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POSITIVE SOLUTIONS OF A NONLOCAL CAPUTO FRACTIONAL BVP

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Dedicated to Professor John R. Graef on the occasion of his seventieth birthday.

ABSTRACT. We discuss the existence of multiple positive solutions for a nonlocal fractional problem recently considered by Nieto and Pimentel. Our approach relies on classical fixed point index.

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1. INTRODUCTION

Recently, Nieto and Pimentel [22] studied the existence of positive solutions of the nonlocal fractional boundary value problem (BVP)

$$\begin{aligned} &{}^{C}D^{\alpha}u(t)+f(t,u(t))=0, \quad t\in(0,1),\\ &u'(0)=0,\ \beta\,{}^{C}D^{\alpha-1}u(1)+u(\eta)=0, \end{aligned} \tag{1.1}$$

where $1<\alpha\le 2$, ${}^{C}D^{\alpha}$ denotes the Caputo fractional derivative of order α, $\beta>0$, $0\le\eta\le 1$ and f is continuous. The reason, given in [22], for studying the BVP (1.1) is that it is seen as a mathematical generalisation of the BVP

$$\begin{aligned} &u''(t)+f(t,u(t))=0, \quad t\in(0,1),\\ &u'(0)=0,\ \beta u'(1)+u(\eta)=0, \end{aligned} \tag{1.2}$$

that was studied by Infante and Webb [14], who were motivated by the previous work of Guidotti and Merino [10]. The BVP (1.2) can be used as a model for heated bar of length 1 with a thermostat, where a controller at $t=1$ adds or removes heat according to the temperature detected by a sensor at $t=\eta$. Heat-flow problems of

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this type have been studied recently, see for example [4, 12, 13, 15, 18, 24, 27, 28, 31] and references therein.

Here we discuss the existence of multiple positive solutions for the nonlocal BVP

$$\begin{gathered} {}^{C}D^{\alpha}u(t) + f(t,u(t)) = 0, \quad t \in (0,1), \\ u'(0) + \lambda[u] = 0,\ \beta\, {}^{C}D^{\alpha-1}u(1) + u(\eta) = 0, \end{gathered} \tag{1.3}$$

where $\lambda[\cdot]$ is a functional given by

$$\lambda[u] = \Lambda_0 + \int_0^1 u(s)\, d\Lambda(s),$$

involving a Stieltjes integral. This type of BCs includes as special cases

$$\lambda[u] = \sum_{i=1}^{m} \lambda_i u(\xi_i) \quad (m\text{-point problems})$$

and

$$\lambda[u] = \int_0^1 \lambda(s)u(s)\, ds \quad \text{(continuously distributed cases)}.$$

Multi-point and integral BCs are widely studied objects, see, for example, Karakostas and Tsamatos [16, 17], Ma [21], Ntouyas [23], Webb [29, 30], Henderson and co-authors [11], Infante and Webb [15, 32], Zima [33].

We mention that John Graef and co-authors have actively contributed to the study of nonlocal and fractional problems with interesting papers, for some of their recent works see [6, 7, 8, 9].

In this note, we use the methodology developed in [15], that is to rewrite the BVP (1.3) as a *perturbed* Hammerstein integral equation of the form

$$u(t) = \gamma(t)\lambda[u] + \int_0^1 k(t,s)f(s,u(s))\, ds, \tag{1.4}$$

and use the classical *theory of fixed point index*, for example see [1, 5], in order to gain the existence of positive solutions of (1.4), by working in a suitable cone of positive functions. The existence of positive solutions of (1.4) provide the existence of *positive solutions* of the BVP (1.3).

2. PRELIMINARIES

We firstly recall the definition of the Caputo derivative. For its properties we refer to the books [2, 3, 25, 26].

Definition 2.1. For a function $y : [0,+\infty) \to \mathbb{R}$, the Caputo derivative of fractional order $\alpha > 0$ is given by

$${}^{C}D^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha+1-n}}\, \mathrm{d}s, \quad n = [\alpha]+1,$$

where Γ denotes the Gamma function, that is

$$\Gamma(s) = \int_0^{+\infty} x^{s-1} e^{-x} dx,$$

and $[\alpha]$ denotes the integer part of a number α.

We now recall some results from [15], regarding the existence of multiple positive solutions of perturbed Hammerstein integral equations

$$u(t) = \gamma(t)\lambda[u] + \int_0^1 k(t,s) f(s, u(s))\, ds := Tu(t), \tag{2.1}$$

We point out that a cone K in a Banach space X, is a closed, convex set such that $\lambda x \in K$ for all $x \in K$ and $\lambda \geq 0$, and $K \cap (-K) = \{0\}$.

We work in the space of continuous functions $C[0,1]$ endowed with the usual supremum norm and we look for fixed points of T in the following cone of non-negative functions

$$K = \left\{ u \in C[0,1], u \geq 0 : \min_{t \in [a,b]} u(t) \geq c\|u\| \right\}, \tag{2.2}$$

with c a positive constant related to the kernel k and the function γ.

The cone (2.2) was first used by Krasnosel'skiĭ, see e.g. [19], and D. Guo, see e.g. [5], and then used by many authors.

The following assumptions on the terms that occur in (2.1) are a special case of the ones in [15].

- $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.
- $k : [0,1] \times [0,1] \to [0,\infty)$ is continuous.
- There exist a function $\Phi : [0,1] \to [0,\infty)$, $\Phi \in L^1[0,1]$, an interval $[a,b] \subset [0,1]$ and a constant $c_1 \in (0,1]$ such that

$$k(t,s) \leq \Phi(s) \text{ for } t, s \in [0,1] \text{ and}$$
$$c_1 \Phi(s) \leq k(t,s) \text{ for } t \in [a,b] \text{ and } s \in [0,1].$$

- λ is an affine functional given by

$$\lambda[u] = \Lambda_0 + \int_0^1 u(s)\, d\Lambda(s), \tag{2.3}$$

  where $\Lambda_0 \geq 0$ and $d\Lambda$ is a *positive* Stieltjes measure with $\Lambda_1 := \int_0^1 d\Lambda(s) < \infty$.
- $\gamma : [0,1] \to [0,\infty)$ is continuous, there exists a constant $c_2 \in (0,1]$ such that

$$\gamma(t) \geq c_2\|\gamma\| \quad \text{for } t \in [a,b].$$

  and

$$\tilde{\lambda}[\gamma] := \int_0^1 \gamma(t)\, d\Lambda(t) < 1,$$

The assumptions above enable us to use the cone (2.2) with $c = \min\{c_1, c_2\}$. A routine argument shows that T maps K into K and is compact.

We make use, for our index calculations, of the following open bounded sets (relative to K):

$$K_\rho = \{u \in K : \|u\| < \rho\}, \quad V_\rho = \left\{u \in K : \min_{t\in[a,b]} u(t) < \rho\right\}.$$

These sets have the key property that

$$K_\rho \subset V_\rho \subset K_{\rho/c}.$$

The first Lemma is a special case of Lemma 2.4 of [15] and ensures that, for a suitable $\rho > 0$, the fixed point index is 0 on the set V_ρ.

Lemma 2.2. *Assume that*

(I_ρ^0) *there exists $\rho > 0$ such that for some $\lambda_0 \geq 0$*

$$\lambda[u] \geq \lambda_0\rho \ \text{ for } u \in \partial V_\rho, \quad c_2\|\gamma\|\lambda_0 + f_{\rho,\rho/c} \cdot \frac{1}{M} > 1, \tag{2.4}$$

where

$$f_{\rho,\rho/c} = \inf\left\{\frac{f(t,u)}{\rho} : (t,u) \in [a,b] \times [\rho, \rho/c]\right\} \ \text{ and } \ \frac{1}{M} = \inf_{t\in[a,b]} \int_a^b k(t,s)\,ds.$$

Then $i_K(T, V_\rho) = 0$.

The next Lemma is a special case of Lemma 2.6 of [15] provides a condition that yields, for a suitable $\rho > 0$, that the index is 1 on a set K_ρ.

Lemma 2.3. *Assume that*

(I_ρ^1) *there exists $\rho > 0$ such that*

$$\frac{\Lambda_0\|\gamma\|}{\rho(1 - \lambda\tilde{[\gamma]})} + \left(\frac{\|\gamma\|}{1 - \tilde{\lambda}[\gamma]}\int_0^1 \mathcal{K}(s)\,ds + \frac{1}{m}\right) f^{0,\rho} < 1, \tag{2.5}$$

where

$$\mathcal{K}(s) = \int_0^1 k(t,s)\,d\Lambda(t),$$

$$f^{0,\rho} = \sup\left\{\frac{f(t,u)}{\rho} : (t,u) \in [0,1] \times [0,\rho]\right\} \ \text{ and }$$

$$\frac{1}{m} = \sup_{t\in[0,1]} \int_0^1 k(t,s)\,ds.$$

Then $i_K(T, K_\rho) = 1$.

The two Lemmas above give the following result on the existence of multiple positive solutions for Eq. (2.1). The proof follows from the properties of fixed point index and is omitted.

Theorem 2.4. *The integral equation* (2.1) *has at least one non-zero solution in* K *if any of the following conditions hold.*

(S_1) *There exist* $\rho_1, \rho_2 \in (0,\infty)$ *with* $\rho_1/c < \rho_2$ *such that* $(\mathrm{I}^0_{\rho_1})$ *and* $(\mathrm{I}^1_{\rho_2})$ *hold.*
(S_2) *There exist* $\rho_1, \rho_2 \in (0,\infty)$ *with* $\rho_1 < \rho_2$ *such that* $(\mathrm{I}^1_{\rho_1})$ *and* $(\mathrm{I}^0_{\rho_2})$ *hold.*

The integral equation (2.1) *has at least two non-zero solutions in* K *if one of the following conditions hold.*

(S_3) *There exist* $\rho_1, \rho_2, \rho_3 \in (0,\infty)$ *with* $\rho_1/c < \rho_2 < \rho_3$ *such that* $(\mathrm{I}^0_{\rho_1})$, $(\mathrm{I}^1_{\rho_2})$ *and* $(\mathrm{I}^0_{\rho_3})$ *hold.*
(S_4) *There exist* $\rho_1, \rho_2, \rho_3 \in (0,\infty)$ *with* $\rho_1 < \rho_2$ *and* $\rho_2/c < \rho_3$ *such that* $(\mathrm{I}^1_{\rho_1})$, $(\mathrm{I}^0_{\rho_2})$ *and* $(\mathrm{I}^1_{\rho_3})$ *hold.*

The integral equation (2.1) *has at least three non-zero solutions in* K *if one of the following conditions hold.*

(S_5) *There exist* $\rho_1, \rho_2, \rho_3, \rho_4 \in (0,\infty)$ *with* $\rho_1/c < \rho_2 < \rho_3$ *and* $\rho_3/c < \rho_4$ *such that* $(\mathrm{I}^0_{\rho_1})$, $(\mathrm{I}^1_{\rho_2})$, $(\mathrm{I}^0_{\rho_3})$ *and* $(\mathrm{I}^1_{\rho_4})$ *hold.*
(S_6) *There exist* $\rho_1, \rho_2, \rho_3, \rho_4 \in (0,\infty)$ *with* $\rho_1 < \rho_2$ *and* $\rho_2/c < \rho_3 < \rho_4$ *such that* $(\mathrm{I}^1_{\rho_1})$, $(\mathrm{I}^0_{\rho_2})$, $(\mathrm{I}^1_{\rho_3})$ *and* $(\mathrm{I}^0_{\rho_4})$ *hold.*

3. THE NONLOCAL BVP

A direct calculation shows that solution of the linear equation

$$^{C}D^{\alpha}u + y = 0,$$

under the BCs

$$u'(0) + \lambda[u] = 0, \quad \beta\, ^{C}D^{\alpha-1}u(1) + u(\eta) = 0,$$

can be written in the form

$$u(t) = \left(\frac{\beta}{\Gamma(3-\alpha)} + \eta - t\right)\lambda[u] + \beta\int_0^1 y(s)ds + \int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds - \int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds.$$

Therefore the solution of the BVP (1.3) is

$$u(t) = \gamma(t)\lambda[u] + \int_0^1 k(t,s)f(s,u(s))ds$$

where

$$\gamma(t) = \left(\frac{\beta}{\Gamma(3-\alpha)} + \eta - t\right)$$

and

$$k(t,s) = \beta + \frac{1}{\Gamma(\alpha)}\begin{cases}(\eta-s)^{\alpha-1}, & s \le \eta\\ 0, & s > \eta\end{cases} - \frac{1}{\Gamma(\alpha)}\begin{cases}(t-s)^{\alpha-1}, & s \le t\\ 0, & s > t.\end{cases}$$

Here we focus on the case

$$\beta\Gamma(\alpha) > (1-\eta)^{\alpha-1}, \quad \beta > (1-\eta)\Gamma(3-\alpha),$$

where $[a,b]$ can be chosen equal to $[0,1]$.

Upper and lower bounds for $k(t,s)$ were given in [22] as follows:

$$\Phi(s) = \frac{\beta\Gamma(\alpha)+\eta^{\alpha-1}}{\Gamma(\alpha)}, \quad c_1 = \frac{\beta\Gamma(\alpha)-(1-\eta)^{\alpha-1}}{\beta\Gamma(\alpha)+\eta^{\alpha-1}}.$$

Furthermore, by direct computation, we have

$$\|\gamma\| = \eta + \frac{\beta}{\Gamma(3-\alpha)}, \quad c_2 = \frac{\beta+(\eta-1)\Gamma(3-\alpha)}{\beta+\eta\Gamma(3-\alpha)}.$$

Hence we work with the cone

$$K = \left\{ u \in C[0,1] : \min_{t\in[0,1]} u(t) \geq c\|u\| \right\},$$

where

$$c = \min\left\{ \frac{\beta\Gamma(\alpha)-(1-\eta)^{\alpha-1}}{\beta\Gamma(\alpha)+\eta^{\alpha-1}}, \frac{\beta+(\eta-1)\Gamma(3-\alpha)}{\beta+\eta\Gamma(3-\alpha)} \right\}. \tag{3.1}$$

Example 3.1. Consider the BVP

$$\begin{gathered} {}^{C}D^{\alpha}u(t) + f(t,u(t)) = 0, \quad t \in (0,1), \\ u'(0) + \lambda u(\xi) = 0, \quad \beta\, {}^{C}D^{\alpha-1}u(1) + u(\eta) = 0, \quad \xi,\eta \in [0,1]. \end{gathered} \tag{3.2}$$

For this BVP we may take in (2.3) $\Lambda_0 = 0$ and $d\Lambda(s)$ the Dirac measure of weight $\lambda > 0$ at ξ. A direct calculation gives

$$m = \frac{\Gamma(\alpha+1)}{\beta\Gamma(\alpha+1)+\eta^{\alpha}} \quad \text{and} \quad \frac{1}{M} = \frac{\Gamma(\alpha+1)}{\beta\Gamma(\alpha+1)+\eta^{\alpha}-1}.$$

Here we may take in (2.4) $\lambda_0 = \lambda$ since, for $u \in \partial V_\rho$, we have

$$\lambda[u] = \lambda u(\xi) \geq \lambda\rho.$$

For these BCs, (2.4) reads

$$\frac{\lambda(\beta+(\eta-1)\Gamma(3-\alpha))}{\Gamma(3-\alpha)} + f_{\rho,\rho/c}\cdot\frac{1}{M} > 1, \tag{3.3}$$

and so, $i_K(T,V_\rho) = 0$.

From Lemma 2.3, we have that $i_K(T,K_\rho) = 1$ if $\tilde{\lambda}[\gamma] < 1$ and

$$\left(\frac{\beta+\eta\Gamma(3-\alpha)}{(1-\tilde{\lambda}[\gamma])\Gamma(3-\alpha)} \int_0^1 \mathcal{K}(s)\,ds + \frac{1}{m} \right) f^{0,\rho} < 1. \tag{3.4}$$

So we need

$$\tilde{\lambda}[\gamma] = \int_0^1 \gamma(t)\,d\Lambda(t) = \lambda\gamma(\xi) = \lambda\left(\frac{\beta}{\Gamma(3-\alpha)} + \eta - \xi \right) < 1.$$

Since $\mathcal{K}(s) = \lambda k(\xi, s)$, we obtain

$$\int_0^1 \mathcal{K}(s)\, ds = \lambda \int_0^1 k(\xi, s)\, ds = \lambda \left(\beta + \frac{\eta^\alpha}{\Gamma(\alpha+1)} - \frac{\xi^\alpha}{\Gamma(\alpha+1)} \right).$$

Note that all the numbers in (3.3) and (3.4) can be computed. For example the choice of

$$\alpha = 3/2, \quad \beta = 4/5, \quad \eta = 3/4, \quad \xi = 1/4, \quad \lambda = 1/2$$

gives $c = 0.132$. Then the $i_K(T, V_\rho) = 0$ condition needs

$$f_{\rho,\rho/c} > 0.218$$

and $i_K(T, K_\rho) = 1$ requires

$$f^{0,\rho} < 1.255.$$

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