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EXISTENCE RESULTS FOR BOUNDARY VALUE PROBLEMS FOR MULTIVALUED FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We study the existence of solutions for a class of boundary value problems for fractional differential inclusions, where the right hand side is a convex or a non-convex multi-valued map. Suitable fixed point theorems are used to prove some new existence results. Examples illustrating the abstract results are also presented.

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1. INTRODUCTION

Recently, there has been a great interest to differential equations and inclusions with non-integer order, since fractional order models are more accurate than integer models. Fractional derivatives provide an excellent instrument for the description of systems with memory and hereditary properties. Many books and monographs are devoted to the development of fractional calculus, see for instance [15, 17, 19, 21, 23, 24] and references therein. For application of fractional calculus to the other fields of science, we refer the reader to [13, 17, 19].

In [12] the authors consider the following boundary value problem:

$$D^{\delta}y(t) - \lambda y(t) = f(t, y(t)), \quad t \in J := [0, 1], \ 1 < \delta < 2,$$

$$y(0) = y(1) = 0,$$

where D^{δ} is the standard Riemann-Liouville fractional derivative, f is continuous and $\lambda \in \mathbb{R}$. In this paper we continue the study in [12] to cover the multi-valued case. More precisely, we shall be concerned with existence of solutions for the following boundary value problem for nonlinear fractional differential inclusion of the form:

$$D^{\delta}y(t) - \lambda y(t) \in F(t, y(t)), \quad t \in J := [0, 1], \ 1 < \delta < 2, \tag{1.1}$$

$$y(0) = y(1) = 0. \tag{1.2}$$

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Here D^{δ} is the standard Riemann-Liouville fractional derivative, and $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), and $\lambda \in \mathbb{R}$ with $\lambda \neq 0$. For more details about multi-valued analysis, see [5, 8, 10, 14]. Differential equations and inclusions with various boundary conditions are widely investigated by many authors, see for instance the papers [1, 2, 6, 22].

This paper is organized as follows. In section 2 we recall some definitions and preliminary results about multi-valued analysis and fractional calculus witch will be used in the sequel. Section 3 is devoted to our main existence results. Illustrative examples are given in the last section.

2. PRELIMINARIES

In this section, we introduce notations and preliminary facts that are used throughout this paper.

2.1. **Multivalued Analysis.** Let $C(J)$ denotes the Banach space of continuous functions from J into $\mathbb{R}$ with the norm

$$\|f\| = \sup\{|f(t)| : t \in J\}.$$

By $L^1(J, \mathbb{R})$ denotes the Banach space of functions $y : J \longrightarrow \mathbb{R}$ which are Lebesgue integrable normed by

$$\|y\|_{L^1} = \int_0^1 |y(t)|dt.$$

Let $(X, |\cdot|)$ be a Banach space. A multi-valued map $F : X \to \mathcal{P}(X)$:

(i) is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$;
(ii) is bounded on bounded sets if $F(B) = \bigcup_{x \in B} F(x)$ is bounded in X for all bounded set B of X i.e. $\sup_{x \in B}\{\sup\{|y| : y \in F(x)\}\} < \infty$;
(iii) is called upper semi-continuous (u.s.c. for short) on X if for each $x_0 \in X$ the set $F(x_0)$ is nonempty, closed subset of X, and for each open set $\mathcal{U}$ of X containing $F(x_0)$, there exists an open neighborhood $\mathcal{V}$ of x_0 such that $F(\mathcal{V}) \subseteq \mathcal{U}$;
(iv) is said to be completely continuous if $F(B)$ is relatively compact for every bounded subset B of X;
(v) has a fixed point if there exists $x \in X$ such that $x \in F(x)$.

For each $y \in C(J)$ the set

$$S_{F,y} = \{f \in L^1(J, \mathbb{R}) : f(t) \in F(t, y) \text{ for a.e. } \ t \in J\}$$

is known as the set of selections of the multi-valued map F.

In the following by $\mathcal{P}_p$ we denote the set of all nonempty subsets of X which have the property "p", where "p" will be bounded (b), closed (cl), concex (c), compact (cp) etc. Thus $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) :$

Y is bounded}, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y$ is compact}, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) :$ Y is compact and convex}.

We define the graph of G to be the set $Gr(G) = \{(x, y) \in X \times Y, y \in G(x)\}$ and recall two useful results regarding closed graphs and upper-semicontinuity.

Lemma 2.1 ([8, Proposition 1.2]). *If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 2.2 ([18]). *Let X be a separable Banach space. Let $F : J \times X \to \mathcal{P}_{cp,c}(X)$ be measurable with respect to t for each $x \in X$ and u.s.c. with respect to x for almost all $t \in J$ and $S_{F,x} \neq \emptyset$ for any $x \in C(J, X)$, and let Θ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator*

$$\Theta \circ S_F : C(J, X) \to \mathcal{P}_{cp,c}(C(J, X)),\ x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x,y})$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

For more details on multi-valued maps and the proof of the known results cited in this section we refer interested reader to the books of Deimling [8], Gorniewicz [10] and Hu and Papageorgiou [14].

2.2. **Fractional Calculus.** Now, we recall the definitions of the Riemann-Liouville fractional primitive and derivative.

Definition 2.3 ([23, 24]). The Riemann-Liouville fractional integral of order $s > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I_0^s f(t) = \frac{1}{\Gamma(s)} \int_0^t (t-\tau)^{s-1} f(\tau) d\tau,$$

where Γ is the gamma function.

Definition 2.4 ([23, 24]). The Riemann-Liouville fractional derivative of order $s > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is given by

$$\begin{aligned} D^s f(t) &= \frac{1}{\Gamma(1-s)} \frac{d}{dt} \int_0^t (t-\tau)^{-s} f(\tau) d\tau \\ &= \frac{d}{dt} I_0^{1-s} f(t). \end{aligned}$$

The following lemma is of great importance in the proof of our main results.

Lemma 2.5 ([12]). *The boundary value problem*

$$D^\delta u(t) - \lambda u(t) = f(t), \quad t \in J := J,\ 1 < \delta < 2,$$

$$u(0) = u(1) = 0,$$

has a unique solution $u \in C(J)$ *given by*

$$u(t) = \int_0^1 G_{\lambda,\delta}(t,s)f(s)ds,$$

where

$$G_{\lambda,\delta}(t,s) = \begin{cases} -\dfrac{e_\delta^{\lambda t} e_\delta^{\lambda(1-s)}}{e_\delta^{\lambda}} + e_\delta^{\lambda(t-s)}, & 0 \le s \le t \le 1; \\ -\dfrac{e_\delta^{\lambda t} e_\delta^{\lambda(1-s)}}{e_\delta^{\lambda}}, & 0 \le t < s \le 1. \end{cases}$$

Here e_δ denotes the δ-exponential function given by:

$$e_\delta^{\lambda t} := t^{\delta-1}E_{\delta,\delta}(\lambda t^\delta),\ t \in \mathbb{C} \setminus \{0\},\ \mathfrak{Re}(\delta) > 0,\ \lambda \in \mathbb{C},$$

and $E_{\delta,\beta}$ is the general Mittag-Leffler function defined by:

$$E_{\delta,\beta}(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\delta j + \beta)},\ t, \beta \in \mathbb{C},\ \mathfrak{Re}(\delta) > 0.$$

For more details about the δ-exponential function, see [15].

3. EXISTENCE RESULTS

Before stating and proving our main existence results for problem (1.1)–(1.2), we will give the definition of its solution.

Definition 3.1. A function $y \in AC^1(J)$ is said to be a solution of the problem (1.1)–(1.2) if there exists a function $f \in L^1(J,\mathbb{R})$ with $f(t) \in F(t,y)$ a.e. on J such that y satisfies the differential equation $D^\delta y(t) - \lambda y(t) = f(t)$ on J and the boundary condition (1.2).

3.1. The Upper Semicontinuous case.

Consider first the case when F has convex values. Our first result is based on Bohnenblust-Karlin fixed point theorem.

Lemma 3.2 (Bohnenblust-Karlin [3])**.** *Let X be a Banach space, D a nonempty subset of X, witch is bounded, closed and convex. Suppose $G : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is u.s.c. with closed, convex values, and $G(D) \subset D$ and $\overline{G(D)}$ is compact. Then G has a fixed point.*

Theorem 3.3. *Assume that:*

(H_1) $F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is Carathéodory i.e.*
- (i) $t \longmapsto F(t,y)$ *is measurable for each* $y \in \mathbb{R}$*;*
- (ii) $y \longmapsto F(t,y)$ *is u.s.c. for almost all* $t \in J$*;*

(H_2) *for each* $r > 0$, *there exists* $\varphi_r \in L^1(J, \mathbb{R}^+)$ *such that*

$$\|F(t,y)\| = \sup\{|v| : v \in F(t,y)\} \leq \varphi_r(t)$$

for all $\|y\| \leq r$ *and for a.e.* $t \in J$ *and*

$$\liminf_{r\to\infty} \frac{1}{r}\int_0^1 \varphi_r(t)dt = \gamma. \tag{3.1}$$

Then the boundary problem (1.1)–(1.2) *has at least one solution on* J *provided that:*

$$\left[\frac{e_\delta^{|\lambda|}}{|e_\delta^{\lambda}|} + 1\right] e_\delta^{|\lambda|}\gamma < 1. \tag{3.2}$$

Proof. We transform the problem (1.1)–(1.2) into a fixed point problem. Consider the multi-valued map: $N : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ defined by

$$N(y) = \left\{ h \in C(J,\mathbb{R}) : h(t) = \int_0^1 G_{\lambda,\delta}(t,s)f(s)ds, f \in S_{F,y} \right\}.$$

It is clear that fixed points of N are solutions of problem (1.1)–(1.2). It turns to show that the operator N satisfies all condition of Lemma 3.2. The proof is given in several steps.

Step 1. $N(y)$ *is convex for each* $y \in C(J,\mathbb{R})$.

Indeed, if h_1, h_2 belongs to $N(y)$, then there exist $f_1, f_2 \in S_{F,y}$ such that for each $t \in J$, we have

$$h_i(t) = \int_0^1 G_{\lambda,\delta}(t,s)f_i(s)ds, \ i = 1, 2.$$

Let $0 \leq \mu \leq 1$. Then for each $t \in J$, we have

$$[\mu h_1 + (1-\mu)h_2](t) = \int_0^1 G_{\lambda,\delta}(t,s)[\mu f_1(s) + (1-\mu)f_2(s)]ds.$$

Since F has convex values, $S_{F,y}$ is convex, then

$$\mu h_1 + (1-\mu)h_2 \in N(y).$$

Step 2. $N(y)$ *maps bounded sets (balls) into bounded sets in* $C(J,\mathbb{R})$.

For a positive number r let $B_r = \{y \in C(J,\mathbb{R}) : \|y\| \leq r\}$ be a bounded ball in $C(J,\mathbb{R})$. We shall prove that there exists a positive number r' such that $N(B_{r'}) \subseteq B_{r'}$. If not, for each positive number r, there exists a function $y_r(\cdot) \in B_r$, $\|N(y_r)\| > r$ for some $t \in J$ and

$$h_r(t) = \int_0^1 G_{\lambda,\delta}(t,s)f_r(s)ds$$

for some $f_r \in S_{F,y_r}$. However, on the other hand, we have:

$$\begin{aligned} r &< \|N(y_r)\| \\ &\leq \frac{|e_\delta^{\lambda t}|}{|e_\delta^\lambda|}\int_0^1 |e_\delta^{\lambda(1-s)}||f_r(s)|ds + \int_0^t |e_\delta^{\lambda(t-s)}||f_r(s)|ds \\ &\leq \left[\frac{e_\delta^{|\lambda|}}{|e_\delta^\lambda|}+1\right] e_\delta^{|\lambda|}\int_0^1 \varphi(s)ds. \end{aligned}$$

Dividing both sides by r and take the lower limit as $r \to \infty$, we get:

$$\left[\frac{e_\delta^{|\lambda|}}{|e_\delta^\lambda|}+1\right] e_\delta^{|\lambda|}\gamma \geq 1,$$

which contradicts (3.2). Hence there exists a positive number r such that $N(B_r) \subseteq B_r$.

Step 3. *$N(y)$ maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.*

Let y be any element in B_r and $h \in N(y)$, then there exists a function $f \in S_{F,y}$ such that for each $t \in J$ we have

$$h(t) = \int_0^1 G_{\lambda,\delta}(t,s)f(s)ds.$$

Let $t_1,\ t_2 \in J,\ t_1 < t_2$. Thus

$$\begin{aligned} |h(t_2)-h(t_1)| &\leq \left| -\frac{e_\delta^{\lambda t_2}}{e_\delta^\lambda}\int_0^1 e_\delta^{\lambda(1-s)}f(s)ds - \int_0^{t_2} e_\delta^{\lambda(t_2-s)}f(s)ds \right. \\ &\quad \left. +\frac{e_\delta^{\lambda t_1}}{e_\delta^\lambda}\int_0^1 e_\delta^{\lambda(1-s)}f(s)ds + \int_0^{t_1} e_\delta^{\lambda(t_1-s)}f(s)ds \right| \\ &\leq \frac{|e_\delta^{\lambda t_1}-e_\delta^{\lambda t_2}|}{|e_\delta^\lambda|}\int_0^1 e_\delta^{|\lambda|(1-s)}\varphi_r(s)ds \\ &\quad + \int_0^{t_1} |e_\delta^{\lambda(t_1-s)} - e_\delta^{\lambda(t_2-s)}|\varphi_r(s)ds \\ &\quad + \int_{t_1}^{t_2} e_\delta^{|\lambda|(t_2-s)}\varphi_r(s)ds. \end{aligned}$$

The right hand of the above inequality tends to zero independently of $y \in B_r$ as $t_1 \to t_2$.

As a consequence of Steps 1–3 together with Arzelá-Ascoli theorem, we conclude that $N : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ is completely continuous.

Step 4. *$N(y)$ is closed for each $y \in C(J,\mathbb{R})$.*

Let $\{u_n\}_{n\geq 0} \in N(y)$ be such that $u_n \to u$ $(n \to \infty)$ in $C(J,\mathbb{R})$. Then $u \in C(J,\mathbb{R})$ and there exists $v_n \in S_{F,y_n}$ such that, for each $t \in J$,

$$u_n(t) = \int_0^1 G_{\lambda,\delta}(t,s)v_n(s)ds$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1(J,\mathbb{R})$. Thus, $v \in S_{F,y}$ and for each $t \in J$, we have

$$u_n(t) \to u(t) = \int_0^1 G_{\lambda,\delta}(t,s)v(s)ds$$

Hence, $u \in N(y)$.

By Lemma 2.1, since $N(y)$ is completely continuous, in order to prove that it is u.s.c. it is enough to prove that it has a closed graph. Thus, in our next step, we show that

Step 5. *N has a closed graph.*

Let $y_n \to y_*$, $h_n \in N(y_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(y_*)$. Now $h_n \in N(y_n)$ implies that there exists $f_n \in S_{F,y_n}$ such that for each $t \in J$,

$$h_n(t) = \int_0^1 G_{\lambda,\delta}(t,s)f_n(s)ds.$$

We must show that there exists $f_* \in S_{F,y_*}$ such that for each $t \in J$,

$$h_*(t) = \int_0^1 G_{\lambda,\delta}(t,s)f_*(s)ds.$$

Consider the continuous linear operator

$$\Theta : L^1(J,\mathbb{R}) \to C(J), \quad f \mapsto \Theta(f)(t) = \int_0^1 G_{\lambda,\delta}(t,s)f(s)ds.$$

From Lemma 2.2, it follows that $\Theta \circ S_{F,y}$ is a closed graph operator. Moreover, we have

$$h_n \in \Theta(S_{F,y_n}).$$

Since $y_n \to y_*$, Lemma 2.2 implies that

$$h_*(t) = \int_0^1 G_{\lambda,\delta}(t,s)f_*(s)ds$$

for some $f_* \in S_{F,y_*}$

Hence, we conclude that N is a compact multivalued map, u.s.c. with convex closed values. As a consequence of Lemma 3.2, we deduce that N has a fixed point which is a solution of the boundary problem (1.1)–(1.2). This completes the proof.

Next, we give an existence result based upon the following form of fixed point theorem applied to completely continuous operators [20].

Lemma 3.4. *Let X a Banach space, and $T : X \to \mathcal{P}(X)$ be a completely continuous multi-valued map. If the set*

$$\mathcal{E} = \{x \in X : \sigma x \in T(x),\ \sigma > 1\}$$

is bounded, then T has a fixed point.

Theorem 3.5. *Assume that the following hypotheses hold:*

(H_3) *$F : J \times \mathbb{R} \to \mathcal{P}_{b,cl,c}(\mathbb{R})$ is a Carathéodory multi-valued map;*
(H_4) *there exists a function $\mu \in L^1(J, \mathbb{R})$ such that*

$$\|F(t, y)\| \le \mu(t), \quad \text{for a.e. } t \in J \text{ and each } y \in \mathbb{R}.$$

Then the boundary problem (1.1)–(1.2) *has at least one solution on J.*

Proof. Define N as in the proof of Theorem 3.3. As in Theorem 3.3 we can prove that N is completely continuous. It remains to show that the set

$$\mathcal{E} = \{y \in C(J) : \sigma y \in N(y),\ \sigma > 1\}$$

is bounded. Let $y \in \mathcal{E}$, then $\sigma y \in N(y)$ for some $\sigma > 1$ and there exists a function $f \in S_{F,y}$ such that

$$y(t) = \sigma^{-1} \int_0^1 G_{\lambda,\delta}(t, s) f(s) ds.$$

For each $t \in J$, we have

$$|y(t)| \ \le\ \frac{e_\delta^{|\lambda| t}}{|e_\delta^\lambda|} \int_0^1 e_\delta^{|\lambda|(1-s)} |f(s)| ds + \int_0^t e_\delta^{|\lambda|(t-s)} |f(s)| ds.$$

Taking the supremum over $t \in J$, we get

$$\|y\| \le \left[\frac{e_\delta^{|\lambda|}}{|e_\delta^\lambda|} + 1 \right] e_\delta^{|\lambda|} \|\mu\|_{L^1} < \infty.$$

Hence the set $\mathcal{E}$ is bounded. As a consequence of Lemma 3.4 we deduce that N has at least a fixed point which gives rise to a mild solution of the boundary value problem (1.1)–(1.2) on J. □

Our final existence result in this subsection is based on Leray-Schauder nonlinear alternative.

Lemma 3.6 (Nonlinear alternative for Kakutani maps [11]). *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{c,cv}(C)$ is a upper semicontinuous compact map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is a $u \in \partial U$ and $\mu \in (0, 1)$ with $u \in \mu F(u)$.*

Theorem 3.7. *Assume that (H_1) holds. In addition we assume that:*

(H_5) *there exists a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and a function $p \in L^1(J, \mathbb{R}^+)$ such that*
$\|F(t, x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t, x)\} \le p(t)\psi(\|x\|)$ *for each $(t, x) \in J \times \mathbb{R}$;*
(H_6) *there exists a constant $M > 0$ such that*

$$\frac{M}{\psi(M) \|p\|_{L^1} \left[\frac{e_\delta^{|\lambda|}}{|e_\delta^\lambda|} + 1 \right] e_\delta^{|\lambda|}} > 1.$$

Then the boundary value problem (1.1)–(1.2) *has at least one solution on* J.

Proof. Define the operator $N(y)$ as in the proof of Theorem 3.3. Let $y \in \mu N(y)$ for some $\mu \in (0,1)$. We show there exists an open set $U \subseteq C(J,\mathbb{R})$ with $y \notin N(y)$ for any $\mu \in (0,1)$ and all $y \in \partial U$. Let $\mu \in (0,1)$ and $y \in \mu N(y)$. Then there exists $f \in L^1(J,\mathbb{R})$ with $f \in S_{F,y}$ such that, for $t \in J$, we have

$$y(t) = \int_0^1 G_{\lambda,\delta}(t,s)f(s)ds.$$

In view of (H_5), we have for each $t \in J$,

$$|y(t)| \le \psi(\|y\|)\left[\frac{e_\delta^{|\lambda|}}{|e_\delta^{\lambda}|}+1\right]e_\delta^{|\lambda|}\int_0^1 p(s)ds.$$

Consequently, we have

$$\frac{\|y\|}{\psi(\|y\|)\|p\|_{L^1}\left[\frac{e_\delta^{|\lambda|}}{|e_\delta^{\lambda}|}+1\right]e_\delta^{|\lambda|}} \le 1.$$

In view of (H_6), there exists M such that $\|y\| \ne M$. Let us set

$$U = \{y \in C(J,\mathbb{R}) : \|y\| < M\}.$$

Proceed as in the proof of Theorem 3.3, we claim that the operator $N : \overline{U} \to \mathcal{P}(C(J,\mathbb{R}))$ is a compact multi-valued map, u.s.c. with convex closed values. From the choice of U, there is no $y \in \partial U$ such that $y \in \mu N(y)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.6), we deduce that N has a fixed point $y \in \overline{U}$ which is a solution of the boundary value problem (1.1)–(1.2). This completes the proof. $\square$

3.2. **The lower semicontinuous case.** As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with this problem is based on the nonlinear alternative of Leray-Schauder type together with the selection theorem of Bressan and Colombo [4] for lower semi-continuous maps with decomposable values.

Definition 3.8. Let A be a subset of $I \times \mathbb{R}$. A is $\mathcal{L}\otimes\mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in I and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$.

Definition 3.9. A subset $\mathcal{A}$ of $L^1(I,\mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset I$, the function $u\chi_{\mathcal{J}} + v\chi_{I-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Lemma 3.10 ([4]). *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1(I,\mathbb{R}))$ be a lower semi-continuous (l.s.c.) multivalued operator with nonempty closed and* *decomposable values. Then N has a continuous selection, that is, there exists a continuous function (single-valued) $h : Y \to L^1(I, \mathbb{R})$ such that $h(x) \in N(x)$ for every $x \in Y$.*

Theorem 3.11. *Assume that (H_5), (H_6) and the following condition holds:*

(H_7) *$F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*
- *(a) $(t,x) \longmapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,*
- *(b) $x \longmapsto F(t,x)$ is lower semicontinuous for each $t \in J$;*

Then the boundary value problem (1.1)–(1.2) *has at least one solution on J.*

Proof. It follows from (H_5) and (H_7) that F is of l.s.c. type [9]. Then from Lemma 3.10, there exists a continuous function $f : AC^1(J, \mathbb{R}) \to L^1(J, \mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $y \in C(J, \mathbb{R})$, where $\mathcal{F} : C(J \times \mathbb{R}) \to \mathcal{P}(L^1(J, \mathbb{R}))$ is the Nemytskii operator associated with F, defined as

$$\mathcal{F}(y) = \{w \in L^1(J, \mathbb{R}) : w(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}.$$

Consider the problem

$$D^\delta y(t) - \lambda y(t) = f(y(t)), \quad t \in J := J, \ 1 < \delta < 2, \tag{3.3}$$

$$y(0) = y(1) = 0. \tag{3.4}$$

Observe that if $y \in AC^1(J, \mathbb{R})$ is a solution of (3.3)–(3.4), then y is a solution to the boundary value problem (1.1)–(1.2). In order to transform the problem (3.3)–(3.4) into a fixed point problem, we define the operator $\overline{N}$ as

$$\overline{N}(y) = \int_0^1 G_{\lambda,\delta}(t,s) f(y(s)) ds.$$

It can easily be shown that $\overline{N}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.7. So we omit it. This completes the proof. □

3.3. **The Lipschitz case.** Now we prove the existence of solutions for the boundary value problem (1.1)–(1.2) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [7].

Let (X, d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

where $d(A, b) = \inf_{a \in A} d(a; b)$ and $d(a, B) = \inf_{b \in B} d(a; b)$. Then $(\mathcal{P}_{b,cl}(X),\ H_d)$ is a metric space (see [16]).

Definition 3.12. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called (**a**) $\theta-$Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \theta d(x, y) \text{ for each } x, y \in X;$$

(**b**) a contraction if and only if it is $\theta-$Lipschitz with $\theta < 1$.

Lemma 3.13 ([7]). *Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Theorem 3.14. *Assume that the following conditions hold:*

(H_8) *$F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, y) : J \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $y \in \mathbb{R}$.*

(H_8) *$H_d(F(t, y), F(t, \bar{y})) \leq m(t)|y - \bar{y}|$ for almost all $t \in J$ and $y, \bar{y} \in \mathbb{R}$ with $m \in L^1(J, \mathbb{R}^+)$ and $d(0, F(t, 0)) \leq m(t)$ for almost all $t \in J$.*

Then the boundary value problem (1.1)–(1.2) *has at least one solution on J if*

$$\left[\frac{e_\delta^{|\lambda|}}{|e_\delta^\lambda|} + 1\right] e_\delta^{|\lambda|} \|m\|_{L^1} < 1.$$

Proof. We transform the boundary value problem (1.1)–(1.2) into a fixed point problem. Consider the operator $N : C(J, \mathbb{R}) \to \mathcal{P}(C(J, \mathbb{R}))$ defined at the begin of the proof of Theorem 3.3. We show that the operator N, satisfies the assumptions of Lemma 3.13. The proof will be given in two steps.

Step I. $N(y)$ is nonempty and closed for every $v \in S_{F,y}$.

Note that since the set-valued map $F(\cdot, y(\cdot))$ is measurable with the measurable selection theorem (e.g., [5, Theorem III.6]) it admits a measurable selection $v : J \to \mathbb{R}$. Moreover, by the assumption (H_8), we have

$$|v(t)| \leq m(t) + m(t)|y(t)|,$$

i.e. $v \in L^1(J, \mathbb{R})$ and hence F is integrably bounded. Therefore, $S_{F,y} \neq \emptyset$. Moreover $N(y) \in \mathcal{P}_{cl}(C(J, \mathbb{R}))$ for each $y \in C(J, \mathbb{R})$, as proved in Step 4 of Theorem 3.3.

Step II. Next we show that there exists $\theta < 1$ such that

$$H_d(N(y), N(\bar{y})) \leq \theta \|y - \bar{y}\| \text{ for each } y, \bar{y} \in AC^1(J, \mathbb{R}).$$

Let $y, \bar{y} \in AC^1(J, \mathbb{R})$ and $h_1 \in N(y)$. Then there exists $v_1(t) \in F(t, y(t))$ such that, for each $t \in J$,

$$h_1(t) = \int_0^1 G_{\lambda,\delta}(t, s) v_1(s) ds.$$

By (H_9), we have

$$H_d(F(t, y), F(t, \bar{y})) \leq m(t)|y(t) - \bar{y}(t)|.$$

So, there exists $w(t) \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w(t)| \leq m(t)|y(t) - \bar{y}(t)|, \ t \in J.$$

Define $U : J \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w(t)| \leq m(t)|y(t) - \bar{y}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{y}(t))$ is measurable (Proposition III.4 [5]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \bar{y}(t))$ and for each $t \in J$, we have $|v_1(t) - v_2(t)| \leq m(t)|y(t) - \bar{y}(t)|$.

For each $t \in J$, let us define

$$h_2(t) = \int_0^1 G_{\lambda,\delta}(t,s)v_2(s)ds.$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| &= \left| -\frac{e_\delta^{\lambda t}}{e_\delta^{\lambda}} \int_0^1 e_\delta^{\lambda(1-s)} v_1(s)ds - \int_0^t e_\delta^{\lambda(t-s)} v_1(s)ds \right. \\ &\quad \left. + \frac{e_\delta^{\lambda t}}{e_\delta^{\lambda}} \int_0^1 e_\delta^{\lambda(1-s)} v_2(s)ds + \int_0^t e_\delta^{\lambda(t-s)} v_2(s)ds \right| \\ &= \left| \frac{e_\delta^{\lambda t}}{e_\delta^{\lambda}} \int_0^1 e_\delta^{\lambda(1-s)} [v_2(s) - v_1(s)]ds + \int_0^t e_\delta^{\lambda(t-s)} [v_2(s) - v_1(s)]ds \right| \\ &\leq \left[\frac{e_\delta^{|\lambda|}}{|e_\delta^{\lambda}|} + 1 \right] e_\delta^{|\lambda|} \|m\|_{L^1} \|y - \overline{y}\|. \end{aligned}$$

Hence,

$$\|h_1 - h_2\| \leq \left[\frac{e_\delta^{|\lambda|}}{|e_\delta^{\lambda}|} + 1 \right] e_\delta^{|\lambda|} \|m\|_{L^1} \|y - \overline{y}\|.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$H_d(N(y), N(\bar{y})) \leq \theta \|x - \bar{x}\| \leq \left[\frac{e_\delta^{|\lambda|}}{|e_\delta^{\lambda}|} + 1 \right] e_\delta^{|\lambda|} \|m\|_{L^1} \|y - \overline{y}\|.$$

Since N is a contraction, it follows by Lemma 3.13 that N has a fixed point y which is a solution of (1.1)–(1.2). This completes the proof. □

4. **EXAMPLES**

Consider the nonlinear fractional boundary value problem

$$D^\delta y(t) - \lambda y(t) \in F(t, y(t)),\ t \in J := [0,1],\ 1 < \delta < 2,\ \lambda \in \mathbb{R}, \tag{4.1}$$

$$y(0) = y(1) = 0. \tag{4.2}$$

(a) Let $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map given by

$$y \to F(t,y) = \left[\frac{y^2 e^{-y^2}}{x^2 + 3},\ \frac{t|y| \sin |y|}{|y| + 1} \right]. \tag{4.3}$$

For $f \in F$, we have

$$|f| \leq \max\left(\frac{y^2 e^{-y^2}}{y^2+3}, \frac{t|y|\sin|y|}{|y|+1}\right) \leq t|y|+1, \quad y \in \mathbb{R}.$$

Thus,

$$\|F(t,y)\|_{\mathcal{P}} := \sup\{|x| \,:\, x \in F(t,y)\} \leq rt+1 = \varphi_r(t), \quad \|y\| \leq r.$$

We can find that $\liminf_{r\to\infty} \frac{1}{r}\int_0^1 \varphi_r(s)ds = \gamma = 1/2$. Therefore, all the conditions of Theorem 3.3 are satisfied. So, problem (4.1)–(4.2) with $F(t,y)$ given by (4.3) has at least one solution on J provided

$$e_\delta^{|\lambda|} = E_{\delta,\delta}(|\lambda|) < 1.$$

(b) If $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$y \to F(t,y) = \left[\frac{y^4}{y^4+2} + e^{-y^2} + t + 2,\ \frac{|y|}{|y|+1} + t^2 + \frac{1}{2}\right]. \tag{4.4}$$

For $f \in F$, we have

$$|f| \leq \max\left(\frac{y^4}{y^4+2} + e^{-y^2} + t + 2,\ \frac{|y|}{|y|+1} + t^2 + \frac{1}{2}\right) \leq 5, \quad x \in \mathbb{R}.$$

Here $\|F(t,y)\|_{\mathcal{P}} := \sup\{|x| \,:\, x \in F(t,y)\} \leq 5 = p(t)\psi(\|y\|)$, $y \in \mathbb{R}$, with $p(t) = 1$, $\psi(\|y\|) = 5$. It is easy to verify that $M > 10e_\delta^{|\lambda|} = 10E_{\delta,\delta}(|\lambda|)$. Then, by Theorem 3.7, the problem (4.1)–(4.2) with $F(t,y)$ given by (4.4) has at least one solution on J.

(c) Consider the multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ given by

$$y \to F(t,y) = \left[0,\ (t+1)\sin y + \frac{2}{3}\right]. \tag{4.5}$$

Then we have

$$\sup\{|u| \,:\, u \in F(t,y)\} \leq (t+1) + \frac{2}{3},$$

and

$$H_d(F(t,y),\, F(t,\overline{y})) \leq (t+1)|y-\overline{y}|.$$

Let $m(t) = t+1$. Then $H_d(F(t,x),\, F(t,\overline{x})) \leq m(t)|y-\overline{y}|$, and $\|m\|_{L^1} = 3/2$. By Theorem 3.14, the problem (4.1)–(4.2) with the $F(t,y)$ given by (4.5) has at least one solution on J if

$$\frac{3}{2}e_\delta^{|\lambda|} = \frac{3}{2}E_{\delta,\delta}(|\lambda|) < 1.$$

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