisfies $1-t \ge \gamma$ for each $t \in [a,b]$. Finally, both recall that $\max_{t\in[0,1]} G(t,s) = G(s,s)$, for each $s \in [0,1]$, and note that $\gamma \in (0,1)$.

We next recall the fixed point theorem, which we shall use in this work. As mentioned in Section 1, the use of Krasnosel'skiĭ's theorem is rather omnipresent in the recent literature. For our existence theorem we eschew this approach and instead use the concept of the Fréchet derivative at $+\infty$ of a suitable operator T. Since we will utilize this result in the context of an order cone, $\mathcal{K}$, we state the result in that form – see [36, §7.9] for more details. It should be noted, as will become apparent in Section 3, that by using this result we make direct use of the "asymptotic linearity" of the functionals φ and ψ appearing in (1.1), as was mentioned earlier in Section 1.

Definition 2.1 ([36, Definition 7.32.b])**.** Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces over $\mathbb{R}$. Set

$$U(+\infty, r) := \{x \in \mathcal{X} : \|x\| \ge r\},$$

where $r > 0$. Let $T : \mathcal{X} \to \mathcal{Y}$ be an operator. If $\mathcal{X}$ has an order cone $\mathcal{K}$, then the operator $T'(+\infty) \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, where $\mathcal{L}(\mathcal{X},\mathcal{Y})$ is the collection of all linear transformations between $\mathcal{X}$ and $\mathcal{Y}$, is called the **positive Fréchet derivative of T at $+\infty$ along the cone $\mathcal{K}$** if and only if there is a fixed $r > 0$ such that for all $x \in U(+\infty, r) \cap \mathcal{K}$ it holds that

$$Tx = T'(+\infty)x + o(\|x\|),\ \|x\| \to +\infty;$$

that is,

$$\frac{\|Tx - T'(+\infty)x\|}{\|x\|} \to 0 \text{ as } \|x\| \to +\infty.$$

Lemma 2.2 ([36, Corollary 7.34])**.** *Suppose that*

1. $T : \mathcal{K} \subseteq \mathcal{X} \to \mathcal{K}$ *is a compact operator on the Banach space* $\mathcal{X}$ *with order cone* $\mathcal{K}$*; and*

2. *$T'(+\infty)$ exists as a positive Fréchet derivative of T at $+\infty$, and its spectral radius, denoted $r\left(T'(+\infty)\right)$, satisfies $r\left(T'(+\infty)\right) < 1$.*

Then T has a fixed point.

Henceforth, we shall work in the cone $\mathcal{K} \subseteq \mathfrak{X}$ defined by

$$\mathcal{K} := \Big\{ (x,y) \in \mathfrak{X} : x(t), y(t) \geq 0, \min_{t\in[a,b]} [x(t)+y(t)] \geq \gamma \|(x,y)\|, \\ \widetilde{L}_1(x) \geq 0, \widetilde{L}_2(y) \geq 0 \Big\}, \tag{2.6}$$

where $\widetilde{L}_1, \ \widetilde{L}_2 : \mathcal{C}([0,1]) \to \mathbb{R}$ are linear functions on which we shall momentarily impose some structural hypothesis – cf., (H1) and (H3) in the sequel. The cone $\mathcal{K}$ is a slight modification of a cone originally introduced by Infante and Webb [27]. Note that $\mathcal{K} \neq \varnothing$ since due to condition (H3) below, it holds that $(1-t, 1-t) \in \mathcal{K}$.

We next provide the structural conditions that we impose on problem (1.1). As already noted one of the principal conditions we utilize is that φ and ψ in some sense behave like the linear functionals L_1 and L_2, respectively, as $\|(x,y)\| \to +\infty$. This notion of asymptotic relatedness is made precise in condition (H1) below. In addition, and as also mentioned in the introduction, we also assume the existence of linear functions $\widetilde{L}_1$ and $\widetilde{L}_2$ that form lower bounds for φ and ψ, respectively.

H1: Assume that there exist linear functionals $\widetilde{L}_1$, $\widetilde{L}_2$, L_1, $L_2 : \mathcal{C}([0,1]) \to \mathbb{R}$, which may be realized in the form

$$L_1(y) := \int_{[0,1]} y(s)\ d\alpha_1(s) \text{ and } L_2(y) := \int_{[0,1]} y(s)\ d\alpha_2(s)$$

and

$$\widetilde{L}_1(y) := \int_{[0,1]} y(s)\ d\widetilde{\alpha}_1(s) \text{ and } \widetilde{L}_2(y) := \int_{[0,1]} y(s)\ d\widetilde{\alpha}_2(s),$$

where α_1, α_2, $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2 \in BV([0,1])$, such that each of

$$\lim_{\substack{\|(x,y)\|\to+\infty \\ (x,y)\in\mathcal{K}}} \frac{|\varphi(x) - L_1(x)|}{\|(x,y)\|} = 0 \text{ and } \lim_{\substack{\|(x,y)\|\to+\infty \\ (x,y)\in\mathcal{K}}} \frac{|\psi(y) - L_2(y)|}{\|(x,y)\|} = 0$$

and

$$\varphi(x) \geq \widetilde{L}_1(x) \text{ and } \psi(y) \geq \widetilde{L}_2(y)$$

holds for each $(x,y) \in \mathcal{K}$.

H2: There are constants C_1, $C_2 \geq 0$ such that

$$|L_1(x)| \leq C_1\|x\| \text{ and } |L_2(y)| \leq C_2\|y\|,$$

for each $(x,y) \in \mathfrak{X}$.

H3: For each $i = 1, 2$ each of

$$\int_{[0,1]} (1-t)\ d\widetilde{\alpha}_i(t) \geq 0$$

and

$$\int_{[0,1]} G(t,s)\ d\widetilde{\alpha}_i(t) \geq 0$$

holds, where the latter holds for each $s \in [0,1]$.

H4: Assume that each of

$$\lim_{x+y\to+\infty} \frac{f(t,x,y)}{x+y} = 0$$

and

$$\lim_{x+y\to+\infty} \frac{g(t,x,y)}{x+y} = 0$$

holds, uniformly for $t \in [0,1]$.

Remark 2.3. We emphasize that the last part of condition (H1) does *not* require that either the map $x \mapsto \varphi(x)$ or the map $y \mapsto \psi(y)$ is nonnegative for all $(x,y) \in \mathcal{X}$. Indeed, this nonnegativity, by virtue of the functionals $\widetilde{L}_1$ and $\widetilde{L}_2$, need only hold for $(x,y) \in \mathcal{K}$.

We conclude by stating Lemma 2.4, which asserts that the cone $\mathcal{K}$ is invariant under the action of the operator T. Although the proof is straightforward, we include it for the sake of completeness.

Lemma 2.4. *Let T be the operator defined in* (2.3). *Provided that conditions* (H1) *and* (H3) *hold, then $T(\mathcal{K}) \subseteq \mathcal{K}$.*

Proof. Given $(x,y) \in \mathcal{K}$ fixed but arbitrary, it is obvious that $\left(T_1(x,y)\right)(t) \geq 0$, $\left(T_2(x,y)\right)(t) \geq 0$ for each $t \in [0,1]$; note that this uses the fact that, for example, $\varphi(x) \geq \widetilde{L}_1(x) \geq 0$, for each $(x,y) \in \mathcal{K}$. In addition we compute

(2.7)

$$\begin{aligned}\min_{t\in[a,b]} \left(T_1(x,y)\right)(t) &\geq \varphi(x) \min_{t\in[a,b]}(1-t) + \lambda_1 \min_{t\in[a,b]} \int_0^1 G(t,s) f(s,x(s),y(s))\ ds \\ &\geq \gamma\varphi(x) \max_{t\in[0,1]}(1-t) + \lambda_1\gamma \max_{t\in[0,1]} \int_0^1 G(t,s) f(s,x(s),y(s))\ ds \\ &\geq \gamma\|T_1(x,y)\|.\end{aligned}$$

Notice that in (2.7) we have again used the fact that $\varphi(x) \geq \widetilde{L}_1(x) \geq 0$ for each $(x,y) \in \mathcal{K}$. Since a similar calculation shows that $\min_{t\in[a,b]} \left(T_2(x,y)\right)(t) \geq \gamma\|T_2(x,y)\|$, we conclude that $\min_{t\in[a,b]} \left(\left(T_1(x,y)\right)(t) + \left(T_2(x,y)\right)(t)\right) \geq \gamma\left\|\left(T_1(x,y), T_2(x,y)\right)\right\|$.

Finally, by condition (H3), the fact that $\varphi(x) \geq \widetilde{L}_1(x) \geq 0$ on $\mathcal{K}$, and the nonnegativity of f we may write

$$\begin{aligned} &\widetilde{L}_1\left(T_1(x,y)\right) \\ &= \varphi(x)\int_{[0,1]}(1-t)\ d\widetilde{\alpha}_1(t) + \lambda_1\int_{[0,1]}\int_0^1 G(t,s)f(s,x(s),y(s))\ ds\ d\widetilde{\alpha}_1(t) \\ &= \varphi(x)\int_{[0,1]}(1-t)\ d\widetilde{\alpha}_1(t) + \lambda_1\int_0^1\left[\int_{[0,1]} G(t,s)\ d\widetilde{\alpha}_1(t)\right]f(s,x(s),y(s))\ ds \\ &\geq 0. \end{aligned} \tag{2.8}$$

Since it similarly holds that $\widetilde{L}_2\left(T_2(x,y)\right) \geq 0$, we conclude that $T(\mathcal{K}) \subseteq \mathcal{K}$, and this completes the proof. □

3. Statement and Proof of Existence Theorem and Discussion

We now provide an existence theorem for problem (1.1). After stating and proving this result, which is Theorem 3.2, we then provide an example, which should help to explicate the use of our result. Finally, we provide some concluding discussion regarding Theorem 3.2. Before proceeding with this program, however, we need an easy preliminary lemma that will play a key role in the proof of Theorem 3.2.

Lemma 3.1. *Suppose that* $f : [0,1]\times[0,+\infty)\times[0,+\infty) \to [0,+\infty)$ *is a continuous function satisfying*

$$\lim_{x+y\to+\infty}\frac{f(t,x,y)}{x+y} = 0,$$

uniformly for $t \in [0,1]$. *Let* $M : [0,+\infty) \to [0,+\infty)$ *be the function defined by*

$$M(r) := \max_{(t,x,y)\in[0,1]\times[0,r]\times[0,r]} f(t,x,y). \tag{3.1}$$

Then it holds that

$$\lim_{r\to+\infty}\frac{M(r)}{r} = 0. \tag{3.2}$$

Proof. Suppose for contradiction that the conclusion of the lemma was false. Then there would be a sequence $\{r_i\}_{i=1}^{\infty} \subseteq [0,+\infty)$ such that $r_i \to +\infty$ and

$$\frac{M\left(r_i\right)}{r_i} \geq \eta > 0, \tag{3.3}$$

for some constant $\eta > 0$. By definition there would then exist a sequence, say $\{(t_i,x_i,y_i)\}_{i=1}^{\infty} \subseteq [0,1]\times[0,r_i]\times[0,r_i]$, such that

$$M\left(r_i\right) = f\left(t_i,x_i,y_i\right); \tag{3.4}$$

that is, f is maximal on $[0,1]\times[0,r_i]\times[0,r_i]$ at the point (t_i,x_i,y_i).

Clearly, the result is trivial if f is bounded. So, let us assume that f is unbounded as $x+y \to +\infty$. Since $r_i \to +\infty$ and f is unbounded as $x+y \to +\infty$, it must hold that $x_i + y_i \to +\infty$. Consequently, either $x_i \to +\infty$ or $y_i \to +\infty$. Regardless, since it holds that $0 \le x_i,\ y_i \le r_i$, we estimate

$$\frac{1}{x_i + y_i} \ge \frac{1}{2r_i}. \tag{3.5}$$

But in observation of (3.5) we then calculate

$$\frac{f\left(t_i, x_i, y_i\right)}{x_i + y_i} = \frac{M\left(r_i\right)}{x_i + y_i} \ge \frac{M\left(r_i\right)}{2r_i} \ge \frac{1}{2}\eta > 0, \tag{3.6}$$

for each $i \in \mathbb{N}$, which contradicts the assumption that

$$\frac{f(t,x,y)}{x+y} \to 0$$

as $x + y \to +\infty$, uniformly for $t \in [0,1]$. And this completes the proof. □

Theorem 3.2. *Suppose that each of conditions* (H1)–(H4) *holds. Suppose, in addition, that at least one of the following conditions holds.*

1. $\varphi(0) > 0$
2. $\psi(0) > 0$
3. *The partial map* $t \mapsto f(t, 0, 0)$ *is not zero for a.e.* $t \in [0, 1]$
4. *The partial map* $g \mapsto g(t, 0, 0)$ *is not zero for a.e.* $t \in [0, 1]$

Finally, assume that

$$\max\left\{C_1, C_2\right\} < 1,$$

where C_1 *and* C_2 *are from condition* (H2). *Then for each* $(\lambda_1, \lambda_2) \in (0, +\infty)\times(0, +\infty)$ *problem* (1.1) *has at least one positive solution.*

Proof. Define the operator $T'(+\infty) : \mathcal{X} \to \mathcal{X}$ by

$$(T'(+\infty)(x,y))(t) := \left(\underbrace{(1-t)L_1(x)}_{:=\left(T_1'(+\infty)\right)(x,y)}, \underbrace{(1-t)L_2(y)}_{:=\left(T_2'(+\infty)\right)(x,y)}\right). \tag{3.7}$$

Observe that $T'(+\infty) \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ – i.e., it is a linear operator from $\mathcal{X}$ to $\mathcal{X}$, owing to the fact that each of L_1 and L_2 is linear. We will demonstrate that

$$T'(+\infty)(x,y) - T(x,y) = o(\|(x,y)\|),\ \|(x,y)\| \to +\infty, \tag{3.8}$$

thus demonstrating that $T'(+\infty)$ is, in fact, the Fréchet derivative of T at $+\infty$ along the cone $\mathcal{K}$.

To this end, fix $\lambda_1,\ \lambda_2 > 0$ and let $\varepsilon > 0$ be fixed but otherwise arbitrary. Let $M_f,\ M_g : [0, +\infty) \to [0, +\infty)$ be the continuous functions defined in (3.1) relative to f and g – that is to say,

$$M_f(r) := \max_{(t,x,y)\in[0,1]\times[0,r]\times[0,r]} f(t,x,y)$$

and

$$M_g(r) := \max_{(t,x,y)\in[0,1]\times[0,r]\times[0,r]} g(t,x,y).$$

We claim that we may choose a number $r_0 > 0$ sufficiently large such that

$$\max\{M_f(r_0), M_g(r_0)\} \leq \frac{\varepsilon}{6\min\{\lambda_1,\lambda_2\}\int_0^1 G(s,s)\,ds} r_0; \tag{3.9}$$

that this is true follows from Lemma 3.1. By choosing r_0 even larger if necessary, condition (H1) implies that we can obtain the estimates

$$|\varphi(x) - L_1(x)| < \frac{\varepsilon}{6}\|(x,y)\| \text{ and } |\psi(y) - L_2(y)| < \frac{\varepsilon}{6}\|(x,y)\|, \tag{3.10}$$

whenever $\|(x,y)\| \geq r_0$. Finally, since condition (H4) holds, we can simultaneously obtain

$$f(t,x,y) \leq \frac{\varepsilon}{6\lambda_1\int_0^1 G(s,s)\,ds}(x+y) \text{ and } g(t,x,y) \leq \frac{\varepsilon}{6\lambda_2\int_0^1 G(s,s)\,ds}(x+y) \tag{3.11}$$

whenever $x + y \geq r_0$ by once again choosing r_0 even larger if necessary; note that inequality (3.11) holds uniformly for each $t \in [0,1]$. Observe that with r_0 fixed as above, we obtain for each $(t,x,y) \in [0,1]\times[0,r_0]\times[0,r_0]$ the estimate

$$0 \leq f(t,x,y) \leq M_f(r_0) \text{ and } g(t,x,y) \leq M_g(r_0), \tag{3.12}$$

by the definitions of M_f and M_g.

Henceforth let r_0 be fixed as above so that (3.9)–(3.11) hold, and for each given $(x,y) \in \mathfrak{X}$ define the measurable sets $\mathcal{N}$ and $[0,1]\setminus\mathcal{N}$ by, respectively,

$$\mathcal{N} := \{s \in [0,1] : 0 \leq x(s) + y(s) \leq r_0\} \tag{3.13}$$

and

$$[0,1]\setminus\mathcal{N} := \{s \in [0,1] : x(s) + y(s) > r_0\}. \tag{3.14}$$

Evidently, for each $s \in \mathcal{N}$ it holds that $0 \leq x(s), y(s) \leq r_0$. With the preceding estimates in hand, we first estimate for each $(x,y) \in \mathcal{K}$ satisfying $\|(x,y)\| \geq r_0$ and each $t \in [0,1]$ that

$$
\begin{aligned}
&\left|(1-t)\varphi(x)-(1-t)L_1(x)+\lambda_1\int_0^1 G(t,s)f(s,x(s),y(s))\ ds\right| \\
&\le |(1-t)\varphi(x)-(1-t)L_1(x)|+\lambda_1\int_0^1 G(s,s)f(s,x(s),y(s))\ ds \\
&\le |(1-t)\varphi(x)-(1-t)L_1(x)| \\
&\quad +\lambda_1\int_{\mathcal{N}} G(s,s)f(s,x(s),y(s))\ ds+\lambda_1\int_{[0,1]\setminus\mathcal{N}} G(s,s)f(s,x(s),y(s))\ ds \\
&\le \frac{\varepsilon}{6}\|(x,y)\|+\lambda_1\int_{\mathcal{N}} G(s,s)M_f\left(r_0\right)\ ds+\lambda_1\int_{[0,1]\setminus\mathcal{N}} G(s,s)f(s,x(s),y(s))\ ds \\
&\le \frac{\varepsilon}{6}\|(x,y)\|+\left(M_f\left(r_0\right)+\frac{\varepsilon}{6\lambda_1\int_0^1 G(s,s)\ ds}\|(x,y)\|\right)\lambda_1\int_0^1 G(s,s)\ ds \\
&\le \frac{\varepsilon}{6}\|(x,y)\| \\
&\quad +\left(\frac{\varepsilon}{6\min\left\{\lambda_1,\lambda_2\right\}\int_0^1 G(s,s)\ ds}r_0+\frac{\varepsilon}{6\lambda_1\int_0^1 G(s,s)\ ds}\|(x,y)\|\right)\lambda_1\int_0^1 G(s,s)\ ds \\
&\le \frac{\varepsilon}{2}\|(x,y)\|.
\end{aligned}
\tag{3.15}
$$

In an entirely similar fashion we also may estimate for each $(x,y)\in\mathcal{K}$ satisfying $\|(x,y)\|\ge r_0$ and each $t\in[0,1]$

$$
\left|(1-t)\psi(y)-(1-t)L_2(y)+\lambda_2\int_0^1 G(t,s)f(s,x(s),y(s))\ ds\right|\le\frac{\varepsilon}{2}\|(x,y)\|. \tag{3.16}
$$

By the arbitrariness of $t\in[0,1]$ in each of estimates (3.15) and (3.16) we thus estimate

$$
\begin{aligned}
&\|T(x,y)-T'(+\infty)(x,y)\| \\
&=\left\|T_1(x,y)-\left(T_1(+\infty)\right)(x,y)\right\|+\left\|T_2(x,y)-\left(T_2(+\infty)\right)(x,y)\right\| \\
&=\max_{t\in[0,1]}\left|(1-t)\varphi(x)-(1-t)L_1(x)+\lambda_1\int_0^1 G(t,s)f(s,x(s),y(s))\ ds\right| \\
&\quad +\max_{t\in[0,1]}\left|(1-t)\psi(y)-(1-t)L_2(y)+\lambda_2\int_0^1 G(t,s)f(s,x(s),y(s))\ ds\right| \\
&\le\frac{\varepsilon}{2}\|(x,y)\|+\frac{\varepsilon}{2}\|(x,y)\| \\
&=\varepsilon\|(x,y)\|.
\end{aligned}
\tag{3.17}
$$

But from (3.17) we obtain that

$$
\frac{\|T(x,y)-T'(+\infty)(x,y)\|}{\|(x,y)\|}\le\varepsilon, \tag{3.18}
$$

whenever $\|(x,y)\| \geq r_0$. Since ε was arbitrary, it follows that (3.8) holds, as claimed. Consequently, by definition we have that $T'(+\infty)$ is the Fréchet derivative of T at $+\infty$ along the cone $\mathcal{K}$.

It remains to show that the operator T has a nontrivial fixed point in $\mathcal{K}$. To this end we shall now appeal to Lemma 2.2. So, suppose for contradiction that there was an eigenvalue $\mu \in [1,+\infty)$ and an associated eigenvector $(x,y) \in \mathfrak{X}$ such that $T'(+\infty)(x,y) = \mu(x,y)$. Since μ is an eigenvalue with associated eigenvector (x,y) we must have $\|(x,y)\| > 0$ and, moreover, for each $t \in [0,1]$ we can write

$$\mu(x(t),y(t)) = \left(\left(T_1'(+\infty)\right)(x,y), \left(T_2'(+\infty)\right)(x,y)\right) = \left((1-t)L_1(x),(1-t)L_2(y)\right), \tag{3.19}$$

which implies that

$$\mu x(t) = (1-t)L_1(x) \text{ and } \mu y(t) = (1-t)L_2(y). \tag{3.20}$$

Each of the equalities in (3.20) must hold for each $t \in [0,1]$. Since x and y are fixed, moreover, $L_1(x)$ and $L_2(y)$ are real-valued constants. We endeavor to show that (3.20) has only the trivial solution whenever $\mu \geq 1$ – thus contradicting the fact that $(\mu,(x,y))$ is an eigenpair.

So, observe that (3.20) implies that

$$x(t) = (1-t)\mu^{-1}L_1(x), \tag{3.21}$$

for each $t \in [0,1]$. Accordingly, we then estimate

$$\begin{aligned}(1-t)\mu^{-1}L_1(x) &= (1-t)\mu^{-1}\int_0^1 x(s)\ d\alpha_1(s)\\ &= (1-t)\mu^{-1}\int_0^1 (1-s)\mu^{-1}L_1(x)\ d\alpha_1(s)\\ &= (1-t)\mu^{-2}L_1(x)\int_0^1 (1-s)\ d\alpha_1(s),\end{aligned} \tag{3.22}$$

whence

$$1-t = (1-t)\mu^{-1}\int_0^1 (1-s)\ d\alpha_1(s), \tag{3.23}$$

for each $t \in [0,1]$. (Note that we assume in passing from (3.22) to (3.23) that $L_1(x) \neq 0$, for if it is, then from (3.21) we immediately obtain that $x(t) \equiv 0$, which is the trivial solution.) Then (3.23) implies the estimate

$$\begin{aligned}1-t = (1-t)\mu^{-1}\int_0^1 (1-s)\ d\alpha_1(s) &\leq (1-t)\mu^{-1}\left|\int_0^1 (1-s)\ d\alpha_1(s)\right|\\ &\leq (1-t)\mu^{-1}C_1 \max_{s\in[0,1]}(1-s)\\ &= (1-t)\mu^{-1}C_1,\end{aligned} \tag{3.24}$$

where to obtain the second inequality in (3.24) we have used condition (H2). But (3.24) must hold for each $t \in [0,1]$. In particular, it must hold when $t = 0$, which implies that

$$C_1 \geq \mu \geq 1,$$

whence $C_1 \geq 1$, which contradicts the assumption, in the statement of the theorem, that $C_1 < 1$. Thus, we conclude that if $\mu x(t) = (1-t)L_1(x)$ and $C_1 < 1$, then it must hold that $x(t) \equiv 0$. A similar argument demonstrates that if $\mu y(t) = (1-t)L_2(y)$ and $C_2 < 1$, then the existence of a nontrivial solution again leads to a contradiction and, hence, it must hold that $y(t) \equiv 0$. Consequently, we conclude that for each $\mu \geq 1$, it follows that (3.19) has only the trivial solution over $\mathcal{X}$. In other words, the operator $T'(+\infty)$ does *not* possess an eigenvalue greater than or equal to unity, as desired.

Finally, we conclude from Lemma 2.2 that the operator T has a fixed point in $\mathcal{K}$, say $(T(x_0, y_0))(t) = (x_0(t), y_0(t))$, for $t \in [0,1]$. Furthermore, that $\|(x_0, y_0)\| \neq 0$ follows from the assumption that at least one of conditions (1)–(4) in the statement of the theorem holds; this ensures that the identity element $(0,0) \in \mathcal{K}$ cannot be a fixed point of T. And this completes the proof. □

Example 3.3. Consider the nonlinear functionals φ, $\psi : \mathcal{C}([0,1]) \to \mathbb{R}$ defined by

$$\varphi(x) := \frac{1}{30} x\left(\frac{2}{5}\right) - \frac{1}{200}\left[1 - e^{-x\left(\frac{1}{2}\right)}\right] x\left(\frac{3}{5}\right) \tag{3.25}$$

and

$$\psi(y) := \left[\frac{2}{50} y\left(\frac{1}{2}\right) - \frac{1}{1000} y\left(\frac{9}{20}\right)\right] e^{-y\left(\frac{1}{3}\right)} + \left[2 - e^{-y\left(\frac{1}{4}\right)}\right]\left[\frac{1}{50} y\left(\frac{1}{5}\right) - \frac{1}{1000} y\left(\frac{1}{3}\right)\right]. \tag{3.26}$$

Fix the interval $[a,b] := \left[\frac{1}{4}, \frac{3}{4}\right]$. Then it can be shown that $\gamma = \frac{1}{4}$. In addition, henceforth let f, $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ be any functions satisfying condition (H4). Finally, define the functionals L_1, L_2, $\widetilde{L}_1$, $\widetilde{L}_2 : \mathcal{C}([0,1]) \to \mathbb{R}$ by the following.

$$\begin{aligned} L_1(x) &:= \frac{1}{30} x\left(\frac{2}{5}\right) - \frac{1}{200} x\left(\frac{3}{5}\right) \\ L_2(y) &:= \frac{2}{50} y\left(\frac{1}{5}\right) - \frac{1}{500} y\left(\frac{1}{3}\right) \\ \widetilde{L}_1(x) &:= \frac{1}{30} x\left(\frac{2}{5}\right) - \frac{1}{200} x\left(\frac{3}{5}\right) \\ \widetilde{L}_2(y) &:= \frac{1}{50} y\left(\frac{1}{5}\right) - \frac{1}{500} y\left(\frac{1}{3}\right) - \frac{1}{1000} y\left(\frac{9}{20}\right) \end{aligned} \tag{3.27}$$

Now, observe from (3.27) that

$$|L_1(x)| \leq \underbrace{\left[\frac{1}{30} + \frac{1}{200}\right]}_{:=C_1} \|(x,y)\| \text{ and } |L_2(y)| \leq \underbrace{\left[\frac{2}{50} + \frac{1}{500}\right]}_{:=C_2} \|(x,y)\|, \tag{3.28}$$

for each $(x, y) \in \mathcal{X}$, so that

$$\max\{C_1, C_2\} = \max\left\{\frac{23}{600}, \frac{21}{500}\right\} < 1.$$

Consequently, the auxiliary condition in the statement of Theorem 3.2 is satisfied, as is condition (H2) for that matter. In addition, letting $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ be the integrators associated, respectively, to the Stieltjes integral realization of L_1 and L_2, it is easy to check numerically that (H3) holds.

On the other hand, the remaining requirements of condition (H1) are also easily checked. For example, we note that

(3.29)
$$\frac{|\psi(y) - L_2(y)|}{\|(x, y)\|} = \frac{\left|\left[\frac{2}{50}y\left(\frac{1}{2}\right) - \frac{1}{1000}y\left(\frac{9}{20}\right)\right]e^{-y\left(\frac{1}{3}\right)} - e^{-y\left(\frac{1}{4}\right)}\left[\frac{1}{50}y\left(\frac{1}{5}\right) - \frac{1}{1000}y\left(\frac{1}{3}\right)\right]\right|}{\|(x, y)\|}.$$

If $\|(x, y)\| \to +\infty$, then either $\|x\| \to +\infty$ or $\|y\| \to +\infty$. In the former case we compute

$$\begin{aligned}&\frac{\left|\left[\frac{2}{50}y\left(\frac{1}{2}\right) - \frac{1}{1000}y\left(\frac{9}{20}\right)\right]e^{-y\left(\frac{1}{3}\right)} - e^{-y\left(\frac{1}{4}\right)}\left[\frac{1}{50}y\left(\frac{1}{5}\right) - \frac{1}{1000}y\left(\frac{1}{3}\right)\right]\right|}{\|(x, y)\|}\\ &\le \frac{\frac{41}{1000}\|y\| + \frac{21}{1000}\|y\|}{\|x\|} \to 0.\end{aligned} \tag{3.30}$$

In the latter case we compute

$$\begin{aligned}&\frac{\left|\left[\frac{2}{50}y\left(\frac{1}{2}\right) - \frac{1}{1000}y\left(\frac{9}{20}\right)\right]e^{-y\left(\frac{1}{3}\right)} - e^{-y\left(\frac{1}{4}\right)}\left[\frac{1}{50}y\left(\frac{1}{5}\right) - \frac{1}{1000}y\left(\frac{1}{3}\right)\right]\right|}{\|(x, y)\|}\\ &\le \frac{41}{1000}e^{-y\left(\frac{1}{3}\right)} + \frac{21}{1000}e^{-y\left(\frac{1}{4}\right)}\\ &\to 0.\end{aligned} \tag{3.31}$$

Note that (3.31) holds seeing as $\frac{1}{4}, \frac{1}{3} \in [a, b]$, and so,

$$\min_{t\in[a,b]}(x + y)(t) \ge \gamma\|(x, y)\| \ge \gamma\|y\| \to +\infty. \tag{3.32}$$

Since in this case $\|x\|$ remains finite, it follows that $\min_{t\in[a,b]} x(t)$ must remain finite, and so, $\min_{t\in[a,b]} y(t) \to +\infty$, whence

$$e^{-y\left(\frac{1}{3}\right)}, e^{-y\left(\frac{1}{4}\right)} \to 0, \tag{3.33}$$

which yields (3.31). Note that if it holds that $\|x\|, \|y\| \to +\infty$, then we may invoke either (3.30) or (3.31). Since these cases are exhaustive, we conclude that

$$\lim_{\substack{\|(x,y)\|\to+\infty\\(x,y)\in\mathcal{K}}} \frac{|\psi(y) - L_2(y)|}{\|(x, y)\|} = 0. \tag{3.34}$$

A similar argument demonstrates that

$$\lim_{\substack{\|(x,y)\|\to+\infty \\ (x,y)\in\mathcal{K}}} \frac{|\varphi(x)-L_1(x)|}{\|(x,y)\|}=0. \tag{3.35}$$

Finally, we observe that $\varphi(x)\geq \widetilde{L}_1(x)\geq 0$ and $\psi(y)\geq \widetilde{L}_2(y)\geq 0$, for each $(x,y)\in\mathcal{K}$. For example, we calculate

$$\begin{aligned}\varphi(x)&=\frac{1}{30}x\left(\frac{2}{5}\right)-\frac{1}{200}\left[1-e^{-x\left(\frac{1}{2}\right)}\right]x\left(\frac{3}{5}\right)\\ &\geq\frac{1}{30}x\left(\frac{2}{5}\right)-\frac{1}{200}x\left(\frac{3}{5}\right)\\ &=\widetilde{L}_1(x)\\ &\geq 0,\end{aligned} \tag{3.36}$$

for each $(x,y)\in\mathcal{K}$, where we use the fact that $x(t)\geq 0$, for each $t\in[0,1]$. Similarly, we compute

(3.37)

$$\begin{aligned}\psi(y)&=\left[\frac{2}{50}y\left(\frac{1}{2}\right)-\frac{1}{1000}y\left(\frac{9}{20}\right)\right]e^{-y\left(\frac{1}{3}\right)}+\left[2-e^{-y\left(\frac{1}{4}\right)}\right]\left[\frac{1}{50}y\left(\frac{1}{5}\right)-\frac{1}{1000}y\left(\frac{1}{3}\right)\right]\\ &\geq\frac{2}{50}y\left(\frac{1}{5}\right)-\frac{1}{500}y\left(\frac{1}{3}\right)-\frac{1}{1000}y\left(\frac{9}{20}\right)e^{-y\left(\frac{1}{3}\right)}-\frac{1}{50}y\left(\frac{1}{5}\right)e^{-y\left(\frac{1}{4}\right)}\\ &\geq\frac{2}{50}y\left(\frac{1}{5}\right)-\frac{1}{500}y\left(\frac{1}{3}\right)-\frac{1}{1000}y\left(\frac{9}{20}\right)-\frac{1}{50}y\left(\frac{1}{5}\right)\\ &=\frac{1}{50}y\left(\frac{1}{5}\right)-\frac{1}{500}y\left(\frac{1}{3}\right)-\frac{1}{1000}y\left(\frac{9}{20}\right)\\ &=\widetilde{L}_2(y)\\ &\geq 0,\end{aligned}$$

for each $(x,y)\in\mathcal{K}$.

Consequently, we conclude that each of conditions (H1)–(H4) holds. Moreover, it holds that $\max\{C_1,C_2\}<1$. Therefore, if either of the partial maps $t\mapsto f(t,0,0)$ or $t\mapsto g(t,0,0)$ is a.e. nonzero, then by Theorem 3.2 it follows that problem (1.1) has at least one positive solution for each $(\lambda_1,\lambda_2)\in(0,+\infty)\times(0,+\infty)$.

Remark 3.4. Note that, in some ways, the result of Theorem 3.2 is a generalization and improvement of [12, Theorem 3.4]. In particular and as mentioned in Section 1, [12, Theorem 3.4] only allowed for $\lambda=1$ in problem (1.5). Moreover, growth conditions on $\frac{g(y)}{y}$ were assumed at both 0 and $+\infty$. By way of contrast, our result here allows for any positive λ_1, λ_2 and, furthermore, imposes a growth condition on the functions f and g only at $+\infty$. In addition, Theorem 3.2 evidently applies to systems rather than only to scalar equations.

Remark 3.5. We note that we believe it possible to replace the growth condition (H4) by a slightly more general condition if we abandon the asymptotic Fréchet derivative approach and instead, say, use a more direct approach by means of the Leray-Schauder degree. We believe that such approach could follow roughly the arguments provided in [24]. However, for the sake of simplicity and keeping the exposition shorter we have elected to focus only on the asymptotic Fréchet derivative approach in the present paper, and so, we leave the possibility of utilizing alternative fixed point theorems for future work.

Remark 3.6. We conclude by noting that we could certainly modify the proof of Theorem 3.2 to allow for a variety of boundary conditions. In particular, it is not difficult to see that by a suitable and straightforward modification we could admit the more general boundary conditions

$$\begin{aligned} x(0) &= \varphi(x) \\ y(0) &= \psi(y) \\ x(1) &= \widetilde{\varphi}(x) \\ y(1) &= \widetilde{\psi}(y), \end{aligned} \tag{3.38}$$

for example, where $\widetilde{\varphi}$, $\widetilde{\psi} : \mathcal{C}([0,1]) \to \mathbb{R}$ are (possibly nonlinear) functionals with the same essential structure as φ and ψ.

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