Dynamic Systems and Applications 23 (2014) 691-698

EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A FRACTIONAL BOUNDARY VALUE PROBLEM WITH A SEPARATED BOUNDARY CONDITION

LINGJU KONG, QINGKAI KONG, AND MIN WANG

Department of Mathematics, University of Tennessee at Chattanooga,
Chattanooga, TN 37403, USA
Department of Mathematics, Northern Illinois University, DeKalb, IL 60115, USA
Department of Mathematics, University of Tennessee at Chattanooga,
Chattanooga, TN 37403, USA

Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. The authors study a nonlinear fractional boundary value problem with a separated boundary condition. The associated Green's function is constructed as a series of functions by applying the spectral theory. A criterion for the existence and uniqueness of solutions is obtained based on it.

AMS (MOS) Subject Classification. primary 34B15; secondary 34B27

1. Introduction

We study the boundary value problem (BVP) consisting of the nonlinear fractional differential equation

$$-D_{0+}^{\alpha}u + a(t)u' = w(t)f(t,u), \quad 0 < t < 1, \tag{1.1}$$

and the boundary condition (BC)

$$u(0) = u'(0) = u(1) = 0, \tag{1.2}$$

where $2 < \alpha \leq 3$, $a \in C^1[0,1]$, $w \in L[0,1]$ such that $w(t) \not\equiv 0$ a.e. on $[0,1]$, $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, and $D_{0+}^{\alpha}h$ is the α-th Riemann-Liouville fractional derivative of h for $h : [0,1] \to \mathbb{R}$ defined by

$$(D_{0+}^{\alpha}h)(t) = \frac{1}{\Gamma(l-\alpha)}\frac{d^l}{dt^l}\int_0^t (t-s)^{l-\alpha-1}h(s)ds, \quad l = \lfloor \alpha \rfloor + 1, \tag{1.3}$$

provided the right-hand side exists with Γ the Gamma function.

The Green's functions play an important role in the study of nonlinear BVPs as the existence of solutions or positive solutions of a nonlinear BVP can be established by constructing a completely continuous operator based on the associated Greens

Received May 2013

function and finding the fixed point of the operator. This idea has been widely used in the study of nonlinear BVPs of both integer and fractional orders; the reader is referred to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and references therein for some recent results. Due to the unusual feature of the fractional calculus, the Green's functions for fractional BVPs have not been well developed. In most existing work in the literature, the Green's functions were constructed only to solve the BVPs consisting of

$$-D^{\alpha}_{0+}u = f(t,u), \quad 0 < t < 1, \tag{1.4}$$

and certain BCs, see for example [1, 2, 5, 6]. When a more general equation such as

$$-D^{\alpha}_{0+}u + a(t)u = f(t,u), \quad 0 < t < 1, \tag{1.5}$$

is involved, the method employed in those papers fail to work due to the complexity caused by the extra term $a(t)u$.

Recently, Graef, Kong, Kong, and Wang [8, 9] used the Green's function method to study the BVPs consisting of Eq. (1.5) with $1 < \alpha \leq 2$ and two different types of BCs. An approach based on the spectral theory is used to construct the Green's functions as series of functions. We refer the reader to[8, Theorem 2.1] and [9, Theorem 2.1] for the detail. It is notable that this approach can be extended to BVPs consisting of Eq. (1.5) and some other BCs; however, it cannot be directly applied to BVP (1.1), (1.2) due to the appearance of the term with u'.

In this paper, by a modified approach, we will first establish the Green's function for the BVP consisting of the equation

$$-D^{\alpha}_{0+}u + a(t)u' = 0, \quad 0 < t < 1, \tag{1.6}$$

and BC (1.2). Based on it, we will then obtain conditions for the existence and uniqueness of solutions for BVP (1.1), (1.2).

This paper is organized as follows: After this introduction, our main results are stated in Section 2. One example is also given therein. All the proofs are given in Section 3.

2. **Main results**

The Green's function for BVP (1.4), (1.2) is given by Feng, Zhang, and Ge [5, Lemma 2.1] as

$$G_0(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.1}$$

It is easy to see that

$$\frac{\partial G_0(t,s)}{\partial s} = \begin{cases} \dfrac{(1-\alpha)(t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-2})}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{(1-\alpha)t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.2}$$

We will use (2.1) and (2.2) to construct the Green's function for BVP (1.6), (1.2).

Let G_0 be defined by (2.1) and $\mathcal{K} : [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{K}(t,s) &= \frac{\partial}{\partial s}\left[a(s)G_0(t,s)\right] \\ &= a'(s)G_0(t,s) + a(s)\frac{\partial G_0(t,s)}{\partial s}, \quad (t,s) \in [0,1]\times[0,1]. \end{aligned} \tag{2.3}$$

Throughout this paper, we assume that $|a(t)|$ and $|a'(t)|$ are small enough such that

(H) $B := \max_{t\in[0,1]} \int_0^1 |\mathcal{K}(t,s)|ds < 1$.

Clearly, when a is constant, $B = |a| \max_{t\in[0,1]} \int_0^1 \left|\frac{\partial G_0(t,s)}{\partial s}\right| ds$.

Define $G : [0,1]\times[0,1] \to \mathbb{R}$ and $\overline{G} : [0,1] \to \mathbb{R}$ by

$$G(t,s) = \sum_{n=0}^{\infty} G_n(t,s) \quad \text{and} \quad \overline{G}(s) = \frac{\overline{G}_0(s)}{1-B}, \tag{2.4}$$

where G_0 is defined by (2.1),

$$G_n(t,s) = \int_0^1 \mathcal{K}(t,\tau)G_{n-1}(\tau,s)d\tau, \quad n = 1,2,\ldots, \tag{2.5}$$

and

$$\overline{G}_0(s) = \begin{cases} \dfrac{(\theta(s)(1-s))^{\alpha-1} - (\theta(s)-s)^{\alpha-1}}{\Gamma(\alpha)}, & s \in (0,1), \\ 0, & s = 0,1, \end{cases} \tag{2.6}$$

with $\theta(s) = \frac{s}{1-(1-s)^{(\alpha-1)/(\alpha-2)}}$, $s \in (0,1)$.

We then have the following result.

Theorem 2.1. *The function $G(t,s)$ defined by* (2.4) *as a series of functions is uniformly convergent for $(t,s) \in [0,1]\times[0,1]$. Furthermore, G is the Green's function for BVP* (1.6), (1.2) *and satisfies $|G(t,s)| \le \overline{G}(s)$ on $[0,1]\times[0,1]$.*

With the Green's function G given in Theorem 2.1, we may study the existence and uniqueness of solutions of BVP (1.1), (1.2).

Theorem 2.2. *Assume f satisfies the Lipschitz condition in x*

$$|f(t,x_1) - f(t,x_2)| \le k|x_1 - x_2| \quad \text{for } (t,x_1),\ (t,x_2) \in [0,1]\times\mathbb{R},$$

with $k \in (0, 1/\int_0^1 \overline{G}(s)|w(s)|ds)$. Then BVP (1.1), (1.2) *has a unique solution. If in addition, $f(t,0) \equiv 0$ on $[0,1]$, then BVP* (1.1), (1.2) *has no nontrivial solution.*

To illustrate the application of our results, we consider the following example.

Example 2.3. Consider the BVP

$$\begin{cases} -D_{0+}^{\alpha}u + au' = p\tan^{-1}u + e^t, \\ u(0) = u'(0) = u(1) = 0, \end{cases} \tag{2.7}$$

where $2 < \alpha \leq 3$ and $|a| < (\max_{t\in[0,1]} \int_0^1 |\partial G_0(t,s)/\partial s| ds)^{-1}$. We claim that BVP (2.7) has a unique solution when $0 < p < 1/\int_0^1 \overline{G}(s)ds$.

In fact, Assumption (H) holds when $|a| < (\max_{t\in[0,1]} \int_0^1 |\partial G_0(t,s)/\partial s| ds)^{-1}$. Let $f(t,x) = p\tan^{-1}x + e^t$ and $w(t) \equiv 1$. It is easy to see that $|f(t,x_1) - f(t,x_2)| \leq p|x_1 - x_2|$ for any (t,x_1), $(t,x_2) \in [0,1]\times\mathbb{R}$. Then by Theorem 2.2, BVP (2.7) has a unique solution. Furthermore, it is easy to see that the solution is nontrivial since $f(t,0) \not\equiv 0$.

3. **Proofs**

The following lemma on the spectral theory in Banach spaces is used to derive the associated Green's function. See [16, page 795, items 57b and 57d] for the detail.

Lemma 3.1. *Let X be a Banach space, $\mathcal{B} : X \to X$ a linear operator with the operator norm $\|\mathcal{B}\|$ and the spectral radius $r(\mathcal{B})$ of $\mathcal{B}$. Then*

(a) $r(\mathcal{B}) \leq \|\mathcal{B}\|$;
(b) *if $r(\mathcal{B}) < 1$, then $(\mathcal{I} - \mathcal{B})^{-1}$ exists and $(\mathcal{I} - \mathcal{B})^{-1} = \sum_{n=0}^{\infty}\mathcal{B}^n$, where $\mathcal{I}$ stands for the identity operator.*

The following lemma is excerpted from [5, Proposition 2.2].

Lemma 3.2. *Let G_0 and $\overline{G}_0$ be defined by* (2.1) *and* (2.6), *respectively. Then $G_0(t,s) \leq \overline{G}_0(s)$ on $[0,1]\times[0,1]$.*

In the sequel we let $X = C[0,1]$ be the Banach space with the standard maximum norm.

Proof of Theorem 2.1. For any $h \in X$, let u be a solution of the BVP consisting of

$$-D_{0+}^{\alpha}u + a(t)u' = h(t),\ 0 < t < 1,$$

and (1.2). By (2.1),

$$u(t) = \int_0^1 G_0(t,s)(h(s) - a(s)u'(s))ds, \tag{3.1}$$

i.e.,

$$u(t) + \int_0^1 a(s)G_0(t,s)u'(s)ds = \int_0^1 G_0(t,s)h(s)ds.$$

By integration by parts and BC (1.2),

$$\int_0^1 a(s)G_0(t,s)u'(s)ds = -\int_0^1 \mathcal{K}(t,s)u(s)ds,$$

where $\mathcal{K}$ is defined by (2.3). Hence

$$u(t) - \int_0^1 \mathcal{K}(t,s)u(s)ds = \int_0^1 G_0(t,s)h(s)ds. \tag{3.2}$$

Define $\mathcal{A}$ and $\mathcal{B}: X \to X$ by

$$(\mathcal{A}h)(t) = \int_0^1 G_0(t,s)h(s)ds \quad \text{and} \quad (\mathcal{B}h)(t) = \int_0^1 \mathcal{K}(t,s)h(s)ds. \tag{3.3}$$

Then Eq. (3.2) can be written as

$$u - \mathcal{B}u = \mathcal{A}h. \tag{3.4}$$

By (H), it is easy to verify that $\|\mathcal{B}\| < 1$. Then by Lemma 3.1, $r(\mathcal{B}) < 1$, and

$$u = (\mathcal{I} - \mathcal{B})^{-1}\mathcal{A}h = \sum_{n=0}^{\infty} \mathcal{B}^n \mathcal{A}h. \tag{3.5}$$

By (3.3), (2.5), and induction, we can prove that for $n \in \mathbb{N}_0$,

$$(\mathcal{B}^n \mathcal{A}h)(t) = \int_0^1 G_n(t,s)h(s)ds, \tag{3.6}$$

and

$$|G_n(t,s)| \le B^n \overline{G}_0(s) \text{ on } [0,1] \times [0,1],$$

where B is defined in (H). Since $B \in [0,1)$, by (2.4), for $(t,s) \in [0,1] \times [0,1]$,

$$|G(t,s)| = |\sum_{n=0}^{\infty} G_n(t,s)| \le \sum_{n=0}^{\infty} |G_n(t,s)| \le \sum_{n=0}^{\infty} B^n \overline{G}_0(s) = \overline{G}(s).$$

Therefore, $G(t,s)$ as a series of functions is uniformly convergent on $[0,1] \times [0,1]$. By (2.4), (3.5), and (3.6),

$$u(t) = \sum_{n=0}^{\infty} \int_0^1 G_n(t,s)h(s)ds = \int_0^1 G(t,s)h(s)ds, \quad t \in [0,1]. \tag{3.7}$$

On the other hand, let u be defined by (3.7). By (2.4) and (3.6), u satisfies (3.5). Hence (3.4) holds. By (3.3) and (2.3), u satisfies (3.1). Therefore, u is a solution of Eq. (3.4).

Thus, G is the Green's function for BVP (1.6), (1.2). □

Now we prove Theorem 2.2 using the contraction mapping principle.

Proof of Theorem 2.2. Define $T: X \to X$ by

$$(Tu)(t) = \int_0^1 G(t,s)w(s)f(s,u(s))ds, \ u \in X.$$

Clearly, T is completely continuous and $u(t)$ is a solution of BVP (1.1), (1.2) if and only if u is a fixed point of T in X.

For any u_1, $u_2 \in X$, and $t \in [0,1]$,

$$\begin{aligned}|(Tu_1 - Tu_2)(t)| &= \left| \int_0^1 G(t,s)w(s)\left(f(s,u_1(s)) - f(s,u_2(s))\right) ds \right| \\ &\leq \int_0^1 \overline{G}(s)|w(s)|\,|f(s,u_1(s)) - f(s,u_2(s))|\, ds \\ &\leq \int_0^1 \overline{G}(s)|w(s)|k\,|u_1(s) - u_2(s)|\, ds \\ &\leq (k \int_0^1 \overline{G}(s)|w(s)|ds)\|u_1 - u_2\|.\end{aligned}$$

Note that $k\int_0^1 \overline{G}(s)|w(s)|ds < 1$. Hence T is a contraction mapping. By the contraction mapping principle, T has a unique fixed point. Thus, BVP (1.1), (1.2) has a unique solution.

If in addition, $f(t,0) \equiv 0$ on $[0,1]$. Then obviously $u(t) \equiv 0$ is a solution of BVP (1.1), (1.2). By the uniqueness of solutions, BVP (1.1), (1.2) has no nontrivial solutions. □

REFERENCES

[1] B. Ahmad and J. J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, *Comput. Math. Appl.* **58** (2009), 1838–1843.

[2] Z. Bai and H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.* **311** (2005), 495–505.

[3] A. Cabada and G. Wang, Positive solutions of nonlinear fractional differential equations with integral boundary value conditions, *J. Math. Anal. Appl.* **389** (2012), 403–411

[4] L. Erbe, Eigenvalue criteria for existence of positive solutions to nonlinear boundary value problems, *Math. Computer Modelling* **32** (2000), 529–539.

[5] M. Feng, X. Zhang, and W. Ge, New existence results for higher-order nonlinear fractional differential equation with integral boundary conditions, *Bound. Value Probl.* **2011**, Art. ID 720702, 20 pp.

[6] C. Goodrich, Existence of a positive solution to a class of fractional differential equations, *Appl. Math. Lett.* **23** (2010), 1050–1055.

[7] J. R. Graef and L. Kong, Existence results for nonlinear periodic boundary value problems, *Proc. Edinburgh Math. Soc.* **52** (2009), 79–95.

[8] J. R. Graef, L. Kong, Q. Kong, and M. Wang, Fractional boundary value problems with integral boundary conditions, *Appl. Anal.* **92** (2013), 2008–2020.

[9] J. R. Graef, L. Kong, Q. Kong, and M. Wang, Existence and uniqueness of solutions for a fractional boundary value problem with Dirichlet boundary condition, *Electron. J. Qual. Theory Differ. Equ.* **2013** No. 55, 11 pp.

[10] J. R. Graef, L. Kong, and H. Wang, Existence, multiplicity, and dependence on a parameter for a periodic boundary value problem, *J. Differential Equations* **245** (2008), 1185–1197.

[11] G. Infante, F. Minhós, and P. Pietramala, Non-negative solutions of systems of ODEs with coupled boundary conditions. Commun. Nonlinear Sci. Numer. Simul. 17 (2012), no. 12, 4952–4960.

[12] D. Jiang and C. Yuan, The positive properties of the Green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its application, *Nonlinear Anal.* **72** (2010), 710–719.

[13] Q. Kong and M. Wang,Positive solutions of nonlinear fractional boundary value problems with Dirichlet boundary conditions, *Electron. J. Qual. Theory Differ. Equ.* **2012**, No. 17, 13 pp.

[14] Q. Kong and M. Wang, Positive solutions of even order periodic boundary value problems, *Rocky Montain J. Math.* **41** (2011), 1907–1931.

[15] Q. Kong and M. Wang, Positive solutions of even order system periodic boundary value problems, *Nonlinear Anal.* **72** (2010), 1778–1791.

[16] E. Zeidler, *Nonlinear Functional Analysis and its Applications I: Fixed-Point Theorems*, Springer-Verlag, New York, 1986.