

# ASYMPTOTIC BEHAVIOR FOR A GENERAL CLASS OF HOMOGENEOUS SECOND ORDER EVOLUTION EQUATIONS IN A HILBERT SPACE

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**ABSTRACT.** We study the asymptotic behavior of solutions to the following general homogeneous second order evolution equation, with suitable assumptions on  $p(t)$  and  $r(t)$ ,

$$\begin{cases} p(t)u''(t) + r(t)u'(t) \in Au(t) & \text{a.e. on } \mathbb{R}^+ \\ u(0) = u_0, & \sup_{t \geq 0} |u(t)| < +\infty \end{cases}$$

where  $A$  is a maximal monotone operator in a real Hilbert space, and present some applications. In the homogeneous case, our results extend those given in [7, 10, 12].

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## 1. Introduction

Let  $H$  be a real Hilbert space with inner product  $(\cdot, \cdot)$  and norm  $|\cdot|$ . We denote weak convergence in  $H$  by  $\rightharpoonup$  and strong convergence by  $\rightarrow$ . A (nonlinear) possibly multivalued operator in  $H$  is a nonempty subset  $A$  of  $H \times H$ .  $A$  is said to be monotone if  $(y_2 - y_1, x_2 - x_1) \geq 0$  for all  $[x_i, y_i] \in A$ ,  $i = 1, 2$ .  $A$  is maximal monotone if  $A$  is monotone and  $R(I + A) = H$ , where  $I$  is the identity operator on  $H$ . Given a function  $\varphi : H \rightarrow ]-\infty, +\infty]$ , its subdifferential is defined as the multivalued operator  $\partial\varphi$  where

$$\partial\varphi(x) = \{w \in H \mid \varphi(x) - \varphi(y) \leq (w, x - y), \quad \forall y \in H\}.$$

$\varphi$  is called proper if there exists  $x \in H$  such that  $\varphi(x) < +\infty$ . It is a well known result that if  $\varphi$  is proper, convex and lower semicontinuous, then  $\partial\varphi$  is a maximal monotone operator. We refer the reader to the books by Barbu [4], Brézis [5] and Morosanu [17] for further details on the properties of monotone operators and subdifferentials of convex functions in Hilbert spaces.

In [22–23], Véron proved the existence of solutions satisfying  $u', u'' \in L^2((0, +\infty); H)$ , to the following second order evolution equation

$$(1) \quad \begin{cases} p(t)u''(t) + r(t)u'(t) \in Au(t) \text{ a.e. on } \mathbb{R}^+ \\ u(0) = u_0, \quad \sup_{t \geq 0} |u(t)| < +\infty, \end{cases}$$

where  $A$  is a maximal monotone operator, provided that the following conditions on  $p(t)$  and  $r(t)$  hold:

$$(2) \quad p \in W^{2,\infty}(0, +\infty), \quad r \in W^{1,\infty}(0, +\infty)$$

$$(3) \quad \exists \alpha > 0, \quad \text{such that } \forall t \geq 0, \quad p(t) \geq \alpha.$$

He showed also the uniqueness of the solution to (1) if moreover:

$$(4) \quad \int_0^\infty e^{-\int_0^t \frac{r(s)}{p(s)} ds} dt = +\infty$$

The existence and asymptotic behavior of solutions to (1) when  $p(t) \equiv 1$  and  $r(t) \equiv 0$  has been studied by Barbu [2–4], Morosanu [17–18], Mitidieri [15–16] and Poffald-Reich [20–21], and by Bruck [6] for nonhomogeneous and periodic forcing case. The authors [8–10] studied the asymptotic behavior of solutions to (1) when  $p(t) \equiv 1$  and  $r(t) \equiv c \leq 0$  for the nonhomogeneous case, without assuming neither  $A^{-1}(0) \neq \emptyset$  nor the maximality of the monotone operator  $A$ , extending previous results on the asymptotic behavior of solutions to (1). In [7], when  $A = \partial\varphi$  where  $\varphi$  is a proper, convex and lower semicontinuous function, we proved an ergodic theorem and a weak convergence theorem for solutions to (1), by assuming (2), (3), (4) and that  $t^2r(t)$  is eventually bounded from below. Véron [24] showed that strong convergence may not occur for  $p(t) \equiv 1$  and  $r(t) \equiv 0$ , even when  $A = \partial\varphi$ . In [11], we proved the strong convergence of solutions to (1) when  $p(t) \equiv 1$  and  $r(t) \equiv c > 0$ , and in [12], we considered (1), with  $p$  and  $r$  being time dependent, and proved the strong convergence of solutions, and determined their rate of convergence. In this paper, we consider (1) with rather general conditions on  $p(t)$  and  $r(t)$ , and prove ergodic, weak, as well as strong convergence theorems for solutions to (1), extending the previous results mentioned above. The paper is divided into five sections. In section 2, we prove an ergodic and a weak convergence theorem. In section 3, we investigate the strong convergence of solutions. Section 4 is devoted to the subdifferential case, i.e. when  $A = \partial\varphi$ . Finally, in section 5 we present some applications of our results to optimization and partial differential equations. We note also that our results are new even for the one dimensional case of ordinary differential equations (where of course weak and strong convergence coincide), as shown e.g. by considering bounded solutions to the following ordinary differential equation:

$$\frac{3}{2}u'' + \frac{2}{t+1}u' = u^3, \quad u(0) = 1.$$

One can easily verify that the solution is given by  $u(t) = \frac{1}{t+1}$ , which converges to zero as  $t \rightarrow +\infty$ , as predicted by our Theorem 2.4.

In the following, we recall some notations and definitions that we need in the sequel.

Throughout the paper, we denote  $u_h(t) := u(t+h)$ ,  $\sigma_T(u) := \frac{1}{T} \int_0^T u(t)dt$ ,  $v(t, h) := u(t+h) - u(t)$ , and  $M := \text{Max}\{\|p\|_{W^{2,\infty}}, \|r\|_{W^{1,\infty}}, \sup_{t \geq 0} |u(t)|\}$ .

**Definition 1.1.** Given a bounded curve  $\{u(t)\}$  in  $H$ , the asymptotic center  $c$  of  $\{u(t)\}$  is defined as follows (see [13]): for every  $x \in H$ , let  $\varphi(x) = \limsup_{t \rightarrow +\infty} |u(t) - x|^2$ . Then  $\varphi$  is a continuous and strictly convex function on  $H$ , satisfying  $\varphi(x) \rightarrow +\infty$  as  $|x| \rightarrow +\infty$ . Thus  $\varphi$  achieves its minimum on  $H$  at a unique point  $c$ , called the asymptotic center of the curve  $\{u(t)\}$ .

## 2. Weak Convergence

In this section, we prove the almost weak convergence of solutions to (1), and then deduce the weak convergence of  $u$  to a zero of  $A$ , without assuming  $A^{-1}(0) \neq \emptyset$ . In the homogeneous case, the following theorem extends [7, Theorem 2.3], as well as [10, Lemma 3.4] and [8–9, Theorem 3.1]. We start with a lemma.

**Lemma 2.1.** *Suppose that  $u(t)$  is a solution to (1). If  $q \in H$  satisfies the following inequality:*

$$(p(t)u''(t) + r(t)u'(t), u(t) - q) \geq 0,$$

*then  $|u(t) - q|^2$  is nonincreasing or eventually increasing. Therefore, there exists  $\lim_{t \rightarrow +\infty} |u(t) - q|^2$ .*

**Proof.** It follows from the assumption that

$$p(t) \frac{d^2}{dt^2} |u(t) - q|^2 + r(t) \frac{d}{dt} |u(t) - q|^2 \geq 0$$

Dividing both sides of the above inequality by  $p(t)$  and multiplying by  $e^{\int_0^t \frac{r(s)}{p(s)} ds}$ , we get:

$$(5) \quad \frac{d}{dt} \left( e^{\int_0^t \frac{r(s)}{p(s)} ds} \frac{d}{dt} |u(t) - q|^2 \right) \geq 0.$$

We consider two cases.

If  $\frac{d}{dt} |u(t) - q|^2 \leq 0$  for each  $t > 0$ , then  $|u(t) - q|^2$  is nonincreasing. Otherwise, there exists  $t_0 > 0$  such that  $\frac{d}{dt} |u(t) - q|^2|_{t=t_0} > 0$ . Integrating (5) from  $t_0$  to  $t$ , we get for each  $t \geq t_0$ ,

$$e^{\int_0^t \frac{r(s)}{p(s)} ds} \frac{d}{dt} |u(t) - q|^2 \geq 2e^{\int_0^{t_0} \frac{r(s)}{p(s)} ds} (u'(t_0), u(t_0) - q) > 0$$

Then

$$\frac{d}{dt}|u(t) - q|^2 \geq 0, \forall t \geq t_0,$$

which shows that  $|u(t) - q|$  is eventually increasing.  $\square$

**Corollary 2.2.** *If  $A^{-1}(0) \neq \emptyset$  and  $q \in A^{-1}(0)$ , then the conclusions of Lemma 2.1 hold.*

**Proof.** If  $q \in A^{-1}(0)$ , then by the monotonicity of  $A$ ,  $q$  satisfies the inequality in Lemma 2.1.  $\square$

**Theorem 2.3.** *Let  $u$  be a solution to (1). Assume that  $p, r$  satisfy the assumptions (2) and (3). If either one of the following conditions hold:*

- i)  $r$  and  $p'$  are monotone,
- ii)  $r'(t) \geq p''(t)$ ,
- iii)  $p''(t) \geq r'(t)$ ,

*then  $\sigma_T(u_h) := \frac{1}{T} \int_0^T u(t+h)dt \rightarrow c \in A^{-1}(0)$  as  $T \rightarrow +\infty$ , uniformly for  $h \geq 0$ .*

*Moreover,  $c$  is the asymptotic center of the curve  $(u(t))_{t \geq 0}$ .*

**Proof.** By the monotonicity of  $A$ , we have

$$(p(t)u''(t) + r(t)u'(t), u(t) - u(s+h)) \geq (p(s+h)u''(s+h) + r(s+h)u'(s+h), u(t) - u(s+h))$$

Integrating from  $s = 0$  to  $s = T$  and dividing by  $T$ , we get:

$$\begin{aligned} (p(t)u''(t) + r(t)u'(t), u(t) - \sigma_T(u_h)) &\geq \frac{1}{T} \int_0^T p(s+h) \frac{d}{ds}(u'(s+h), u(t) - u(s+h)) ds \\ &- \frac{1}{T} \int_0^T \frac{r(s+h)}{2} \frac{d}{ds}|u(t) - u(s+h)|^2 ds = \frac{1}{T} p(T+h)(u(t) - u(T+h), u'(T+h)) \\ &- \frac{1}{T} p(h)(u'(h), u(t) - u(h)) + \frac{p'(T+h)}{2T}|u(t) - u(T+h)|^2 - \frac{1}{2T} p'(h)|u(t) - u(h)|^2 \\ &- \frac{1}{2T} \int_0^T p''(s+h)|u(t) - u(s+h)|^2 ds - \frac{r(T+h)}{2T}|u(t) - u(T+h)|^2 \end{aligned} \tag{6}$$

$$+ \frac{r(h)}{2T}|u(t) - u(h)|^2 + \frac{1}{2T} \int_0^T r'(s+h)|u(t) - u(s+h)|^2 ds$$

Obviously, by (2) we have:

$$\frac{p'(T+h)}{2T}|u(t) - u(T+h)|^2 \rightarrow 0 \tag{7}$$

$$\frac{-r(T+h)}{2T}|u(t) - u(T+h)|^2 \rightarrow 0 \tag{8}$$

as  $T \rightarrow +\infty$ . Also, clearly the second, fourth and seventh terms on the right hand side tend to zero. The first term on the right hand side converges to zero because:

$$\begin{aligned}
& \frac{p(T+h)}{T} |(u(t) - u(T+h), u'(T+h))| \leq \frac{2Mp(T+h)}{T} |u'(h)| \\
& \quad + \frac{2Mp(T+h)}{T} |u'(T+h) - u'(h)| \\
& = \frac{2Mp(T+h)}{T} |u'(h)| + \frac{2Mp(T+h)}{T} \left| \int_0^T u''(s+h) ds \right| \\
& \leq \frac{2Mp(T+h)}{T} |u'(h)| + \frac{2Mp(T+h)}{\sqrt{T}} \|u''\|_{L^2((0,+\infty);H)} \\
(9) \quad & \leq \frac{2Mp(T+h)}{T} |u'(0)| + \frac{4Mp(T+h)}{\sqrt{T}} \|u''\|_{L^2((0,+\infty);H)} \xrightarrow{T \rightarrow +\infty} 0,
\end{aligned}$$

uniformly for  $h \geq 0$ . Now assume that (i) holds. We will show that the fifth and eighth terms on the right hand side are bigger than expressions tending to zero as  $T \rightarrow +\infty$ . If  $p''(t) \leq 0$ , then:

$$(10) \quad -\frac{1}{2T} \int_0^T p''(s+h) |u(t) - u(s+h)|^2 ds \geq 0.$$

Otherwise,  $p''(t) \geq 0$  and by (2) we have:

$$(11) \quad -\frac{1}{2T} \int_0^T p''(s+h) |u(t) - u(s+h)|^2 ds \geq 4M^2 \frac{1}{2T} (p'(h) - p'(T+h)) \rightarrow 0$$

as  $T \rightarrow +\infty$ , uniformly for  $h \geq 0$ . If  $r'(t) \geq 0$ , then

$$(12) \quad \frac{1}{2T} \int_0^T r'(s+h) |u(t) - u(s+h)|^2 ds \geq 0.$$

Otherwise,  $r'(t) \leq 0$  and we get:

$$(13) \quad \frac{1}{2T} \int_0^T r'(s+h) |u(t) - u(s+h)|^2 ds \geq 4M^2 \frac{1}{2T} (r(T+h) - r(h)) \rightarrow 0$$

as  $T \rightarrow +\infty$ , uniformly for  $h \geq 0$ . Suppose  $q$  is a weak cluster point of  $\sigma_T(u_h)$ . Then for any two sequences  $T_n$  and  $h_n$  of positive real numbers such that  $T_n \rightarrow +\infty$  and  $\sigma_{T_n}(u_{h_n}) \rightharpoonup q$ , by replacing  $T$  by  $T_n$  and  $h$  by  $h_n$  in (6)–(13), and letting  $n \rightarrow +\infty$ , we get:

$$(14) \quad (p(t)u''(t) + r(t)u'(t), u(t) - q) \geq 0$$

If  $p(t)$  and  $r(t)$  satisfy condition (ii), then we get again (14) from (6)–(9). If  $p(t)$  and  $r(t)$  satisfy condition (iii), then we have

$$\begin{aligned}
& -\frac{1}{2T} \int_0^T p''(s+h)|u(t) - u(s+h)|^2 ds + \frac{1}{2T} \int_0^T r'(s+h)|u(t) - u(s+h)|^2 ds \\
& = \frac{1}{2T} \int_0^T (r'(s+h) - p''(s+h))|u(t) - u(s+h)|^2 ds \\
& \geq 4M^2 \frac{1}{2T} \int_0^T (r'(s+h) - p''(s+h)) ds \\
& = \frac{4M^2}{2T} (r(T+h) - r(h) - p'(T+h) + p'(h)),
\end{aligned}$$

which converges to zero as  $T \rightarrow +\infty$ . Then we get again (14) from (6)–(9). Now by Lemma 2.1, there exists  $\lim_{t \rightarrow +\infty} |u(t) - q|$ . If  $c$  is another weak cluster point of  $\sigma_T(u_h)$ , then there exists  $\lim_{t \rightarrow +\infty} (|u(t) - q|^2 - |u(t) - c|^2)$ . This implies that there exists  $\lim_{t \rightarrow +\infty} (u(t), c - q)$ , then there exists  $\lim_{T \rightarrow +\infty} (\sigma_T(u_h), c - q)$  uniformly for  $h \geq 0$ . It follows that  $(c, c - q) = (q, c - q)$ . Therefore  $c = q$ , and hence  $\sigma_T(u_h) \rightarrow c$  as  $T \rightarrow +\infty$ , uniformly for  $h \geq 0$ , which shows the almost weak convergence of  $u(t)$  to  $c$  as  $t \rightarrow +\infty$ . Now we prove that  $c \in A^{-1}(0)$ . Let  $y \in Ax$ . By the monotonicity of  $A$ , we have:

$$(x - u(t), y) = (x - u(t), y - Au(t)) + (x - u(t), Au(t)) \geq (x - u(t), p(t)u''(t) + r(t)u'(t))$$

Integrating from  $t = 0$  to  $T$ , dividing by  $T$ , and letting  $T \rightarrow +\infty$ , by a similar proof as above we get:  $(x - c, y) \geq 0$ . Now the maximality of  $A$  implies that  $c \in A^{-1}(0)$ . Finally, we show that  $c$  is the asymptotic center of the curve  $u(t)$ . Let  $x \in H$ , with  $x \neq c$ . Then:

$$|u(t) - x|^2 = |u(t) - c|^2 + |x - c|^2 + 2(u(t) - c, c - x).$$

Integrating from 0 to  $T$ , and dividing by  $T$ , then taking limsup when  $T \rightarrow +\infty$ , since  $\sigma_T \rightarrow c$ , we get:

$$\limsup_{t \rightarrow +\infty} |u(t) - x|^2 \geq \limsup_{t \rightarrow +\infty} |u(t) - c|^2 + |x - c|^2 > \limsup_{t \rightarrow +\infty} |u(t) - c|^2.$$

Hence  $c$  is the asymptotic center of  $u(t)$  as desired. The proof is now complete.  $\square$

**Theorem 2.4.** *Let  $u$  be a solution to (1). If (2), (3) and the assumptions of Theorem 2.3 are satisfied, then  $u(t) \rightarrow c \in A^{-1}(0)$  as  $t \rightarrow +\infty$ , where  $c$  is the asymptotic center of the curve  $(u(t))_{t \geq 0}$ .*

**Proof.** Since  $u' \in L^2((0, +\infty); H)$ , then  $u$  is asymptotically regular (i.e.  $u(t+h) - u(t) \rightarrow 0$ , as  $t \rightarrow +\infty$ ,  $\forall h \geq 0$ ). Now the result follows from Theorem 2.3 and G. G. Lorentz' Tauberian condition for almost convergence (see [14]).  $\square$

**Remark 2.1.** The conclusions of Theorems 2.3 and 2.4 still hold if the assumptions (i), (ii) or (iii) in Theorem 2.3 are satisfied only for large enough  $t$  (i.e. for  $t \geq t_0$ ).

**Remark 2.2.** Since  $u'$  and  $u''$  belong to  $L^2((0, +\infty); H)$ , by using the demiclosedness of  $A$ , and an argument similar to [19, Lemma 2.1], it can be easily shown that the zero set of  $A$  is nonempty. However, the weak convergence of  $u(t)$  cannot be directly derived from this fact, and we need the arguments in Theorems 2.3 and 2.4.

**Example 2.1.** Consider the following second order evolution equation:

$$(15) \quad \begin{cases} (q(t)u'(t))' \in Au(t) \text{ a.e. on } \mathbb{R}^+ \\ u(0) = u_0, \quad \sup_{t \geq 0} |u(t)| < +\infty \end{cases}$$

where  $A$  is a maximal monotone operator and  $q \in W^{2,\infty}((0, +\infty); H)$ . Then (2) and condition (ii) of Theorem 2.3 are satisfied. (3) is satisfied if  $q(t) \geq \alpha > 0, \forall t \geq 0$ . Then  $u(t)$  converges weakly to a zero of  $A$  as  $t \rightarrow +\infty$ . In this case, condition (4) is equivalent to the following condition:

$$(16) \quad \int_0^\infty \frac{1}{q(t)} = +\infty.$$

For example,  $q(t) = \sin t + 2$  satisfies all of the above conditions.

### 3. Strong Convergence

In this section, we extend our results in [12], and prove the strong convergence of solutions to (1) to a zero of the maximal monotone operator  $A$ , with weaker assumptions on the coefficients. Throughout this section, we assume (2) and (3).

**Lemma 3.1.** *Let  $u$  be a solution to (1). Then  $\liminf_{t \rightarrow +\infty} p(t) \frac{d}{dt} |v(t, h)|^2 \leq 0$  and  $\limsup_{t \rightarrow +\infty} r(t) |v(t, h)|^2 \leq 0$ , where  $v(t, h) = u(t + h) - u(t)$ .*

**Proof.** Assume by contradiction that  $\liminf_{t \rightarrow +\infty} p(t) \frac{d}{dt} |v(t, h)|^2 \geq c > 0$ . Since  $0 < \alpha \leq p(t) \leq M$ , integrating from  $t = s$  to  $T$ , we get:

$$|v(T, h)|^2 - |v(s, h)|^2 > \frac{c}{M}(T - s).$$

Letting  $T \rightarrow +\infty$ , we get a contradiction since  $u$  is bounded. On the other hand,

$$|r(t) |v(t, h)|^2 = |r(t)| \left| \int_t^{t+h} u'(s) ds \right|^2 \leq Mh \left( \int_t^{t+h} |u'(s)|^2 ds \right) \rightarrow 0$$

as  $t \rightarrow +\infty$ , since  $u' \in H^1((0, +\infty); H)$ . □

**Theorem 3.1.** *Let  $u(t)$  be a solution to (1). Assume that the following conditions (i), (ii) and (iii) are satisfied:*

- (i)  $\int_0^\infty e^{-\int_0^s \frac{r(\tau)}{2p(\tau)} d\tau} ds < +\infty$ ,
- (ii)  $M_1 := \int_0^\infty \left[ \int_0^t e^{-\int_0^s \frac{r(\tau)}{p(\tau)} d\tau} R(s) ds \right]^{1/2} dt < +\infty$

where  $R(s) := \text{Max}\{0, \sup_{t \geq s} r'(t)\}$ . Then  $u(t) \rightarrow p \in H$ , as  $t \rightarrow +\infty$ .

**Proof.** Take  $v(t) = u(t+h) - u(t)$ . By the monotonicity of  $A$  and (1), we get:

$$\begin{aligned} & p(t)(v''(t), v(t)) + (p(t+h) - p(t))(u''(t+h), v(t)) \\ & \quad + r(t)(v'(t), v(t)) + (r(t+h) - r(t))(u'(t+h), v(t)) \geq 0 \\ \Rightarrow & \frac{1}{2}p(t)\frac{d^2}{dt^2}|v(t)|^2 + (p(t+h) - p(t))(u''(t+h), v(t)) \\ & \quad + \frac{1}{2}r(t)\frac{d}{dt}|v(t)|^2 + (r(t+h) - r(t))(u'(t+h), v(t)) \geq 0 \end{aligned}$$

Integrating by parts from  $s$  to  $t$ , we get:

$$\begin{aligned} & \frac{1}{2}p(t)\frac{d}{dt}|v(t)|^2 - \frac{1}{2}p(s)\frac{d}{ds}|v(s)|^2 \\ & \quad - \frac{1}{2}\int_s^t p'(\tau)\frac{d}{d\tau}|v(\tau)|^2 d\tau + \int_s^t (p(\tau+h) - p(\tau))(u''(\tau+h), v(\tau))d\tau \\ & \quad + \frac{1}{2}r(t)|v(t)|^2 - \frac{1}{2}r(s)|v(s)|^2 \\ & \quad - \frac{1}{2}\int_s^t r'(\tau)|v(\tau)|^2 d\tau + \int_s^t (r(\tau+h) - r(\tau))(u'(\tau+h), v(\tau))d\tau \geq 0 \end{aligned}$$

By Lemma 3.1, there is a sequence  $t_n \rightarrow +\infty$  such that  $\lim_{n \rightarrow +\infty} p(t_n)(v(t_n), v'(t_n)) \leq 0$ . Now replacing  $t$  by  $t_n$  in the above inequality, and letting  $n \rightarrow +\infty$ , we get:

$$\begin{aligned} & \frac{1}{2}p(s)\frac{d}{ds}|v(s)|^2 + \frac{1}{2}r(s)|v(s)|^2 \leq \\ & \quad - \frac{1}{2}\int_s^\infty p'(\tau)\frac{d}{d\tau}|v(\tau)|^2 d\tau + \int_s^\infty (p(\tau+h) - p(\tau))(u''(\tau+h), v(\tau))d\tau \\ & \quad - \frac{1}{2}\int_s^\infty r'(\tau)|v(\tau)|^2 d\tau + \int_s^\infty (r(\tau+h) - r(\tau))(u'(\tau+h), v(\tau))d\tau \end{aligned}$$

Dividing by  $h^2$  and letting  $h \rightarrow 0$ , by an application of Fatou's Lemma, it follows from the assumptions that:

$$(17) \quad p(s)\frac{d}{ds}|u'(s)|^2 + r(s)|u'(s)|^2 \leq \int_s^\infty r'(\tau)|u'(\tau)|^2 d\tau \leq R(s) \int_s^\infty |u'(\tau)|^2 d\tau.$$

Then:

$$\frac{d}{ds}|u'(s)|^2 e^{\int_{t_0}^s \frac{r(\tau)}{p(\tau)} d\tau} + \frac{r(s)}{p(s)}|u'(s)|^2 e^{\int_{t_0}^s \frac{r(\tau)}{p(\tau)} d\tau} \leq \frac{e^{\int_{t_0}^s \frac{r(\tau)}{p(\tau)} d\tau}}{p(s)} R(s) \int_s^\infty |u'(\tau)|^2 d\tau.$$

Therefore:

$$\frac{d}{ds}(|u'(s)|^2 e^{\int_{t_0}^s \frac{r(\tau)}{p(\tau)} d\tau}) \leq \frac{e^{\int_{t_0}^s \frac{r(\tau)}{p(\tau)} d\tau}}{p(s)} R(s) \int_s^\infty |u'(\tau)|^2 d\tau.$$



Integrating from  $t_0$  to  $t$  with respect to  $s$ , we get:

$$\begin{aligned} |u'(t)|^2 &\leq |u'(t_0)|^2 e^{-\int_{t_0}^t \frac{r(\tau)}{p(\tau)} d\tau} + \frac{e^{-\int_{t_0}^t \frac{r(\tau)}{p(\tau)} d\tau}}{\alpha} \int_{t_0}^t \left( e^{\int_{t_0}^s \frac{r(\tau)}{p(\tau)} d\tau} R(s) \int_s^\infty |u'(\tau)|^2 d\tau \right) ds \\ &= |u'(t_0)|^2 e^{-\int_{t_0}^t \frac{r(\tau)}{p(\tau)} d\tau} + \frac{1}{\alpha} \int_{t_0}^t \left( e^{-\int_s^t \frac{r(\tau)}{p(\tau)} d\tau} R(s) \int_s^\infty |u'(\tau)|^2 d\tau \right) ds. \end{aligned}$$

Therefore:

$$(18) \quad |u'(t)| \leq |u'(t_0)| e^{-\int_{t_0}^t \frac{r(\tau)}{2p(\tau)} d\tau} + \frac{1}{\sqrt{\alpha}} \left[ \int_{t_0}^t \left( e^{-\int_s^t \frac{r(\tau)}{p(\tau)} d\tau} R(s) \int_s^\infty |u'(\tau)|^2 d\tau \right) ds \right]^{\frac{1}{2}}.$$

Hence:

$$\begin{aligned} |u(T') - u(T)| &\leq \int_T^{T'} |u'(t)| dt \leq |u'(t_0)| \int_T^{T'} e^{-\int_{t_0}^t \frac{r(\tau)}{2p(\tau)} d\tau} dt \\ &\quad + \frac{1}{\sqrt{\alpha}} \int_T^{T'} \left[ \int_{t_0}^t \left( e^{-\int_s^t \frac{r(\tau)}{p(\tau)} d\tau} R(s) \int_s^\infty |u'(\tau)|^2 d\tau \right) ds \right]^{\frac{1}{2}} dt. \end{aligned}$$

Assume  $u(T'_n) \rightarrow p$ . Then we get:

$$\begin{aligned} |u(T) - p| &\leq |u'(t_0)| \int_T^\infty e^{-\int_{t_0}^t \frac{r(\tau)}{2p(\tau)} d\tau} dt \\ &\quad + \frac{1}{\sqrt{\alpha}} \int_T^\infty \left[ \int_{t_0}^t \left( e^{-\int_s^t \frac{r(\tau)}{p(\tau)} d\tau} R(s) \int_s^\infty |u'(\tau)|^2 d\tau \right) ds \right]^{\frac{1}{2}} dt. \end{aligned}$$

Given  $\epsilon > 0$ , choose  $t_0$  big enough such that:  $\int_s^\infty |u'(\tau)|^2 d\tau \leq \epsilon^2$ ,  $\forall s \geq t_0$ . Then we have:

$$|u(t) - p| \leq |u'(t_0)| \int_t^\infty e^{-\int_{t_0}^s \frac{r(\tau)}{2p(\tau)} d\tau} ds + \frac{M_1 \epsilon}{\sqrt{\alpha}}.$$

Therefore  $\limsup_{t \rightarrow +\infty} |u(t) - p| \leq \frac{M_1 \epsilon}{\sqrt{\alpha}}$ . Since  $\epsilon > 0$  is arbitrary, we conclude that  $u(t) \rightarrow p$  as  $t \rightarrow +\infty$ .  $\square$

**Corollary 3.2.** *Let  $u(t)$  be a solution to (1). Assume that  $r(t) \geq 0$ . In addition, assume that (i) and the following stronger condition (ii)' is satisfied:*

$$(ii)' \quad M'_1 := \int_0^{+\infty} e^{\int_0^s \frac{r(\tau)}{p(\tau)} d\tau} R(s) ds < +\infty.$$

*Then  $u(t) \rightarrow p \in A^{-1}(0)$  as  $t \rightarrow +\infty$ .*

**Proof.** First of all, since:  $-\int_s^t \frac{r(\tau)}{p(\tau)} d\tau = -2 \int_0^t \frac{r(\tau)}{2p(\tau)} d\tau + \int_0^s \frac{r(\tau)}{p(\tau)} d\tau$ , it is clear that (i) and (ii)' imply (ii). Therefore by Theorem 3.1,  $u(t) \rightarrow p \in H$  as  $t \rightarrow +\infty$ . Now since  $r(t) \geq 0$ , it follows from (i), (ii)' and (18) that  $|u'(t)| \rightarrow 0$  as  $t \rightarrow +\infty$ . Since  $u'' \in L^2((0, +\infty); H)$ , there is a sequence  $t_n \rightarrow +\infty$  such that  $u''(t_n) \rightarrow 0$  as  $n \rightarrow +\infty$ . Since  $p(t)$  and  $r(t)$  are bounded, the closedness of  $A$  implies that  $p \in A^{-1}(0)$ .  $\square$

**Corollary 3.3** (12, Theorem 2.1). *Let  $u(t)$  be a solution to (1). Assume that  $r'(t) \leq 0$ . If (i) holds, then  $u(t) \rightarrow p \in A^{-1}(0)$  as  $t \rightarrow +\infty$ .*

**Proof.** Since  $r'(t) \leq 0$ , then (i) implies that  $r(t) \geq 0$ , and  $R(s) = 0$  so that (ii)' is clearly satisfied. The conclusion follows now from Corollary 3.2.  $\square$

**Example 3.1.** Let  $p(t) \equiv 1$ ,  $r(t) = \frac{4}{t+1}$ . Then the assumptions of Corollary 3.3 are satisfied; therefore  $u(t) \rightarrow p \in A^{-1}(0)$  as  $t \rightarrow +\infty$ . Moreover, the proof of Theorem 3.1 shows that we have the following rate of convergence:  $|u(t) - p| = O(\frac{1}{t+1})$ .

**Example 3.2.** Let  $p(t) \equiv 1$  and  $r(t) = 1 - e^{-t} \geq 0$ . Then  $r'(t) = e^{-t} > 0$  and  $r'(t) \rightarrow 0$  as  $t \rightarrow +\infty$ . In this case, (i) and (ii) are satisfied, but (ii)' does not hold. Therefore, it follows from Theorem 3.1 that  $u(t) \rightarrow p \in H$ , as  $t \rightarrow +\infty$ .

**Open Problem 3.1.** Say for  $p(t) \equiv 1$ , is it possible to get the strong convergence of  $u(t)$  by assuming that  $\limsup_{t \rightarrow +\infty} r'(t) \leq 0$ , and is there any relationship between this condition and condition (ii) in Theorem 3.1?

#### 4. Subdifferential Case

In this section, we consider the evolution equation (1) when the monotone operator  $A$  is the subdifferential  $\partial\varphi$  of a proper, convex and lower semicontinuous function  $\varphi : H \rightarrow ]-\infty, +\infty]$ . We prove a weak convergence theorem with suitable assumptions on  $p(t)$  and  $r(t)$ , as well as a strong convergence theorem with additional assumptions on  $\varphi$ , extending our results in [7].

**Proposition 4.1.** *Let  $u(t)$  be a solution to (1). Assume that (2) and (3) hold. If  $\int_0^\infty R(t)dt < +\infty$ , then  $\lim_{t \rightarrow +\infty} \varphi(u(t))$  exists.*

**Proof.** By Lemma 2.1, (1) and (17), we have:

$$\begin{aligned} \frac{d}{dt}\varphi(u(t)) &= (\partial\varphi(u(t)), u'(t)) = (p(t)u''(t) + r(t)u'(t), u'(t)) \\ &= \frac{1}{2}p(t)\frac{d}{dt}|u'(t)|^2 + r(t)|u'(t)|^2 \\ &\leq \frac{1}{2}\int_t^\infty r'(s)|u'(s)|^2 ds + \frac{1}{2}r(t)|u'(t)|^2 \\ &\leq \frac{1}{2}R(t)\int_t^\infty |u'(s)|^2 ds + \frac{1}{2}r(t)|u'(t)|^2. \end{aligned}$$

Therefore:

$$\begin{aligned} \varphi(u(T')) - \varphi(u(T)) &\leq \frac{1}{2}\int_T^{T'} R(t)\int_t^\infty |u'(s)|^2 ds dt \\ &\quad + \frac{1}{2}\int_T^{T'} r(t)|u'(t)|^2 dt \\ &\leq \frac{1}{2}\left(\int_T^\infty |u'(t)|^2 dt\right)\left[\int_T^{T'} R(t)dt + M\right] \end{aligned}$$

This implies that:  $\limsup_{t \rightarrow +\infty} \varphi(u(t)) \leq \varphi(u(T)) + C \int_T^\infty |u'(s)|^2 ds$ , for some constant  $C$ . Now since  $u' \in L^2((0, +\infty); H)$ , letting  $T \rightarrow +\infty$ , we get:

$\limsup_{t \rightarrow +\infty} \varphi(u(t)) \leq \liminf_{t \rightarrow +\infty} \varphi(u(t))$ , which completes the proof of the proposition.  $\square$

**Remark 4.1.** In Proposition 4.1, if  $r(t) \leq 0$  and  $r'(t) \leq 0$ , then  $\varphi(u(t))$  is nonincreasing.

**Lemma 4.2.** *Suppose that  $u(t)$  is a solution to (1), and  $q \in H$ . Then*  
 $\liminf_{t \rightarrow +\infty} t \frac{d}{dt} |u(t) - q|^2 \leq 0$ .

**Proof.** Assume by contradiction that  $\liminf_{t \rightarrow +\infty} t \frac{d}{dt} |u(t) - q|^2 > 0$ . Then There exist  $t_0 > 0$  and  $c > 0$  such that for each  $t > t_0$ , we have  $t \frac{d}{dt} |u(t) - q|^2 \geq c > 0$ . Dividing both sides by  $t$  and then integrating from  $t = t_0$  to  $T$ , we get

$$|u(T) - q|^2 - |u(t_0) - q|^2 \geq c(\ln T - \ln t_0).$$

Since  $u(t)$  is bounded, we get a contradiction by letting  $T \rightarrow +\infty$ .  $\square$

**Theorem 4.3.** *Let  $u(t)$  be a solution to (1). Suppose that the assumptions of Proposition 4.1 are satisfied. Then  $u(t)$  converges weakly to some  $c \in A^{-1}(0)$ , as  $t \rightarrow +\infty$ .*

**Proof.** By Remark 2.2, we know that  $A^{-1}(0) \neq \emptyset$ . Let  $q \in A^{-1}(0)$ . By the subdifferential inequality and (1), we get

$$\begin{aligned} \varphi(u(t)) - \varphi(q) &\leq (p(t)u''(t) + r(t)u'(t), u(t) - q) \\ (19) \quad &\leq \frac{1}{2}p(t) \frac{d^2}{dt^2} |u(t) - q|^2 + \frac{1}{2}r(t) \frac{d}{dt} |u(t) - q|^2 \end{aligned}$$

Integrating from  $t = 0$  to  $T$ , we get:

$$\begin{aligned} \int_0^T (\varphi(u(t)) - \varphi(q)) dt &\leq \frac{p(T)}{2} \frac{d}{dT} |u(T) - q|^2 - p(0)(u'(0), u(0) - q) \\ &\quad - \frac{1}{2} \int_0^T p'(t) \frac{d}{dt} |u(t) - q|^2 + \frac{1}{2} \int_0^T r(t) \frac{d}{dt} |u(t) - q|^2 dt \end{aligned}$$

By Lemma 2.1,  $|u(t) - q|$  is nonincreasing or eventually increasing. If  $|u(t) - q|$  is nonincreasing, then

$$(20) \quad \int_0^T (\varphi(u(t)) - \varphi(q)) dt \leq -p(0)(u'(0), u(0) - q) + M(|u(0) - q|^2 - |u(T) - q|^2).$$

If  $|u(t) - q|$  is eventually increasing, then there exists  $t_0$  such that for each  $t \geq t_0$ ,  $\frac{d}{dt} |u(t) - q| > 0$ . Integrating (19) from  $t = t_0$  to  $T$ , we get:

$$\begin{aligned} \int_{t_0}^T (\varphi(u(t)) - \varphi(q)) dt &\leq \frac{p(T)}{2} \frac{d}{dT} |u(T) - q|^2 - p(t_0)(u'(t_0), u(t_0) - q) \\ &\quad - \frac{1}{2} \int_{t_0}^T p'(t) \frac{d}{dt} |u(t) - q|^2 + \frac{1}{2} \int_{t_0}^T r(t) \frac{d}{dt} |u(t) - q|^2 dt \end{aligned}$$

Therefore:

$$(21) \quad \int_{t_0}^T (\varphi(u(t)) - \varphi(q))dt \leq \frac{M}{2} \frac{d}{dT} |u(T) - q|^2 - p(t_0)(u'(t_0), u(t_0) - q) \\ + M(|u(T) - q|^2 - |u(t_0) - q|^2).$$

Taking  $\liminf$  of (20) and (21) as  $T \rightarrow +\infty$ , by Lemma 4.2, we get

$$\int_{t_0}^{\infty} (\varphi(u(t)) - \varphi(q))dt < +\infty.$$

Therefore  $\liminf_{t \rightarrow +\infty} \varphi(u(t)) \leq \varphi(q)$ . By Proposition 4.1,  $\lim_{t \rightarrow +\infty} \varphi(u(t)) = \varphi(q) = \text{Min}\{\varphi(z); z \in H\}$ . If  $u(t_n) \rightarrow s$  as  $n \rightarrow +\infty$ , then

$$\varphi(s) \leq \liminf_{n \rightarrow +\infty} \varphi(u(t_n)) = \lim_{t \rightarrow +\infty} \varphi(u(t)) = \varphi(q).$$

Hence  $s \in A^{-1}(0)$ , and therefore by Corollary 2.2, there exists  $\lim_{t \rightarrow +\infty} |u(t) - s|$ . Now the proof is completed by a similar argument as in Theorem 2.3.  $\square$

**Theorem 4.4.** *Let  $u(t)$  be a solution to (1). Suppose (2) and (3) hold, and that  $r(t) \leq 0$  and  $r'(t) \leq 0$ . Let  $\varphi : H \rightarrow ]-\infty, +\infty]$  be a proper, convex and lower semicontinuous function satisfying the following conditions:  $D(\varphi) = -D(\varphi)$ , and  $\varphi(x) - \varphi(0) \geq a(|x|)(\varphi(-x) - \varphi(0))$ ,  $\forall x \in D(\varphi)$ , where  $a : \mathbb{R}^+ \rightarrow (0, 1)$  is a continuous function. Then  $u(t) \rightarrow q \in A^{-1}(0)$  as  $t \rightarrow +\infty$ , which is a minimum point of  $\varphi$ .*

**Proof.** By Remark 2.2, we know that  $A^{-1}(0) \neq \emptyset$ , and therefore  $\varphi$  has a minimum point. Without loss of generality, we may assume that  $\varphi(0) = 0$  and 0 is a minimum point of  $\varphi$ . For  $t \leq s$ , by the assumptions, Proposition 4.1 and Remark 4.1, we get:

$$\begin{aligned} \varphi(u(t)) &\geq \varphi(u(s)) \geq a(|u(s)|)\varphi(-u(s)) + (1 - a(|u(s)|))\varphi(0) \\ &\geq \varphi(-a(|u(s)|)u(s)) \geq \varphi(u(t)) + (\partial\varphi(u(t)), -a(|u(s)|)u(s) - u(t)) \\ &= \varphi(u(t)) - (p(t)u''(t) + r(t)u'(t), a(|u(s)|)u(s) + u(t)). \end{aligned}$$

Therefore:

$$(p(t)u''(t) + r(t)u'(t), a(|u(s)|)u(s) + u(t)) \geq 0.$$

Let:

$$g(t) = (1 + a(|u(s)|))(|u(t)|^2 - |u(s)|^2) - a(|u(s)|)|u(t) - u(s)|^2$$

then  $g'(t) = 2(u'(t), u(t) + a(|u(s)|)u(s))$  and  $g''(t) = 2(u''(t), u(t) + a(|u(s)|)u(s)) + 2|u'(t)|^2$ . Hence  $p(t)g''(t) + r(t)g'(t) \geq 0$ . Now the same argument as in Lemma 2.1, with  $|u(t) - q|^2$  replaced by  $g(t)$ , shows that  $g(t)$  is either nonincreasing or eventually increasing. Since  $r(t) \leq 0$ , condition (1.3) in [7] is satisfied. Therefore by [7, Lemma 2.2], we conclude that  $g(t)$  is nonincreasing. Then  $g(t) \geq g(s) = 0$  for  $t \leq s$ . It follows that:

$$|u(t) - u(s)|^2 \leq \frac{1 + a(|u(s)|)}{a(|u(s)|)}(|u(t)|^2 - |u(s)|^2) < \frac{2}{a(|u(s)|)}(|u(t)|^2 - |u(s)|^2), \quad \forall s \geq t$$

By Corollary 2.2, there exists  $\lim_{s \rightarrow +\infty} |u(s)|$ . If  $|u(s)| \rightarrow 0$  as  $s \rightarrow +\infty$ , then  $u(s) \rightarrow 0$  and this yields the theorem. Otherwise, if  $|u(s)| \rightarrow r > 0$ , from the continuity of  $a$ , we have  $\lim_{s \rightarrow +\infty} a(|u(s)|) = a(\lim_{s \rightarrow +\infty} |u(s)|) = a(r) > 0$ . Therefore  $\{u(t)\}$  is a cauchy sequence in  $H$ , hence  $u(t) \rightarrow q$  as  $t \rightarrow +\infty$ , and  $q \in A^{-1}(0)$  by Theorem 4.3.  $\square$

### 5. Applications

1. When  $A = \partial\varphi$ , where  $\varphi$  is a proper, convex and lower semicontinuous function satisfying the assumptions of Theorem 4.4, the solution to (1) converges strongly to a zero of the maximal monotone operator  $A$  which is a minimum point of  $\varphi$ .

2. Let  $H = L^2(\Omega)$ , where  $\Omega \subseteq \mathbb{R}^n$  is a bounded domain with smooth boundary  $\Gamma$ . Let  $j : \mathbb{R} \rightarrow (-\infty, +\infty]$  be a proper, convex and lower semicontinuous function, and  $\beta = \partial j$ . We assume for simplicity that  $0 \in \beta(0)$ . Define

$$Au = -\Delta u = -\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$$

with

$$D(A) = \left\{ u \in H^2(\Omega), \frac{-\partial u}{\partial \eta}(x) \in \beta(u(x)), \text{ a.e. on } \Gamma \right\}$$

where  $(\frac{\partial u}{\partial \eta}(x))$  is the outward normal derivative to  $\Gamma$  at  $x \in \Gamma$ . It is known that  $A = \partial\phi$ , where  $\phi : L^2(\Omega) \rightarrow (-\infty, +\infty]$  is the functional:

$$\phi(u) = \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \int_{\Gamma} \beta(u(x)) d\sigma, & \text{if } u \in H^1(\Omega) \text{ and } \beta(u) \in L^1(\Gamma) \\ +\infty, & \text{otherwise.} \end{cases}$$

Now consider the following equation:

$$\begin{cases} p(t) \frac{\partial^2 u}{\partial t^2}(t, x) + r(t) \frac{\partial u}{\partial t}(t, x) + \sum_i \frac{\partial^2 u}{\partial x_i^2}(t, x) = 0 & \text{a.e. on } \mathbb{R}^+ \times \Omega \\ -\frac{\partial u}{\partial \eta}(t, x) \in \beta u(t, x) & \text{a.e. on } \mathbb{R}^+ \times \Gamma \\ u(0, x) = u_0(x) & \text{a.e. on } \Omega \\ \sup_{t \geq 0} |u(t, x)|_{L^2(\Omega)} < +\infty \end{cases}$$

where  $p(t)$ , and  $r(t)$  satisfy the assumptions of Theorems 2.4 or 4.3. Then Theorems 2.4 and 4.3 imply the weak convergence of  $u(t, \cdot)$  to a minimizer of  $\phi$ .

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