Dynamic Systems and Applications 23 (2014) 665-676

# CONTRACTION MAPPING AND STABILITY IN AN INTEGRO-DIFFERENTIAL EQUATION OF NONCONVOLUTION TYPE

BO ZHANG

Department of Mathematics and Computer Science
Fayetteville State University
Fayetteville, NC 28301, USA

Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

**ABSTRACT.** In this paper, we consider a scalar integro-differential equation of nonconvolution type

$$x' = -\int_{t-r}^{t} a(t,s)g(x(s))ds$$

and give conditions on $a$ and $g$ to ensure that the zero solution is asymptotically stable by applying the Contraction Mapping Principle. These conditions do not require a fixed sign of the coefficient function $a(t,s)$, nor do they involve the sign of any derivative of $a(t,s)$. An asymptotic stability theorem with a necessary and sufficient condition is proved.

**AMS (MOS) Subject Classification.** 34K20, 47H10.

## 1. INTRODUCTION

The purpose of this paper is to study the stability properties of the scalar equation

$$x' = -\int_{t-r}^{t} a(t,s)x(s)ds \tag{1.1}$$

as well as its nonlinear analogue

$$x' = -\int_{t-r}^{t} a(t,s)g(x(s))ds \tag{1.2}$$

by means of contraction mappings. Here $r$ is a positive constant, $a : [0,\infty) \times [-r,\infty) \to R$ is piecewise continuous, $R = (-\infty,\infty)$, and $g : R \to R$ is continuous with $xg(x) > 0$ for $x \neq 0$. We set

$$A(t,s) =: \int_{t-s}^{r} a(u+s,s)du \quad \text{for} \quad t \geq 0 \quad \text{and} \quad t-r \leq s \leq t$$

and present our stability results in terms of $A(t,s)$. With other conditions, we show that $\int_0^\infty A(s,s)ds = \infty$ is a necessary and sufficient condition for asymptotic stability.

Received September 2013 1056-2176 $15.00 

Equation (1.2) and its nonlinear perturbations including Volterra equations have been the center of investigation for a very long time. In early 1950s, Brownell and Ergen [1] studied a form of (1.2) in connection with reactor dynamics. Levin and Nohel ([9],[11]) expended the investigation in 1960s. The work continues with new methods and results (see Burton [2], [3]). Levin and Nohel [11] are able to show that the zero solution of (1.2) with $a(t,s) = a(t-s)$ is globally asymptotically stable if $xg(x) > 0$ for $x \neq 0$ and

$$a(r) = 0, \quad a(t) \geq 0, \quad a'(t) \leq 0, \quad a''(t) \geq 0, \quad \text{for} \quad 0 \leq t \leq r \tag{1.3}$$

by constructing a Liapunov functional. The technique is also extended to equations of nonconvolution type (Levin [10]). For more historical background and discussion of applications to dynamical models, we refer the reader to, for example, the work of Burton [4], Hale [7], Graef, Qian, and Zhang [6], Krasovskii [8], Yoshizawa [13], and the references contained therein.

In this part of investigation, we derive stability criteria for (1.1) and (1.2) with integral conditions by means of contraction mapping without asking the sign of $a(t,s)$ or the sign of any derivative of $a(t,s)$. We list two early theorems of Burton here for reference.

**Theorem A** (Burton [2]). Suppose that $r > 0$ and there exists a constant $\alpha < 1$ such that

$$\int_{t-r}^{t} |a(s+r)|ds + \int_0^t e^{-\int_s^t a(u+r)du} |a(s+r)| \int_{s-r}^{s} |a(u+r)|duds \leq \alpha \tag{1.4}$$

for all $t \geq 0$ and $\int_0^\infty a(s)ds = \infty$. Then for every continuous initial function $\psi : [-r,0] \to R$, the solution $x(t,0,\psi)$ of $x' = -a(t)x(t-r)$ is bounded and tends to zero as $t \to \infty$.

A similar result with variable delays is also obtained in Zhang [15], and it is shown that the condition $\int_0^\infty a(s)ds = \infty$ is necessary and sufficient for asymptotic stability.

**Theorem B** (Burton [3]). Suppose that $A(t,t) \geq 0$ and there exists a constant $\alpha < 1$ such that

$$2\int_{t-r}^{t} |A(t,u)|du \leq \alpha \tag{1.5}$$

for all $t \geq 0$. If $\int_0^t A(s,s)ds \to \infty$ as $t \to \infty$, then the zero solution of (1.1) is asymptotically stable. The same assertion holds for (1.2) with additional conditions on $g$.

Our aim here is to generalize Theorem B without asking $A(t,t) \geq 0$ and improve condition (1.5). We show that $\int_0^t A(s,s)ds \to \infty$ as $t \to \infty$ is a necessary and sufficient condition for asymptotic stability. We discuss equations (1.1) and (1.2) in Section 2 and Section 3, respectively. We ask that there is a constant $\alpha < 1$ with

$$\int_{t-r}^{t} |A(t,s)|ds + \int_{0}^{t} e^{-\int_s^t A(u,u)du} |A(s,s)| \int_{s-r}^{s} |A(s,\tau)|d\tau ds \leq \alpha \tag{1.6}$$

for all $t \geq 0$. We see that (1.6) is satisfied if (1.5) holds with $A(t,t) \geq 0$. If the equation is of convolution type, then (1.6) can be easily verified. In fact, for $a(t,s) = a(t-s)$, we have

$$A(t,s) = A(t-s) = \int_{t-s}^{r} a(u)du$$

and

$$\int_{t-r}^{t} |A(t,s)|ds = \int_{t-r}^{t} \Big| \int_{t-s}^{r} a(u)du \Big| ds = \int_{0}^{r} \Big| \int_{s}^{r} a(u)du \Big| ds.$$

If $\int_0^r a(u)du > 0$, then $\int_0^\infty A(s,s)ds = \infty$ and (1.6) becomes

$$\int_{0}^{r} \Big| \int_{s}^{r} a(u)du \Big| ds < 1/2.$$

We also notice that if $A(t,s) = (\sin t)^n(2s+1)/(3s+1)$ and $3/4 < r < 1$, then (1.6) holds for sufficiently large even integer $n$, but (1.5) fails since the value of the first integral of (1.6) is in $(1/2, 1)$, while the second integral tends to zero as $n \to \infty$ even if $A(t,t) \geq 0$.

## 2. THE LINEAR EQUATION

We return to Equation (1.1) from Section 1, which we rewrite for reference

$$x' = -\int_{t-r}^{t} a(t,s)x(s)ds.$$

Here $r$ is a positive constant, $a : [0,\infty) \times [-r,\infty) \to R$ is piecewise continuous. The elegant theory derived for (1.1) here will provide bases for much of the study of nonlinear equations such as (1.2). We state the stability results on $[0,\infty)$ and always look at a solution $x(t) = x(t, t_0, \psi)$ for $t_0 \geq 0$, where $\psi : [t_0 - r, t_0] \to R$ is a continuous initial function and $x(t, t_0, \psi) = \psi(t)$ on $[t_0 - r, t_0]$.

**Theorem 2.1.** *Suppose that* (1.6) *holds and*

$$\int_{0}^{t} A(s,s)ds \to \infty \quad \text{as} \quad t \to \infty. \tag{2.1}$$

*Then the zero solution of* (1.1) *is asymptotically stable.*

*Proof.* Let $t_0 \geq 0$ and $\psi : [t_0 - r, t_0] \to R$ be a given continuous initial function. We denote by $C$ the set of continuous functions and define

$$M = \{\phi : [t_0 - r, \infty) \to R \mid \phi_{t_0} = \psi, \phi \in C, \phi(t) \to 0 \text{ as } t \to \infty\} \tag{2.2}$$

so that if $\|\cdot\|$ is the supremum metric $\|\phi\| = \sup\{|\phi(s)| : s \geq t_0 - r\}$, then $(M, \|\cdot\|)$ is a complete metric space. Here $\phi_{t_0} = \psi$ means that $\phi(t) = \psi(t)$ for $t_0 - r \leq t \leq t_0$.

We will also use $\|\psi\|$ to denote the supremum norm of $\psi$ on $[t_0 r, t_0]$ if there is no confusion occurs.

Write (1.1) as

$$x'(t) = -A(t,t)x(t) + \frac{d}{dt}\int_{t-r}^{t} A(t,s)x(s)ds \tag{2.3}$$

or

$$\begin{aligned}\frac{d}{dt}\big[x(t) - \int_{t-r}^{t} A(t,s)x(s)ds\big] =& - A(t,t)[x(t) - \int_{t-r}^{t} A(t,s)x(s)ds\big]\\ &- A(t,t)\int_{t-r}^{t} A(t,s)x(s)ds.\end{aligned}$$

By variation of parameters formula, we obtain for $t \geq t_0$,

$$\begin{aligned}x(t) =& e^{-\int_{t_0}^{t} A(s,s)ds}\big[\psi(t_0) - \int_{t_0-r}^{t_0} A(t_0,s)\psi(s)ds\big]\\ &+ \int_{t-r}^{t} A(t,s)x(s)ds - \int_{t_0}^{t} e^{-\int_s^t A(u,u)du} A(s,s)\int_{s-r}^{s} A(s,\tau)x(\tau)d\tau ds.\end{aligned} \tag{2.4}$$

Use (2.4) to define a mapping $P : M \to M$ as follows: for $\phi \in M$, let $(P\phi)(t) = \psi(t)$ if $t_0 - r \leq t \leq t_0$ and if $t > t_0$, let

$$\begin{aligned}(P\phi)(t) =& e^{-\int_{t_0}^{t} A(s,s)ds}\big[\psi(t_0) - \int_{t_0-r}^{t_0} A(t_0,s)\psi(s)ds\big]\\ &+ \int_{t-r}^{t} A(t,s)\phi(s)ds - \int_{t_0}^{t} e^{-\int_s^t A(u,u)du} A(s,s)\int_{s-r}^{s} A(s,\tau)\phi(\tau)d\tau ds\end{aligned} \tag{2.5}$$

A fixed point of $P$ is a solution of (1.1).

We see that $\phi \in M$ implies that $P\phi$ is continuous on $[t_0 - r, \infty)$. The first two terms of $P\phi$ tend to zero as $t \to \infty$ since

$$\int_0^{\infty} A(s,s)ds = \infty, \quad \int_{t-r}^{t} |A(t,s)|ds < 1, \text{ and } \phi(t) \to 0 \text{ as } t \to \infty.$$

Let $I_3$ denote the last term of $P\phi$. To see $I_3$ tends to zero as $t \to \infty, \forall\varepsilon > 0$, find $T_1 > t_0$ such that $|\phi(s - r)| < \varepsilon$ for $s \geq T_1$. Thus, for $t \geq T_1$, we have

$$\begin{aligned}|I_3| &= \big|\int_{t_0}^{t} e^{-\int_s^t A(u,u)du}\, A(s,s)\int_{s-r}^{s} A(s,\tau)\phi(\tau)d\tau ds\big|\\ &\leq \int_{t_0}^{T_1} e^{-\int_s^t A(u,u)du}\, |A(s,s)|\int_{s-r}^{s} |A(s,\tau)\phi(\tau)|d\tau ds\\ &\quad + \int_{T_1}^{t} e^{-\int_s^t A(u,u)du}\, |A(s,s)|\int_{s-r}^{s} |A(s,\tau)\phi(\tau)|d\tau ds\\ &\leq \|\phi\|\, e^{-\int_{T_1}^{t} A(u,u)du}\int_{t_0}^{T_1} e^{-\int_s^{T_1} A(u,u)du}\, |A(s,s)|\int_{s-r}^{s} |A(s,\tau)|d\tau ds\\ &\quad + \varepsilon\int_{T_1}^{t} e^{-\int_s^t A(u,u)du}|A(s,s)|\int_{s-r}^{s} |A(s,\tau)|d\tau ds\end{aligned}$$

$$\leq \|\phi\| \, e^{-\int_{T_1}^{t} A(u,u)du} + \alpha\varepsilon.$$

By (2.1), there exists $T_2 > T_1$ such that $t \geq T_2$ implies

$$\|\phi\| e^{-\int_{T_1}^{t} A(u,u)du} < (1-\alpha)\varepsilon.$$

This yields $|I_3| < (1-\alpha)\varepsilon + \alpha\varepsilon = \varepsilon$, and therefore, $I_3 \to 0$ as $t \to \infty$. We now have $(P\phi)(t) \to 0$ as $t \to \infty$ and $P\phi \in M$.

To see $P$ is a contraction, consider $\phi, \eta \in M$. For $t \geq t_0$, we have by (1.6) that

$$\begin{aligned}
|(P\phi)(t) - (P\eta)(t)| &\leq \int_{t-r}^{t} |A(t,s)||\phi(s) - \eta(s)|ds \\
&\quad + \int_{t_0}^{t} e^{-\int_s^t A(u,u)du} \, |A(s,s)| \int_{s-r}^{s} |A(s,\tau)||\phi(\tau) - \eta(\tau)|d\tau ds \\
&\leq \|\phi - \eta\| \left[ \int_{t-r}^{t} |A(t,s)|ds + \int_0^t e^{-\int_s^t A(u,u)du} |A(s,s)| \int_{s-r}^{s} |A(s,\tau)|d\tau ds \right] \\
&\leq \alpha \|\phi - \eta\|.
\end{aligned}$$

By the Contraction Mapping Principle (Smart [12, p. 2]), $P$ has a unique fixed point $x \in M$ which is a solution of (1.1) with $x(s) = \psi(s)$ for $t_0 - r \leq s \leq t_0$ and $x(t) = x(t, t_0, \psi) \to 0$ as $t \to \infty$.

To obtain the asymptotic stability, we need to show that the zero solution of (1.1) is stable. To this end, let $\varepsilon > 0$ be given and choose $\delta > 0$ ($\delta < \varepsilon$) satisfying

$$2\delta K e^{\int_0^{t_0} A(s,s)ds} + \alpha\varepsilon < \varepsilon$$

where

$$K = \sup_{t \geq 0} e^{-\int_0^t A(s,s)ds}. \tag{2.6}$$

If $x(t) = x(t, t_0, \psi)$ is a solution of (1.1) with $\|\psi\| < \delta$, then $x(t)$ satisfies (2.4). We claim that $|x(t)| < \varepsilon$ for all $t \geq t_0$. Note that $|x(t)| < \varepsilon$ on $[t_0 - r, t_0]$. If there exists $t^* > t_0$ such that $|x(t^*)| = \varepsilon$ and $|x(s)| < \varepsilon$ for $t_0 \leq s < t^*$, then it follows from (2.4) that

$$\begin{aligned}
|x(t^*)| &\leq \|\phi\| \left[ 1 + \int_{t_0-r}^{t_0} |A(t_0,s)|ds \right] e^{-\int_{t_0}^{t^*} A(u,u)du} \\
&\quad + \varepsilon \left[ \int_{t^*-r}^{t^*} |A(t^*,s)|ds + \int_{t_0}^{t^*} e^{-\int_s^{t^*} A(u,u)du} |A(s,s)| \int_{s-r}^{s} |A(s,\tau)|d\tau ds \right] \\
&\leq 2\delta K e^{\int_0^{t_0} A(s,s)ds} + \alpha\varepsilon < \varepsilon \qquad (2.7)
\end{aligned}$$

which contradicts the definition of $t^*$. Thus, $|x(t)| < \varepsilon$ for all $t \geq t_0$, and the zero solution of (1.1) is stable. This shows that the zero solution of (1.1) is asymptotically stable on $[0, \infty)$. The proof is complete. □

Next we rewrite condition (2.1) below for reference and show that it is a necessary condition for asymptotic stability if $\int_0^t A(u,u)du$ is bounded below. The technique used here has its root in Zhang [14].

**Theorem 2.2.** *Suppose that* (1.6) *holds and*

$$\liminf_{t\to\infty} \int_0^t A(s,s)ds > -\infty. \tag{2.8}$$

*Then the zero solution of* (1.1) *is asymptotically stable if and only if*

$$\int_0^t A(s,s)ds \to \infty \text{ as } t \to \infty.$$

*Proof.* It follows from Theorem 2.1 that if (1.6) holds, then (2.1) is sufficient for the asymptotic stability of the zero solution of (1.1). Thus, we only need to show that (2.1) is a necessary condition under (2.8). Since (2.8) holds, we define the constant $K$ as in (2.6). Now suppose that (2.1) fails. Then by (2.8), there exists a sequence $\{t_n\}, t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \int_0^{t_n} A(s,s)ds = \ell$$

for some $\ell \in R$. We may also choose a positive number $J$ satisfying

$$-J \le \int_0^{t_n} A(s,s)ds \le J$$

for all $n \ge 1$. To simplify expressions, we define

$$\omega(s) = |A(s,s)| \int_{s-r}^{s} |A(s,u)|du$$

for all $s \ge 0$. By (1.6), we have

$$\int_{t_n-r}^{t_n} |A(t_n,s)|ds + \int_0^{t_n} e^{-\int_s^{t_n} A(u,u)du}\omega(s)ds \le \alpha.$$

This yields

$$\int_0^{t_n} e^{\int_0^s A(u,u)du}\omega(s)ds \le \alpha e^{\int_0^{t_n} A(u,u)du} \le e^J$$

The sequence $\{\int_0^{t_n} e^{\int_0^s A(u,u)du}\omega(s)ds\}$ is bounded, so there exists a convergent subsequence. For brevity in notation, we may assume

$$\lim_{n\to\infty} \int_0^{t_n} e^{\int_0^s A(u,u)du}\omega(s)ds = \gamma$$

for some $\gamma \in R^+ = [0,\infty)$ and choose a positive integer $\bar{k}$ so large that

$$\int_{t_{\bar{k}}}^{t_n} e^{\int_0^s A(u,u)du}\omega(s)ds < \delta_0/4K$$

for all $n \ge \bar{k}$, where $\delta_0 > 0$ satisfies $4\delta_0 K e^J + \alpha < 1$.

By (2.8), the number $K$ in (2.6) is well defined. We now consider the solution of $x(t) = x(t, t_{\bar{k}}, \psi)$ of (1.1) with $\psi(t_{t_{\bar{k}}}) = \delta_0$ and $|\psi(s)| \le \delta_0$ for $t_{\bar{k}} - r \le s \le t_{\bar{k}}$. An argument similar to that in (2.7) shows $|x(t)| \le 1$ for $t \ge t_{\bar{k}}$. We may choose $\psi$ so that

$$\psi(t_{\bar{k}}) - \int_{t_{\bar{k}}-r}^{t_{\bar{k}}} A(t_{\bar{k}}, s)\psi(s)ds \ge \frac{1}{2}\delta_0. \tag{2.9}$$

It follows from (2.4) that for $n \ge \bar{k}$,

$$\begin{aligned}
&\left| x(t_n) - \int_{t_n-r}^{t_n} A(t_n, s)x(s)ds \right| \\
&\ge \frac{1}{2}\delta_0 e^{-\int_{t_{\bar{k}}}^{t_n} A(u,u)du} - \int_{t_{\bar{k}}}^{t_n} e^{-\int_s^{t_n} A(u,u)du}\omega(s)ds \\
&= \frac{1}{2}\delta_0 e^{-\int_{t_{\bar{k}}}^{t_n} A(u,u)du} - e^{-\int_0^{t_n} A(u,u)du}\int_{t_{\bar{k}}}^{t_n} e^{\int_0^s A(u,u)du}\omega(s)ds \\
&= e^{-\int_{t_{\bar{k}}}^{t_n} A(u,u)du}\left[\frac{1}{2}\delta_0 - e^{-\int_0^{t_{\bar{k}}} A(u,u)du}\int_{t_{\bar{k}}}^{t_n} e^{\int_0^s A(u,u)du}\omega(s)ds\right] \\
&\ge e^{-\int_{t_{\bar{k}}}^{t_n} A(u,u)du}\left[\frac{1}{2}\delta_0 - K\int_{t_{\bar{k}}}^{t_n} e^{\int_0^s A(u,u)du}\omega(s)ds\right] \\
&\ge \frac{1}{4}\delta_0 e^{-\int_{t_{\bar{k}}}^{t_n} A(u,u)du} \ge \frac{1}{4}\delta_0 e^{-2J} > 0. \qquad (2.10)
\end{aligned}$$

On the other hand, if the zero solution of (1.1) is asymptotically stable, then $x(t) = x(t, t_{\bar{k}}, \psi) \to 0$ as $t \to \infty$. Since $t_n - r \to \infty$ as $n \to \infty$ and (1.6) holds, we have

$$x(t_n) - \int_{t_n-r}^{t_n} A(t_n, s)x(s)ds \to 0 \text{ as } t \to \infty$$

which contradicts (2.10). Hence, it is necessary that (2.1) holds if the zero solution of (1.1) is asymptotically stable. The proof is complete. □

**Corollary 1.** *Suppose that* (1.6) *holds with* $A(t,t) \ge 0$ *for* $t \ge 0$. *Then the zero solution of* (1.1) *is asymptotically stable if and only if* (2.1) *holds.*

## 3. A NONLINEAR EQUATION

We return to Equation (1.2) from Section 1, which we rewrite for reference

$$x' = -\int_{t-r}^{t} a(t,s)g(x(s))ds.$$

Here $r$ is a positive constant, $a : [0,\infty) \times [-r,\infty) \to R$ is piecewise continuous, and $g : R \to R$ is continuous.

**Remark.** We assume that there exists an $L > 0$ such that on $[-L, L]$, $g$ is Lipschitz continuous, $xg(x) > 0$ for $x \ne 0$, and $\lim_{x\to 0} g(x)/x = g^*(0)$ exists. To simplify expressions, we may redefine $g$ and $a$ by setting $a(t,s)g(x) = a(t,s)D(g(x)/D)$ for a positive constant $D$. In case that $g^*(0) \neq 0$, we may redefine $g$ and $a$ so that $g^*(0) = 1$. We note that if this set of conditions holds for one $L > 0$, then it holds for all smaller $L$. Define $g^*(x) = g(x)/x$ for $x \neq 0$.

**Theorem 3.1.** *Suppose that there exists an $L > 0$ such that on $[-L, L]$, $g$ is Lipschitz continuous and $xg(x) > 0$ for $x \neq 0$ with $g^*(0) = 1$. If (1.6) is satisfied with $A(t,t) \geq 0$ for $t \geq 0$, then the zero solution of (1.2) is asymptotically stable if and only if (2.1) holds.*

*Proof.* First, suppose that (2.1) holds, that is, $\int_0^t A(s,s)ds \to \infty$ as $t \to \infty$. Write (1.2) as

$$x'(t) = -A(t,t)g(x(t)) + \frac{d}{dt}\int_{t-r}^{t} A(t,s)g(x(s))ds.$$

Let $t_0 \geq 0$ and $\psi : [t_0 - r, t_0] \to R$ be a continuous initial function. Since $g$ is Lipschitz continuous on $[-L, L]$, there exists a unique local solution $x(t, t_0, \psi) =: z(t)$ of (1.2). This solution exists on $[t_0, \infty)$ since $z(t)$ is bounded on $[t_0, \infty)$ (see proof below). If we set $A^*(t,s) = A(t,s)g^*(z(s))$, then $z(t)$ is the unique solution of

$$x'(t) = -A^*(t,t)x(t) + \frac{d}{dt}\int_{t-r}^{t} A^*(t,s)x(s)ds. \tag{3.1}$$

By variation of parameters formula, we write (3.1) as

$$\begin{aligned} x(t) = {} & e^{-\int_{t_0}^t A^*(s,s)ds}\big[\psi(t_0) - \int_{t_0-r}^{t_0} A^*(t_0,s)\psi(s)ds\big] \\ & + \int_{t-r}^{t} A^*(t,s)x(s)ds \\ & - \int_{t_0}^{t} e^{-\int_s^t A^*(u,u)du} A^*(s,s)\int_{s-r}^{s} A^*(s,\tau)x(\tau)d\tau ds \end{aligned} \tag{3.2}$$

for $t \geq t_0$. Define

$$S = \{\phi : [t_0 - r, \infty) \to R \mid \phi_{t_0} = \psi, \phi \in C, |\phi(t)| \leq L, \phi(t) \to 0 \text{ as } t \to \infty\} \tag{3.3}$$

where the magnitude of $\psi$ and the size of $L$ will be restricted later. We see that $S$ is a complete metric space with the supremum norm.

Use (3.2) to define a mapping $P : S \to S$ as follows: for $\phi \in S$, let $(P\phi)(t) = \psi(t)$ if $t_0 - r \leq t \leq t_0$ and if $t > t_0$, let

$$\begin{aligned} (P\phi)(t) = {} & e^{-\int_{t_0}^t A^*(s,s)ds}\big[\psi(t_0) - \int_{t_0-r}^{t_0} A^*(t_0,s)\psi(s)ds\big] \\ & + \int_{t-r}^{t} A^*(t,s)\phi(s)ds \\ & - \int_{t_0}^{t} e^{-\int_s^t A^*(u,u)du} A(s,s)\int_{s-r}^{s} A(s,\tau)\phi(\tau)d\tau ds \end{aligned} \tag{3.4}$$

A fixed point of $P$ is a solution of (3.1).

Since $A(t,t) \geq 0$, $\int_{t-r}^{t} A(t,s)ds \leq \alpha$, and $\int_0^t A(s,s)ds \to \infty$ as $t \to \infty$, we see

$$\int_{t_0}^{t} e^{-\int_s^t A^*(u,u)du} A^*(s,s) \int_{s-r}^{s} |A(s,\tau)| d\tau ds$$
$$\to 0 \int_{t_0}^{t} e^{-\int_s^t A(u,u)du} A(s,s) \int_{s-r}^{s} |A(s,\tau)| d\tau ds$$

in the supremum norm as $\|g^*(z(s)) - 1\| \to 0$.

We now find a constant $\mu > 0$ with $\mu(1+\alpha) < (1-\alpha)$ so that

$$\int_{t-r}^{t} |A(t,s)|ds + \int_{0}^{t} e^{-\int_s^t A^*(u,u)du} A^*(s,s) \int_{s-r}^{s} |A(s,\tau)| d\tau ds \leq \frac{1+\alpha}{2} \tag{3.5}$$

for all $t \geq t_0$ whenever $\|g^*(z(s)) - 1\| \leq \mu$. Next, we find a sufficiently small $L > 0$ so that $|u| \leq L$ implies $|g^*(u) - 1| < \mu$. Note that

$$\frac{(1+\mu)(1+\alpha)}{2} < \left(1 + \frac{1-\alpha}{1+\alpha}\right) \frac{(1+\alpha)}{2} = 1.$$

We now find $\delta > 0$ with

$$(1+2\alpha)\delta + \frac{(1+\mu)(1+\alpha)}{2} L < L.$$

Let $\|\psi\| < \delta$. We first claim that $|z(t)| < L$ for all $t \geq t_0$. Note that $z(t)$ satisfies (3.2) with $z(t) = x(t)$ and $|z(t)| < \varepsilon$ on $[t_0 - r, t_0]$. If there exists $t^* > t_0$ such that $|z(t^*)| = L$ and $|z(s)| < L$ for $t_0 \leq s < t^*$, then it follows from (3.2) that

$$\begin{aligned}
|z(t^*)| \leq{}& \|\psi\| \left[1 + (1+\mu) \int_{t_0-r}^{t_0} |A(t_0,s)|ds\right] e^{-(1-\mu)\int_{t_0}^{t^*} A(u,u)du} \\
&+ (1+\mu)L \left[\int_{t^*-r}^{t^*} |A(t^*,s)|ds\right] \\
&+ (1+\mu)L \left[\int_{t_0}^{t^*} e^{-\int_s^{t^*} A^*(u,u)du} A^*(s,s) \int_{s-r}^{s} |A(s,\tau)| d\tau ds\right] \\
\leq{}& (1+2\alpha)\delta + \frac{(1+\mu)(1+\alpha)}{2} L < L.
\end{aligned} \tag{3.6}$$

which contradicts the definition of $t^*$. Thus, $|z(t)| < L$ for all $t \geq t_0$. These estimates will work for $(P\phi)(t)$, yielding $|(P\phi)(t)| < L$ whenever $\phi \in S$. We again see that $\phi \in S$ implies that $P\phi$ is continuous on $[t_0 - r, \infty)$. An argument similar to that in the proof of Theorem 1.1 shows that $(P\phi)(t) \to 0$ as $t \to \infty$, and therefore, $P\phi \in S$.

To see $P$ is a contraction, consider $\phi, \eta \in S$. For $t \geq t_0$, we have by (1.6) that

$$\begin{aligned}
|(P\phi)(t) - (P\eta)(t)| \leq{}& \int_{t-r}^{t} |A^*(t,s)||\phi(s) - \eta(s)|ds \\
&+ \int_{t_0}^{t} e^{-\int_s^t A^*(u,u)du} |A^*(s,s)| \int_{s-r}^{s} |A^*(s,\tau)||\phi(\tau) - \eta(\tau)| d\tau ds \\
\leq{}& (1+\mu)\|\phi - \eta\| \int_{t-r}^{t} |A(t,s)|ds
\end{aligned}$$

$$
\begin{aligned}
&+(1+\mu)\|\phi-\eta\| \int_0^t e^{-\int_s^t A^*(u,u)du} A^*(s,s)\left| \int_{s-r}^{s} |A(s,\tau)| d\tau ds \right. \\
&\leq \frac{(1+\mu)(1+\alpha)}{2}\|\phi-\eta\| \\
&= \beta\|\phi-\eta\|.
\end{aligned}
$$

Since $\beta < 1$, we see that $P$ is a contraction and has a unique fixed point $x \in S$ which is a solution of (3.1) with $x(s) = \psi(s)$ for $t_0 - r \leq s \leq t_0$ and $x(t) \to 0$ as $t \to \infty$. Since $z(t)$ is the unique solution of (3.1) with $z_{t_0} = \psi$, we have $z(t) = x(t)$ and $x(t, t_0, \psi) = z(t) \to 0$ as $t \to \infty$.

To see we have obtained stability, substitute $\varepsilon$ for $L$ in the argument above and conclude that $\|\psi\| < \delta$ implies $|x(t)| < \varepsilon$ for all $t \geq t_0$. Thus, the zero solution of (1.2) is asymptotically stable. The proof is complete.

Conversely, suppose that (2.1) fails. Since $A(t,t) \geq 0$, this implies that

$$
J = \sup_{t \geq 0} \int_0^t A(s,s)ds < \infty.
$$

Let $\delta, \mu$ and $L$ be defined above. We choose $t_0 > 0$ so large that

$$
(1+\mu)L\left[e^{(1+\mu)\int_{t_0}^t A(u,u)du} - 1\right] < \frac{\delta}{4}
$$

for $t \geq t_0$ and consider the solution $x(t) = x(t, t_0, \psi)$ of (1.2) with $\psi(t_{\bar{t}_k}) = \delta$ and $|\psi(s)| \leq \delta$ for $t_0 - r \leq s \leq t_0$. An argument similar to that in (3.6) shows $|x(t)| \leq L$ for $t \geq t_0$. We may choose $\psi$ so that

$$
\psi(t_0) - \int_{t_0-r}^{t_0} A^*(t_0,s)\psi(s)ds \geq \frac{1}{2}\delta. \tag{3.7}
$$

It follows from (3.2) with $z(t) = x(t)$ that for $t \geq t_0$,

$$
\begin{aligned}
&\left| x(t) - \int_{t-r}^{t} A^*(t,s)x(s)ds \right| \\
&\geq \frac{1}{2}\delta e^{-(1+\mu)\int_{t_0}^t A(u,u)du} - (1+\mu)L\int_{t_0}^t e^{-\int_s^t A^*(u,u)du} A^*(s,s)ds \\
&= \frac{1}{2}\delta e^{-(1+\mu)\int_{t_0}^t A(u,u)du} - (1+\mu)L\left[1 - e^{-\int_{t_0}^t A^*(u,u)du}\right] \\
&\geq \frac{1}{2}\delta e^{-(1+\mu)\int_{t_0}^t A(u,u)du} - (1+\mu)L\left[1 - e^{-(1+\mu)\int_{t_0}^t A(u,u)du}\right] \\
&= e^{-(1+\mu)\int_{t_0}^t A(u,u)du}\left[\frac{1}{2}\delta - (1+\mu)L\left(e^{(1+\mu)\int_{t_0}^t A(u,u)du} - 1\right)\right] \\
&\geq \frac{1}{4}\delta e^{-(1+\mu)\int_{t_0}^t A(u,u)du} \geq \frac{1}{4}\delta e^{-(1+\mu)J} > 0. \qquad (3.8)
\end{aligned}
$$

On the other hand, if the zero solution of (1.2) is asymptotically stable, then $x(t) = x(t, t_0, \psi) \to 0$ as $t \to \infty$. Since (1.6) holds and $|g^*(x(t))| \leq 1 + \mu$, we have

$$x(t) - \int_{t-r}^{t} A^*(t,s)x(s)ds \to 0 \text{ as } t \to \infty$$

which contradicts (3.8). Hence, it is necessary that (2.1) holds if the zero solution of (1.2) is asymptotically stable. The proof is complete. □

**Remark.** Constructing a mapping function for a nonlinear equation presents a significant challenge for investigators. The method used here has its root in Burton and Furumochi [5]. We have avoided having a term

$$\int_{t_0}^{t} e^{-\int_s^t A(u,u)du} A(s,s)[\phi(s) - g(\phi(s))]ds$$

in the definition of $(P\phi)$. Such a term may require additional conditions on $g$ and $A$ (see Burton [3]).

**Acknowledgments:** This research was supported in part by ISAS Program grant and Office of Summer School, Fayetteville State University, 2012–2013.

## REFERENCES

[1] F. H. Brownell and W. K. Ergen, A theorem on rearrangements and its applications to certain delay differential equations, *J. Rational Mech. Anal.* , 3(1954), 565–579.

[2] T. A. Burton, Stability by fixed point theory or Liapunov's theory: a comparison, *Fixed Point Theory*, 4(2003), 15–32.

[3] T. A. Burton, Fixed points and stability of a nonconvolution equation, *Proc. Amer. Math. Sci.*, 132(2004), 3679–3687.

[4] T. A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, Dover, Mineola, New York, 2006.

[5] T. A. Burton and T. Furumochi, Fixed points and problems in stability theory for ordinary and functional differential equations, *Dynamic Systems and Applications*, 10(2001), 89–116.

[6] J.R. Graef, C. Qian, and B. Zhang, Asymptotic behavior of solutions of differential equations with variable delays, *Proc. London Math. Soc.*, 81(2000), 72–92.

[7] J. K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[8] N. N. Krasovskii, *Stability of Motion*, Stanford Univ. Press, Stanford, CA, 1963.

[9] J. J. Levin, The asymptotic behavior of the solution of a Volterra equation, *Proc. Amer. Math. Soc.*, 14(1963), 534–541.

[10] J. J. Levin, A nonlinear Volterra equation not of convolution type, *J. of Differential Equations*, 4(1968), 176–186.

[11] J. J. Levin and J. A. Nohel, On a nonlinear delay equation. *J. Math. Anal. Appl.*, 8(1964), 31–44.

[12] D. R. Smart, *Fixed Point Theorems*, Cambridge Univ. Press, Cambridge, 1980.

[13] T. Yoshizawa, *Stability Theory by Liapunov's Second Method*, Math. Soc. Japan, Tokyo, 1966.

[14] B. Zhang, Contraction mapping and stability in a delay-differential equation, *Proceedings of Dynamic Systems and Applications*, 4(2004), 183–190.

[15] B. Zhang, Fixed points and stability in differential equations with variable delays, *Nonlinear Analysis*, 63(2005), e233–e242.