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CHAOS IN A TWO-STAGE DISCRETE MODEL WITH STRONG NONLINEARITIES

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. We develop a discrete juvenile-adult population model with Ricker-type survivorship functions. We first show that the extinction equilibrium, or trivial equilibrium, is locally asymptotically stable when the inherent net reproductive number is less than one. When it is greater than one, we show that the system is persistent. Given the inherent complexity of the system, several numerical examples are used to convey the rich chaotic behavior exhibited. Using bifurcation analysis, the effect of the birth rate on the system's dynamics is explored. It is shown that for certain birth rates the system exhibits chaotic behavior. In addition to bifurcation diagrams, phase portraits of the system's attractors are used to develop a deeper understanding of the model's dynamics. Approximations of the system's Lyapunov exponents are then used to show that for certain birth rates sensitivity to initial conditions is present. Then the case of periodic birth rates is considered.

1. Introduction

Although a few decades ago chaos was seen by some to be insignificant in the study of population dynamics [17], today the importance of chaos throughout mathematics and ecology is recognised. Sir Robert May was one of the first to suggested that erratic fluctuations in population models could arise from deterministic processes [20, 21, 22]. Prior to his work, it was generally accepted that such fluctuations were the result of environmental noise or errors in collected data sets [12]. In recent years chaos has been investigated in a wide range of areas including the study of plants [28, 29], rodents [30], infectious diseases [14, 25], and insects [4, 13, 27]. There has even been nonlinear demographic models developed that have successfully predicted chaotic behavior that was observed in laboratory experiments involving flour beetles [7, 8, 10].

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In this paper, we develop a discrete juvenile-adult model that is inspired by a paper written by Ackleh and Chiquet which explored the dynamics of a population of green tree frogs, *Hyla Cinera* [3]. Their paper was motivated by the work done in a joint effort between the University of Louisiana at Lafayette and the United States Geological Survey National Wetlands Research Center to study an urban population of green tree frogs [26]. There are two main reasons why the populations of these frogs merit investigation. First, global frog populations are declining and by examining a small population of frogs one may be able to find the cause of this decline and possibly shed light on the long-term behavior of global frog populations. Second, frogs act as an indicator species in the sense that by examining how their populations react to certain environmental stresses we can gain insight into how other wildlife populations may be affected. The model studied in [3] is given by

$$\begin{cases} J_{t+1} = (1-\gamma)S_1(J_t)J_t + b(t)A_t \\ A_{t+1} = \gamma S_1(J_t)J_t + S_2(A_t)A_t \\ (J_0, A_0) \in \mathbb{R}_+^2 \backslash (0,0), \end{cases} \tag{1.1}$$

where J_t and A_t represent the population of juveniles and adults, respectively, and γ is the percentage of juveniles that mature to adulthood after each time-step. The survivorship rates of the juvenile class and adult class are given by $S_1(J_t)$ and $S_2(A_t)$, respectively. Note that we have no inter-class competition as is characteristic of amphibious populations, such as tadpoles and adult frogs.

In their work, Ackleh and Chiquet utilized survivorship functions which satisfied the following conditions:

(C1) $S_k(x) \in C^1[0,\infty)$, $S_k'(x) < 0$, $\lim_{x\to\infty} S_k(x) = 0$, $S_k(0) = a_k (0 < a_k < 1)$,
(C2) $(S_k(x)x)' > 0$, and $\lim_{x\to\infty} S_k(x)x = \hat{a_k} < \infty$.

An example of such a function would be the Beverton-Holt function given by

$$S_i(x) = \frac{a_i}{1 + k_i x}.$$

Many ecological models use different types of survivorship functions. As in [7, 8, 10], we will exam the effects of using Ricker-type survivorship functions on a population using a model similar to (1.1). Using the Ricker-type functions will yield much richer dynamics than that presented in [3] including chaos.

The paper is organized as follows: in Section 2, we develop a discrete juvenile-adult model with Ricker-type survivorship functions. We show that the extinction equilibrium, or trivial equilibrium, is locally asymptotically stable when the inherent net reproductive number is less than one, and when it is greater than one, we show that the system is persistent. In Section 3, we investigate the chaotic tendencies of the model via bifurcation diagrams, phase portraits, and Lyapunov exponents. Then in Section 4, the model is altered to study the effect periodic birthing strategies have

on the dynamics of our model. A summary of the results and concluding remarks are given in Section 5.

2. Discrete model with strong nonlinearities

Throughout this paper, we will utilize Ricker-type survivorship functions for our model. Thus, our survivorship functions will be of the form

$$S_i(x) = a_i e^{-k_i x}, \text{ for } x = J_i, A_i.$$

Note that these functions only have properties given in (C1). Both Ricker and Beverton-Holt nonlinearities behave in similar fashions, but there are significant differences between them, mainly the Ricker-type functions are overcompensatory while Beverton-Holt functions are compensatory. Let J_t and A_t represent the population of juveniles and adults, respectively. We denote the time-dependent birth rate as $b(t)$. Hence, the model is given by

$$\begin{cases} J_{t+1} = (1-\gamma)a_1 e^{-k_1 J_t} J_t + b(t) A_t \\ A_{t+1} = \gamma a_1 e^{-k_1 J_t} J_t + a_2 e^{-k_2 A_t} A_t \\ (J_0, A_0) \in \mathbb{R}_+^2 \backslash (0,0), \end{cases} \tag{2.1}$$

where γ retains the same meaning as in (1.1). For the remainder of this section and Section 3, we will assume that $b(t) = b$, a positive constant.

2.1. Extinction equilibrium.

Before examining the inherent chaos of system (2.1), we first classify the stability of the extinction equilibrium $E_0 = (0,0)$. It is clear that E_0 is indeed an equilibrium of system (2.1). In order to show the stability of E_0, we first find the inherent net reproductive number, R_0, by implementing methods used in [3, 11]. The inherent net reproductive number is the total number of offspring an individual is expected to produce, on average, over the course of its lifespan. This number is intimately related to the stability of equilibria and will be given in terms of the system's parameters.

First system (2.1) can be written as

$$X_{t+1} = P(X_t) X_t,$$

where $P(X_t)$ is the projection matrix and is defined by

$$P(X_t) = \begin{pmatrix} (1-\gamma)a_1 e^{-k_1 J_t} & b \\ \gamma a_1 e^{-k_1 J_t} & a_2 e^{-k_2 A_t} \end{pmatrix}. \tag{2.2}$$

Evaluating (2.2) at E_0, we obtain

$$P(E_0) = \begin{pmatrix} (1-\gamma)a_1 & b \\ \gamma a_1 & a_2 \end{pmatrix}. \tag{2.3}$$

In order to calculate R_0, we decompose (2.3) into a fertility matrix given by

$$F = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

and a transition matrix given by

$$T = \begin{pmatrix} (1-\gamma)a_1 & 0 \\ \gamma a_1 & a_2 \end{pmatrix}.$$

The inherent net reproductive number is thus given by the positive strictly dominant eigenvalue of the matrix $F(I-T)^{-1}$. Calculating this we get

$$\mathcal{R}_0 \equiv \frac{\gamma b a_1}{(1-a_2)(1-(1-\gamma)a_1)}.$$

The following theorem summarizes the stability of E_0, the proof of which is similar to [24].

Theorem 2.1. *Let $b(t) = b > 0$. If $R_0 < 1$, system (2.1) has only the trivial steady state $E_0 = (0,0)$, and E_0 is locally asymptotically stable.*

Proof. Assume $R_0 < 1$. Define the map $P : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ to be the right hand side of system (2.1). Linearizing system (2.1) and evaluating the Jacobian at E_0 gives us

$$D(E_0) = \begin{pmatrix} (1-\gamma)a_1 & b \\ \gamma a_1 & a_2 \end{pmatrix}. \tag{2.4}$$

We can establish local asymptotic stability of E_0 by showing the following inequality holds (see Theorem 2.37 in [15]):

$$|\operatorname{tr}(D(E_0))| < 1 + \det(D(E_0)) < 2. \tag{2.5}$$

From (2.4), we see that $\operatorname{tr}(D(E_0)) > 0$ and that $\det(D(E_0)) < 1$. Thus, we need only show $\operatorname{tr}(D(E_0)) < 1 + \det(D(E_0))$. From our assumption that $R_0 < 1$, we obtain

$$1 > (1-\gamma)(1-a_2)a_1 + a_2 + \gamma b a_1.$$

Hence, we have

$$\begin{aligned} 1 + \det(D(E_0)) &> (1-\gamma)(1-a_2)a_1 + a_2 + a_2\gamma b a_1 + (1-\gamma)a_1 a_2 - \gamma a_1 b \\ &= (1-\gamma)a_1 + a_2 \\ &= \operatorname{tr}(D(E_0)). \end{aligned}$$

Therefore, E_0 is locally asymptotically stable. □

2.2. **Uniform Persistence.** We now want to show that system (2.1) is uniformly persistent when $R_0 > 1$. We will be using an argument similar to that in [1, 3]. We will need the following information from [19]. This setup is taken directly from [19].

Let $\mathcal{H}$ be a metric space with metric d, $f : \mathcal{H} \to \mathcal{H}$ be a continuous map and $\mathcal{Y} \in \mathcal{H}$ is closed with $f(\mathcal{H} \setminus \mathcal{Y}) \subset \mathcal{H} \setminus \mathcal{Y}$. Assume that $\mathcal{H}$ has a global attractor X, that is, X is the maximal compact invariant subset of $\mathcal{H}$ and $d(f^n(x), X) \to 0$ as $n \to \infty$, for all $x \in \mathcal{H}$. Note that $\mathcal{Y}$ is in general not a positively invariant set. Let M be the maximal compact invariant set in $\mathcal{Y}$. Then $M \subset \mathcal{Y}$. Let the stable set of M, denoted $W^s(M)$, be defined as follows:

$$W^s(M) := \{x \in X : f^n(x) \to M \text{ as } n \to +\infty\}.$$

Then we have the following theorem from [19].

Theorem 2.2. *f is uniformly persistent (w.r.t $\mathcal{Y}$) if and only if*
(1) M is isolated in X, and
(2) $W^s(M) \subset \mathcal{Y}$.

We will also need the following lemma.

Lemma 2.3. *There exists a compact set $K \in \mathbb{R}^2_+$ such that every forward solution sequence of system* (2.1) *enters K in at most 2 time steps and remains in K forever after.*

Proof. Clearly, one can see that $\mathbb{R}^2_+$ is positively invariant. Also we know that $f(J_t) = a_1 e^{-k_1 J_t} J_t$ and $f(A_t) = a_2 e^{-k_2 A_t} A_t$ are bounded, i.e.,

$$f(J_t) \le m_1 \text{ and } f(A_t) \le m_2 \quad \forall t = 0, 1, \ldots, \tag{2.6}$$

for some $m_1 > 0$ and $m_2 > 0$. Now from (2.6) and (C1) it follows that

$$A(t+1) = \gamma a_1 e^{-k_1 J_t} J_t + a_2 e^{-k_2 A_t} A_t \le \gamma m_1 + m_2 \le m_1 + m_2$$

$\forall t = 0, 1, \ldots$. Therefore, we have

$$A(t) \le m_1 + m_2 \ \forall t = 1, 2, \ldots.$$

Using this we get

$$J(t+1) \le (1-\gamma) m_1 + b(m_1 + m_2) \ \forall t = 1, 2, \ldots,$$

hence

$$J(t) \le (b+1) m_1 + b m_2 \ \forall t = 2, 3, \ldots.$$

Thus, every forward solutions enters the following compact set in at most two time steps and remain there forever:

$$K = \{(J, A) \in \mathbb{R}^2_+ | J \in [0, (b+1)m_1 + b m_2], \quad A \in [0, m_1 + m_2]\}. \tag{2.7}$$

□

The following result will establish the uniform persistence of system (2.1) when $R_0 > 1$ and show the origin is unstable.

Theorem 2.4. *Let $b(t) = b > 0$. If $R_0 > 1$, the trivial steady state $E_0 = (0, 0)$ of system* (2.1) *is unstable and system* (2.1) *is uniformly persistent.*

Proof. Assume $R_0 > 1$. From Theorem 1.1.3 in [9], $P(E_0)$ has a positive strictly dominant eigenvalue $\lambda > 1$. Therefore, we conclude that E_0 is unstable. Now, let $\mathcal{H} = \mathbb{R}^2_+$, $\mathcal{Y} = bd(\mathbb{R}^2_+)$, and define the map $f : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ to be the right hand side of system (2.1). Since $\text{int}(\mathbb{R}^2_+)$ is positively invariant for system (2.1), we have $f(\mathcal{H} \setminus \mathcal{Y}) \subset \mathcal{H} \setminus \mathcal{Y}$. Using Lemma 2.3 and Theorem 2.1 in [16], it follows that there exists a global attractor X in $\mathcal{H}$. Let M be the maximal compact invariant set in $\mathcal{Y}$, which in our context is $M = \{E_0\}$. Now we can use Theorem 2.2 to prove uniform persistence. As in [1], we will actually prove a stronger result that M is a repeller. Once we show that M is a repeller, we have that by Theorem 2.1 of [19], this is equivalent to showing that M is isolated in $\mathcal{H}$ and $W^s(M) \subset M$. Hence, system (2.1) is uniformly persistent. Now we show M is a repeller using an argument similar to one in [2] .

Since $P(E_0)$ is non-negative, irreducible, and primitive, its dominant eigenvalue $\lambda > 1$ has a corresponding left eigenvector $v > 0$. That is,

$$v^T P(E_0) = \lambda v^T.$$

Pick $\lambda^* \in (1, \lambda)$ such that $v^T P(E_0) - \lambda^* v^T > 0$. Then there exists a neighborhood U of $M \in \mathbb{R}^2_+$ such that

$$v^T P(x) - \lambda^* v^T > 0$$

for every $x \in U$ since $P(x)$ is continuous. Define $L : \mathbb{R}^2_+ \to \mathbb{R}_+$ to be

$$L(x) = v^T x.$$

Now, $L(x) = 0$ for $x \in U$ if and only if $x \in M$ and is positive elsewhere in U. Also, we have

$$L(f(x)) = v^T P(x)x > \lambda^* v^T x > L(x)$$

for all $x \in U \setminus M$. Therefore, M is a repeller. It follows that system (2.1) is uniformly persistent. □

3. Chaotic dynamics

We now want to explore the chaotic dynamics exhibited by our system brought about as a result of the Ricker-type survivorship rates. We will first create a bifurcation diagram associated with the total population with our bifurcation parameter being the birth rate. We will then examine some phase portraits associated with

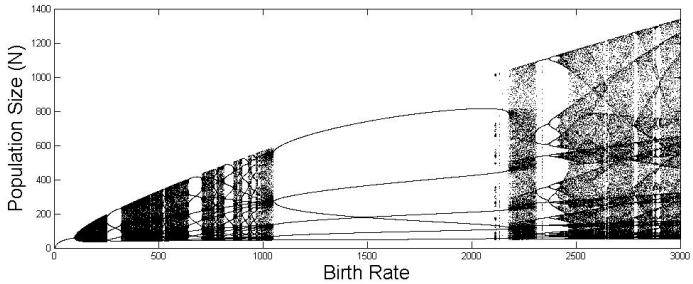
the bifurcation diagram. Then we will approximate the Lyapunov exponents for our system. The parametric values used for all of the analysis in Section 3 for model (2.1) are given in Table 1. These values are loosely based on the ones used in [3].

Parameter	Numerical Value
γ	0.5
a_1	0.2
k_1	0.05
a_2	0.5
k_2	0.05

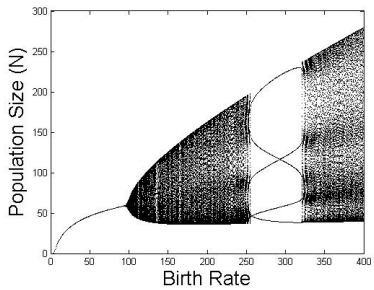
TABLE 1. Parametric values used in numerical simulations

3.1. **Bifurcation analysis and attractors.** We will only plot the bifurcation diagram associated with the total population because the results obtain by examining the juvenile and adult population separately are similar. Figure 1 is the bifurcation diagram of system (2.1) with respect to the birthrate b. It appears our system undergoes a discrete Hopf bifurcation. We see that for lower values of b, the system settles into an equilibrium state, but once b reaches a critical value an invariant loop bifurcates from the equilibria. Figure 1(a) shows the systems behavior for $0 < b < 3000$, while (b) is local amplification to the interval $[0, 400]$ and (c) a further amplification to $[75, 120]$. For small values of b, the system has an interior equilibrium, but as the birth rate is increased, more complex behaviors are exhibited, including periodic solutions of different periods, invariant loops, and chaos.

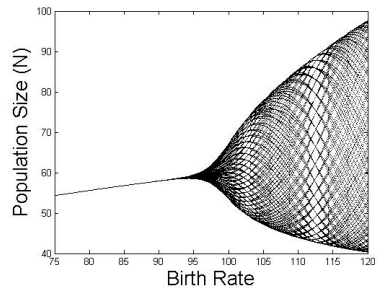
We now want to look at our system's attractor in phase space. The phase portraits associated with Figure 1 are given in Figure 2. For small values of b the system is drawn inward towards an interior equilibria. As b is varied the figures clearly show the process of how a smooth invariant curve bifurcates from the stable fixed point. Also present is the temporary deterioration of the invariant curve and the emergence of a period five solution. As the birth rate is increased further, an invariant loop reemerges. Once the birth rate becomes large enough and the onset of chaos occurs, the system's attractor become increasingly deformed. An attractor associated with a chaotic birth rate is shown in the bottom, rightmost diagram of Figure 2.



(a) Bifurcation diagram of System (2.1) with respect to the birth rate



(b)



(c)

FIGURE 1. Bifurcation diagram of system (2.1) with respect to the birth rate (a), and local amplifications (b) and (c)

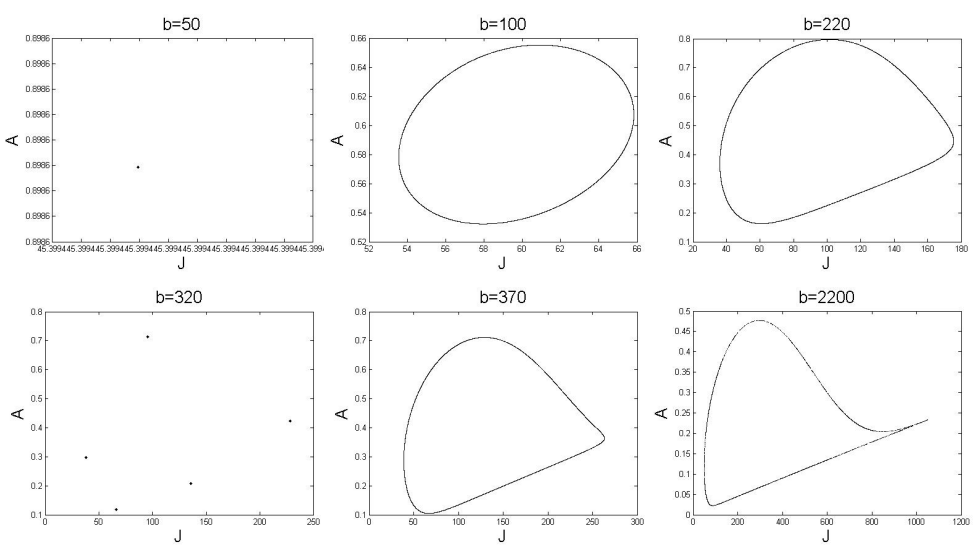


FIGURE 2. Phase portraits for various values of b, which correspond to Figure 1

3.2. **Lyapunov exponents.** The simplest and most intuitive definition of chaos is extreme sensitivity to initial conditions [18]. Lyapunov exponents are commonly used to measure this sensitivity. Alligood et. al. explain that for an m dimensional system, the Lyapunov exponents measure the rate of separation from the current

orbit point along m orthogonal directions [6]. Hence, whenever a Lyapunov exponent is positive, the system is exhibiting sensitivity to initial conditions, i.e. exhibiting chaotic behavior. The Lyapunov exponent which determines sensitivity for a system of difference equations is defined as,

$$\lambda(x_0, y_0) = \lim_{t\to\infty} \frac{1}{t} \ln \rho \left(\prod_{k=0}^{t-1} D\left(x_k, y_k\right) \right),$$

where $\rho(\cdot)$ is the spectral radius of a matrix, and $D(x_k, y_k)$ is the Jacobian of the system evaluated at (x_k, y_k) [5]. Note that the Lyapunov exponents depend upon the initial condition (x_0, y_0) chosen.

Since system (2.1) is two dimensional, each Jacobian has two eigenvalues associated with it and thus has two Lyapunov exponents. While only the Lyapunov exponent based on the spectral radius is needed to determine where a system is chaotic, we extend Allen's definition in order to account for both Lyapunov exponents. Letting the eigenvalues of $\prod_{k=0}^{t-1} D\left(x_k, y_k\right)$ be represented by Λ_1, Λ_2, define

$$\rho_1 \left(\prod_{k=0}^{t-1} D\left(x_k, y_k\right) \right) = \max\left\{|\Lambda_1|, |\Lambda_2|\right\}$$

and

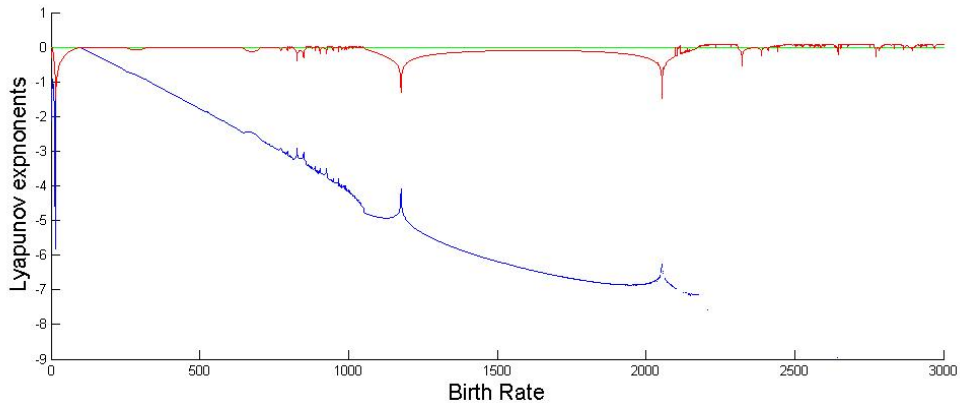
$$\rho_2 \left(\prod_{k=0}^{t-1} D\left(x_k, y_k\right) \right) = \min\left\{|\Lambda_1|, |\Lambda_2|\right\}.$$

Thus, the two Lyapunov exponents of system (2.1) are given by,

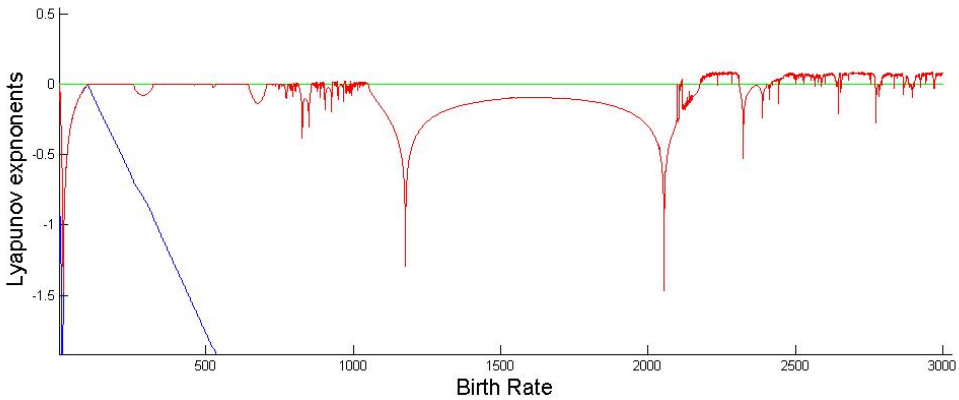
$$\lambda_i(J_0, A_0) = \lim_{t\to\infty} \frac{1}{t} ln \left(\rho_i \left(\prod_{k=0}^{t-1} D(J_k, A_k) \right) \right) \qquad \text{for } i \in \{1, 2\}. \tag{3.1}$$

In order to calculate our Lyapunov exponents numerically we make use of the algorithm outlined in [6], which uses the Gram-Schmidt orthogonalization procedure.

Since we are interested in what effect varying the birth rate has on our system's dynamics, we approximate the Lyapunov exponents for several values of b. The values of the Lyapunov exponents as a function of b are plotted in Figure 3. Here the top graph represents λ_1, while the bottom graph represents λ_2.



(a)



(b)

FIGURE 3. Lyapunov exponents for model composed of Ricker survivorship functions, where (b) is a local amplification of (a)

Examining Figure 3, we see that it is remarkably reminiscent of Figure 1. We see that for low values of the birth rate the system does not exhibit sensitivity to initial conditions, but once it reaches a certain birth rate, it has regions of sensitivity and insensitivity to initial conditions. For a set range of values, we see that the system looses sensitivity for an extended period of time before returning to its previous behavior. These regions correspond to the regions noted in Figure 1 where our system enters into periodic solutions of lower period. While Figure 3 shows that transition between chaotic and non-chaotic regions is indeed common, it does not show to what extent. A periodic attractor is present when $\lambda_1 = 0$ and $\lambda_2 < 0$, while a chaotic attractor is present when $\lambda_1 > 0$. By keeping track of the number of investigated birth rates that fall into these respective classes, we can estimate the frequency that our system possesses a periodic and chaotic attractor. These results are further summarized in Table 2.

Type of Attractor	Lyapunov Exponents	Frequencies
Chaotic	$\lambda_1 > 0$	28.67%
Quasiperiodic	$\lambda_1 = \lambda_2 = 0$	58.18%
Periodic	$\lambda_1 = 0, \lambda_2 < 0$	12.99%
Fixed Point	$\lambda_1 < 0, \lambda_2 < 0$	0.17%

TABLE 2. Observed frequencies for different types of attractors

Figure 4 shows how the approximation of Lyapunov exponents behave for different types of attractors. For quasiperiodic attractors, the approximation converges to zero (Figure 4(a)). When the system posseses a periodic attractor, the Lyapunov exponent approximations converge to a negative numbers (Figure 4(b)). The final example illustrates how the Lyapunov exponents converge to a positive number when the system is exhibiting chaotic dynamics (Figure 4(c)).

4. **Periodic birthrates**

In many ecosystems, species do not reproduce year round. Instead they have a specific breeding season. Thus, it is important to examine the effect periodic birth rates have on a system. In this section, we consider the case where the breeding in system (2.1) is seasonal. We shall examine three separate birthing strategies. First we consider the case where the birth rate is periodic of period 2. For example, when the time step is taken to be one year, this corresponds to 6 months of sexual reproductivity and 6 months sexual dormancy. Next, we consider the case when the birth rate is of period 3. For this type of birth rate, we will consider two different birthing strategies, the case where the population is sexually active for an 8 month interval every year and the scenario where the population reproduces for only 4 months out of the year.

We now examine the case where the birth rate is periodic of period 2. Thus, the birth rate, $b(t)$, is defined by

$$b(t) = \begin{cases} 0 & \text{if } t = 0, 2, 4, \ldots \\ \hat{b} & \text{if } t = 1, 3, 5, \ldots, \end{cases} \tag{4.1}$$

where $\hat{b}$ is positive constant. Figure 5 shows the dynamics exhibited when the birthrate is periodic of period 2. Notice that while the system does still exhibit chaotic behavior, the onset of it is delayed. The route to chaos is also markedly different than the continuous case. Instead of undergoing a Hopf bifurcation and reaching chaos via the quasiperiodic route, the system reaches chaos via the period doubling route. The transitions in and out of chaos are still present in the system with period two birth rates.

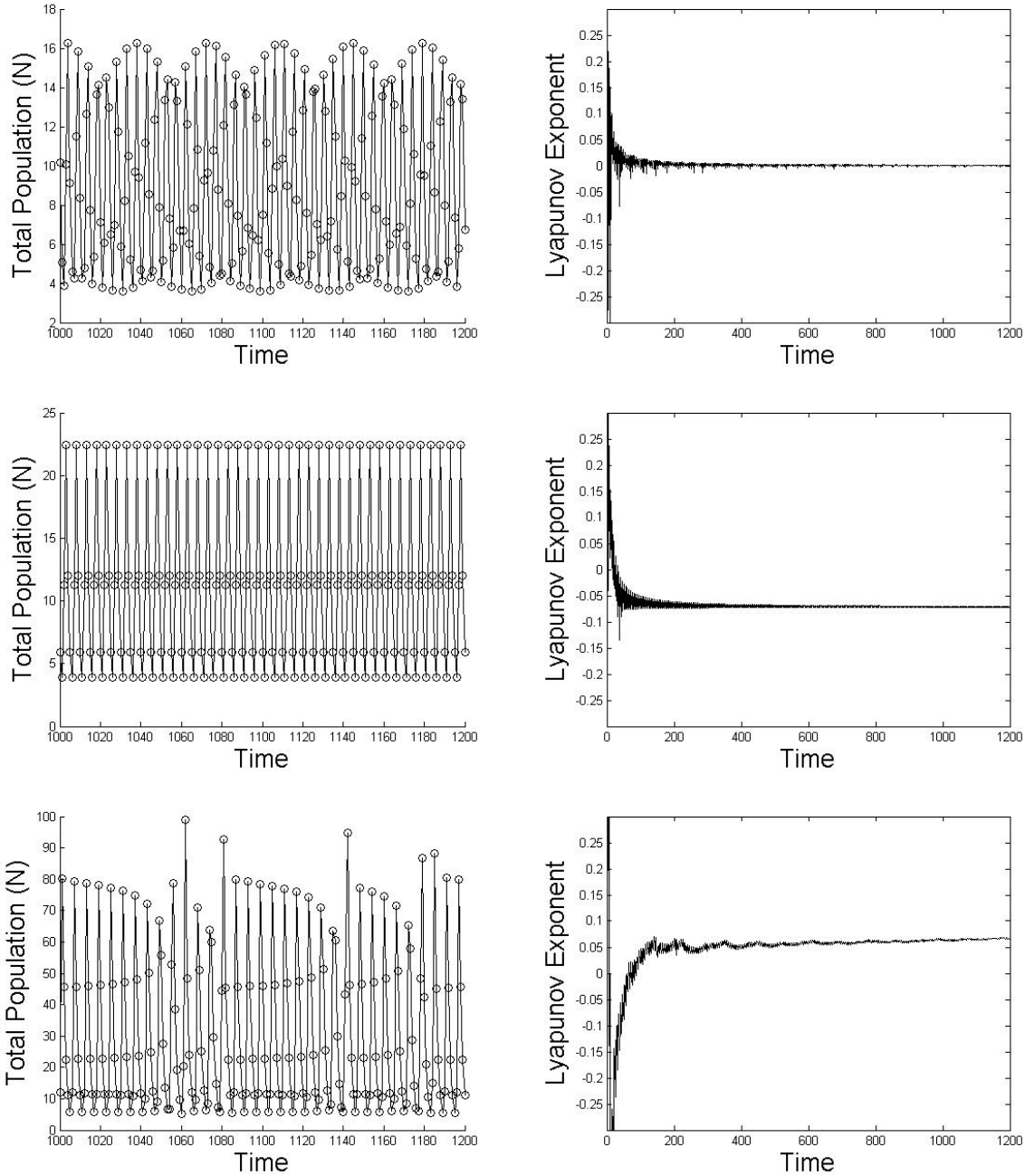


FIGURE 4. Convergence of the Lyapunov exponent for dynamics of (top) quasiperiodic, (mid) periodic, and (bottom) chaotic attractor when $b = 200,\ b = 300,\ b = 2200$ respectively.

It can be shown, through means similar to those implemented in Section 2.1, that the extinction equilibria is locally asymptotically stable when the inherent net reproductive number is less than one and the system is persistent when the inherent net reproductive number is greater than one. Next, we examine the two distinct birthing strategies associated with periodic birth rates of period 3. First, we consider

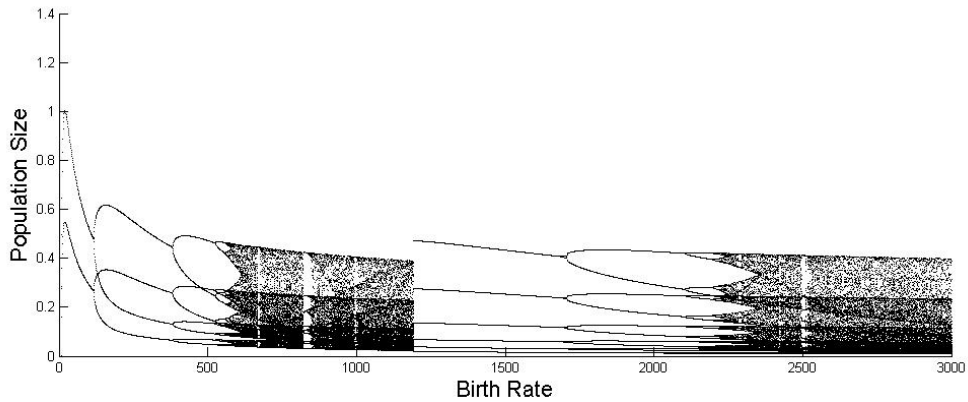


FIGURE 5. Bifurcation diagram with birth strategy given in (4.1)

the scenario where the population breeds for 8 months out of the year. Thus, our birthrate, $b(t)$ is now given by

$$b(t) = \begin{cases} 0 & \text{if } t = 0, 3, 6, \ldots \\ \hat{b} & \text{if } t = 1, 4, 7, \ldots \\ \hat{b} & \text{if } t = 2, 5, 8, \ldots, \end{cases} \tag{4.2}$$

where $\hat{b}$ is a positive constant. The associated bifurcation diagram is given in Figure 6(a). Once again we see a delay of chaos, when compared to system (2.1) and that chaos is reached via the period doubling route.

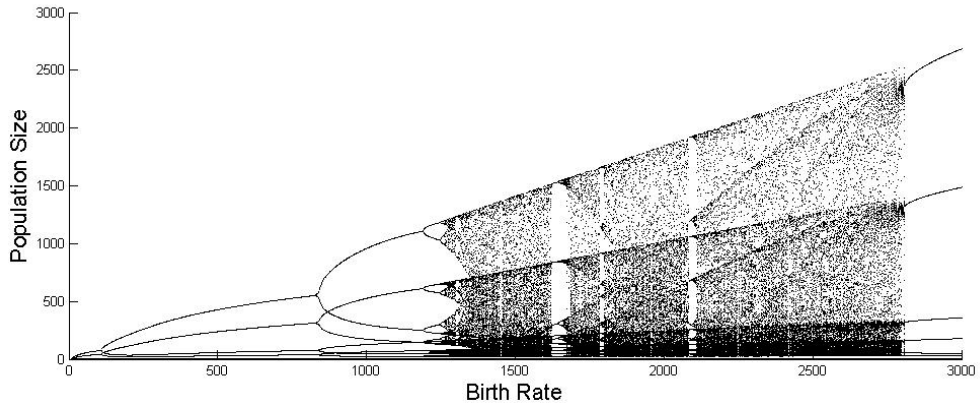
Finally, we consider the case where the population is reproductive for only a 4 month period out of the year. The birth rate is then given by,

$$b(t) = \begin{cases} 0 & \text{if } t = 0, 3, 6, \ldots \\ 0 & \text{if } t = 1, 4, 7, \ldots \\ \hat{b} & \text{if } t = 2, 5, 8, \ldots, \end{cases} \tag{4.3}$$

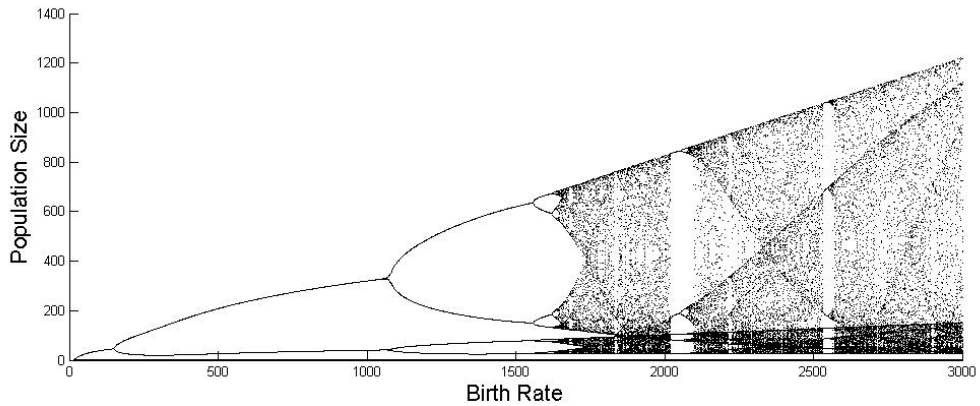
where $\hat{b}$ is again defined to be a positive constant. Here again we have a delay in the onset of chaotic dynamics and the system takes the period doubling route to chaos.

5. **Concluding remarks**

In conclusion, we develop and analyze a discrete juvenile-adult model with Ricker-type survivorship functions. We first show when R_0 is less than one, the extinction equilibrium is locally asymptotically stable. We then show that when R_0 is greater than one, our system is persistent. The presence of chaotic tendencies in our model is then explored via bifurcation diagrams. These diagrams suggest that when the birth rate is varied the system undergoes a discrete Hopf bifurcation and transitions to chaos via the quasiperiodic route. The bifurcation diagrams also show that after the initial onset of chaos there exists large range of birth rates for which chaos is absent.



(a)



(b)

FIGURE 6. Bifurcation diagrams associated with the two birthing strategies for period 3 birth rates where (a) looks at birth strategy given in (4.2), and (b) looks at the birth strategy given in (4.3)

Over the range of the birth rates where chaos disappears it is characteristic of the model to possess low periodic solutions.

We then employ numerical methods to approximate the Lyapnunov exponents for system (2.1) thereby confirming the results of the bifurcation diagrams and phase portraits. It is shown that for low birth rates system (2.1) does not exhibit any sensitivity to initial condition, but when the birth rates are increased past a certain point, system (2.1) did respond sensitively to small perturbations in initial conditions. We then briefly examine the influence of periodic birth rates, with period 2 and 3. It is conjectured that built-in periodicity had a delay effect on the occurrence of chaos. More significant is the change in the route to chaos. Instead of the previous observed quasiperiodic route, the system reaches chaos via the more common period doubling route. In the future, it would be interesting to see how other breeding strategies would influence the behavior of a population.

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