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RAZUMIKHIN-TYPE STABILITY THEOREMS FOR IMPULSIVE DISCRETE SYSTEMS WITH TIME DELAY

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ABSTRACT. This paper studies impulsive discrete systems with time delay. Several criteria on uniform stability and uniform asymptotic stability are established by utilizing the Razumikhin technique. Both linear and nonlinear impulsive discrete systems with time delay are investigated. These stability criteria show that impulses can be used to stabilize a unstable system. Some numerical examples are presented to illustrate the stability criteria.

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1. Introduction

Time delay occurs frequently in many processes, such as network controlled systems [1], high-speed communication networks [2], teleoperated systems [3] and parallel computation [4] just to name a few. Thus, Stability analysis of time-delay systems has attracted increasing attention for the past three decades. We refer the reader to [5] and [6] for the basic theory of time-delay systems.

There are several methods available to study stability problems of differential systems with time delay such as the Lyapunov functional method, the comparison principle, and the Razumikhin technique. Each method has its own advantages and disadvantages. The Lyapunov functional method works similar to the classical Lyapunov method for systems without time delay when evaluating its derivative along solutions of the system, but it requires to construct an appropriate Lyapunov functional which is often more challenging than a Lyapunov function, see [7, 8] and relevant references therein. The comparison principle technique is very general and popular but needs an additional comparison system with known stability properties, see [9, 10] and relevant references therein. On the other hand, the Razumikhin technique requires a standard Lyapunov function which is relatively easier to construct, but evaluation of the derivative of the Lyapunov function must be restricted to a minimal set, see [5, 11, 12] and relevant references therein.

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Impulsive systems occur in many applied fields, such as control technology, communication networks and biological population management, see [11, 13, 14, 15] and references therein. On the other hand, impulsive control has attracted the interest of many researchers in recent years. Such control arises naturally in a wide variety of applications, such as orbital transfer of satellite [16, 17], ecosystem management [18]. It has shown in many cases that impulsive control can give better performance than continuous control.

In recent years, there have appeared several papers devoted to the study of the impulsive systems with time delay where the Razumikhin technique is adopted, see [15, 19] and references therein. Zhang and Chen studied a class of discrete systems with time delay and established a backward Razumikhin-type uniformly asymptotic stability theorem in [20]. Liu and Marquez developed the forward Razumikhin-type uniformly asymptotic stability theorems in [21], which is less restrictive than that in [20].

In [19], the author only considered continuous impulsive systems. And in [21], the author only considered delay-free discrete systems. Therefore, the objective of this paper is to extend the results in [19] and [21] to impulsive discrete systems with time delay. We shall establish in this paper some Razumikhin-type stability criteria for impulsive discrete systems with time delay. To the best of our knowledge, very few results on Razumikhin-type uniform asymptotical stability have been reported for impulsive discrete systems with time delay.

The rest of this paper is organized as follows. In Section 2, we shall introduce some notations and definitions. Then in Section 3, we shall establish criteria on Razumikhin-type uniform stability and uniform asymptotic stability for impulsive discrete systems with time delay. In Section 4, we shall obtain two stability criteria for linear impulsive discrete systems with time delay. In Section 5, we shall discuss some examples to illustrate our results.

2. **Preliminaries**

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^m$ the m-dimensional Euclidean space, $\mathbb{R}^+$ the interval $[0, +\infty)$, $\mathbb{N}$ the natural numbers, i.e., $\mathbb{N} = \{0, 1, 2, \dots\}$. For some positive integer r, let $\mathbb{N}_{-r} = \{-r, -r+1, \dots, -1, 0\}$ and let $\mathbb{N}_{-} = \{0, -1, -2, \dots\}$. A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be belong to class $\mathcal{K}$ ($\varphi \in \mathcal{K}$) if it is continuous, $\varphi(0) = 0$ and strictly increasing. Let $C = \{\phi : \mathbb{N}_{-r} \to \mathbb{R}^m\}$ with some integer $r > 0$. For any $\phi \in C$ and any positive integer s, we define $\|\phi\|_s = \max_{\theta \in \mathbb{N}_{-s}} \{\|\phi(\theta)\|\}$. Let $C_\rho = \{\phi \in C : \|\phi\| < \rho\}$ for some $\rho > 0$.

Consider the following impulsive discrete system with time delay

$$
\begin{cases}
x(n+1) = f(n, \bar{x}_n), & n_0 \in \mathbb{N}, n \geq n_0, \\
\bar{x}(n) = \begin{cases} x(n), & n \neq N_k, \\ x(N_k) + I_k(x(N_k)), & n = N_k, k \in \mathbb{N}, \end{cases} \\
x_{n_0} = \phi,
\end{cases}
\tag{2.1}
$$

where $x \in \mathbb{R}^m$, $f : \mathbb{N} \times C_\rho \to R^m$, $\phi \in C$, $\bar{x}_n \in C_\rho$ is defined by $\bar{x}_n(s) = \bar{x}(n+s)$ for any $s \in \mathbb{N}_{-r}$, and $0 < N_{k+1} - N_k < +\infty$ for any $k \geq 0$.

We assume $f(n, 0) \equiv 0$ so that system (2.1) admits the trivial solution. Denote by $x(n) = x(n, n_0, \phi)$ the solution of system (2.1), for any given initial data: $n_0 \in \mathbb{N}$ and $\phi : \mathbb{N}_{-r} \to \mathbb{R}^m$. We further assume that there exists a $\rho_1 > 0$ such that $\|x + I_k(x)\| < \rho$ if $\|x\| < \rho_1$. It can be seen that $x(N_k)$ denotes the state of x before the impulse at time N_k, and $\bar{x}(N_k)$ the state of x after the impulse at time N_k.

Definition 2.1. The trivial solution of system (2.1) is said to be uniformly stable (US) if, for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$, for any given initial data: $n_0 \in \mathbb{N}$, $x_{n_0} = \phi$, such that $\|\phi\|_r \leq \delta$ implies

$$
\|\bar{x}(n, n_0, \phi)\| \leq \epsilon, \text{ for all } n \geq n_0, \ n \in \mathbb{N}. \tag{2.2}
$$

Definition 2.2. The trivial solution of system (2.1) is said to be uniformly attractive if, there exist a positive real number $\sigma > 0$, for any $\eta > 0$, there exists a positive integer $K = K(\eta) > 0$, for each given initial data: $n_0 \in \mathbb{N}$, $x_{n_0} = \phi$, such that when $\|\phi\|_r < \sigma$ and $n \geq n_0 + K$, the following inequality holds:

$$
\|\bar{x}(n, n_0, \phi)\| < \eta. \tag{2.3}
$$

Definition 2.3. The trivial solution of system (2.1) is said to be uniformly asymptotically stable (UAS) if, the trivial solution of system (2.1) is US and uniformly attractive.

3. **Main results**

In this section, we shall state and prove two stability criteria, US and UAS, for system (2.1).

Theorem 3.1. *Assume $V : \mathbb{N} \times B_\rho \to \mathbb{R}^+$ is a positive definite and continuous in x, where $B_\rho = \{x \in \mathbb{R}^m : \|x\| < \rho\}$. There exist scalar nondecreasing functions $q(\cdot)$, $p_1(\cdot)$, $p_2(\cdot)$, $g(\cdot)$ with $q(w) > 0$, $0 < p_1(w) < w$, $0 < p_2(w) < w$, $g(w) > 0$ for any $w > 0$ such that*

(i) $c_1(\|x\|) \leq V(n, x) \leq c_2(\|x\|)$, $x \in B_\rho, c_1, c_2 \in \mathcal{K}$;

(ii) *for any* $n \in \mathbb{N}$, *if* $V(n, \bar{x}(n)) \geq p_1(V(n+s, \bar{x}(n+s)))$ *for all* $s \in \mathbb{N}_{-r}$, *then*

$$V(n+1, f(n, \bar{x}_n)) - V(n, \bar{x}(n)) \leq q(V(n, \bar{x}(n))),$$

otherwise

$$V(n+1, f(n, \bar{x}_n)) \leq p_2(\max_{s\in\mathbb{N}_{-r}}\{V(n+s, \bar{x}(n+s))\});$$

(iii) $V(N_k, \bar{x}(N_k)) \leq g(V(N_k, x(N_k)))$, *for all* $k \in \mathbb{N}$*;*

(iv) $p_2(w) + \mu q(w) \leq w$ *and* $g(w + q(w)) \leq p_2(w)$ *for any* $w > 0$ *with* $\mu = \max_{k\in\mathbb{N}}\{N_{k+1} - N_k\}$.

Then, the trivial solutions of system (2.1) *is US. It is UAS if*

$$p_2(w) + \mu q(w) \leq (1-\gamma)w \tag{3.1}$$

and

$$g(w + q(w)) \leq (1-\gamma)p_2(w) \tag{3.2}$$

for some $\gamma \in (0, 1)$.

Proof. Denote $U(n) = \max_{\theta\in\mathbb{N}_{-r}}\{V(n+\theta, \bar{x}(n+\theta))\}$.

For any $n_0 \in \mathbb{N}$, and any $\epsilon > 0$, choose $\delta = \delta(\epsilon) > 0$ so that $0 < \delta < \epsilon$ and $c_2(\delta) < c_1(\epsilon)$. If $\|\phi\|_r \leq \delta$, we claim that

$$V(n, \bar{x}(n)) < c_1(\epsilon), \text{ for any } n \geq n_0, n \in \mathbb{N}.$$

If this is not true, since $V(n, \bar{x}(n)) < c_1(\epsilon)$ for all $n < n_0$, then there exists $n^* \in [N_k, N_{k+1})$ for some $k \geq 0$ such that

$$V(n^*, \bar{x}(n^*)) \geq c_1(\epsilon) \text{ and } V(n, \bar{x}(n)) < c_1(\epsilon) \text{ for all } n_0 - r \leq n < n^*. \tag{3.3}$$

If $n^* = N_k$, since

$$\begin{aligned} V(N_k, \bar{x}(N_k)) &\leq g(\max\{V(N_k-1, \bar{x}(N_k-1)) \\ &\qquad +q(V(N_k-1, \bar{x}(N_k-1))), p_2(U(N_k-1))\}) \\ &\leq g(U(N_k-1) + q(U(N_k-1))) \\ &\leq p_2(U(N_k-1)), \end{aligned}$$

then we get

$$V(n^*, \bar{x}(n^*)) \leq p_2(U(N_k-1)) \leq U(N_k-1) \leq c_1(\epsilon),$$

which contradicts (3.3).

If $n^* \in (N_k, N_{k+1})$, first, we will show for any $n \in (N_k, n^*]$,

$$V(n, x(n)) \leq p_2(U(N_k-1)) + \sum_{i=N_k}^{n-1} q(V(i, \bar{x}(i))). \tag{3.4}$$

Since

$$\begin{aligned} V(N_k+1, x(N_k+1)) &\leq \max\{V(N_k, \bar{x}(N_k)) + q(V(N_k, \bar{x}(N_k))), p_2(U(N_k))\} \\ &\leq \max\{p_2(U(N_k-1)) + q(V(N_k, \bar{x}(N_k))), p_2(U(N_k-1))\} \\ &\leq p_2(U(N_k-1)) + q(V(N_k, \bar{x}(N_k))), \end{aligned}$$

which establishes the base case. We proceed by induction and assume for any $n \in (N_k, n^*)$, $U(n) \leq p_2(U(N_k-1)) + \sum_{i=N_k}^{n-1} q(V(N_k, \bar{x}(N_k)))$, then

$$\begin{aligned} V(n+1, x(n+1)) &\leq \max\{V(n, \bar{x}(n)) + q(V(n, \bar{x}(n))), p_2(U(n))\} \\ &\leq p_2(U(N_k-1)) + \sum_{i=N_k}^{n} q(V(i, \bar{x}(i))). \end{aligned}$$

Thus, inequality (3.4) holds for all $n \in (N_k, n^*]$. Hence, we have

$$\begin{aligned} V(n^*, x(n^*)) &\leq p_2(U(N_k-1)) + \sum_{i=N_k}^{n^*-1} q(V(N_k)) \\ &\leq p_2(U(N_k-1)) + \mu q(U(N_k-1)) \\ &\leq U(N_k-1) < c_1(\epsilon) \end{aligned}$$

which contradict with our assumption on n^*.

Therefore, we obtain that

$$c_1(\|\bar{x}(n)\|) \leq V(n, \bar{x}(n)) < c_1(\epsilon), \text{ for all } n \geq n_0, n \in \mathbb{N},$$

which implies $\|\bar{x}(n)\| \leq \epsilon$. Hence, the trivial solution of system (2.1) is US.

In the following, we will prove that the trivial solution of the system (2.1) is uniformly attractive if (3.1) and (3.2) hold. Since the trivial solution of system (2.1) is US, for any fixed positive number $H > 0$ and $H < \rho_1$, there exists a positive number $0 < \delta \leq H$ with $c_2(\delta) \leq c_1(H)$ such that for any $\phi : \mathbb{N}_{-r} \to \mathbb{R}^n$, if $\|\phi\|_r \leq \delta$, we have

$$\|\bar{x}(n)\| \leq H, \ n \geq n_0, \tag{3.5}$$

and

$$V(n, \bar{x}(n)) \leq c_2(\delta) \leq c_1(H), \ n \geq n_0. \tag{3.6}$$

It follows from (3.6) that $U(n) \leq c_2(\delta)$ for any $n \in \mathbb{N}$. In order to show the uniform attractivity of the trivial solution, we need to prove that, for any positive real number η satisfying $0 \leq \eta \leq H$, for any $n_0 \in \mathbb{N}$, $\phi \in C$, there exists a positive integer $K = K(\eta)$ independent of n_0 and δ, such that when $\|\phi\|_r \leq \delta$ and $n \geq K + n_0$, we have

$$\|\bar{x}(n)\| = \|\bar{x}(n, n_0, \phi)\| \leq \eta. \tag{3.7}$$

If for any $n \geq K$, we have

$$V(n, \bar{x}(n)) \leq c_1(\eta). \tag{3.8}$$

Then, by condition (i), (3.7) can be induced from (3.8). Hence, in the following, we just prove that (3.8) holds.

Since $g(w + q(w)) \leq p_2(w)$ and $V(N_k, x(N_k)) \leq \max\{V(N_k - 1) + q(V(N_k - 1)), p_2(U(N_k - 1))\} \leq U(N_k - 1) + q(U(N_k - 1))$, we have

$$\begin{aligned} V(N_k, \bar{x}(N_k)) &\leq g(V(N_k, x(N_k))) \\ &\leq g(U(N_k - 1) + q(U(N_k - 1))) \\ &\leq (1 - \gamma)U(N_k - 1). \end{aligned}$$

We claim that $V(n, \bar{x}(n)) \leq (1 - \gamma)U(N_k - 1)$ for all $n \in [N_k, N_{k+1})$. If this is not true, then there exists $\hat{n} \in [N_k, N_{k+1})$ such that $V(\hat{n}) > (1 - \gamma)U(N_k - 1)$. Notice that $V(N_k, \bar{x}(N_k)) < (1 - \gamma)U(N_k - 1)$, there exist $n^* = \min\{n \,| V(n, x(n)) > (1 - \gamma)U(N_k - 1),\ N_k < n \leq \hat{n}\}$.

From (3.4), we have

$$\begin{aligned} V(n^*, x(n^*)) &\leq p_2(U(N_k - 1)) + \sum_{i=N_k}^{n} q(V(N_k, \bar{x}(N_k))) \\ &\leq p_2(U(N_k - 1)) + \mu q((1 - \gamma)U(N_k)) \\ &\leq (1 - \gamma)U(N_k - 1), \end{aligned}$$

which contradict with our assumption on n^*. Thus, for all $n \in [N_k, N_{k+1})$, we have $V(n, \bar{x}(n)) \leq (1 - \gamma)U(N_k - 1)$.

For all $n \in [N_k, N_{k+1})$, we have

$$U(n) \leq (1 - \gamma)U(N_k - 1) \leq \cdots \leq (1 - \gamma)^k U(N_1 - 1) \leq (1 - \gamma)^k (U(n_0) + \mu q(U(n_0))).$$

Thus, for $\delta = \delta(H) > 0$ with $c_2(\delta) \leq c_1(H)$, for any $\eta > 0$, there exists a $K = K(\eta) > 0$ with $(1 - \gamma)^{K/\mu}(c_1(H) + \mu q(c_1(H))) < c_1(\eta)$, for any $n_0 \in \mathbb{N}$ and $\|\phi\|_r \leq \delta$, when $n \geq n_0 + K$, by condition (i), we have

$$\begin{aligned} V(n, \bar{x}(n)) &\leq U(n) \leq (1 - \gamma)^k U(n_0) \\ &\leq (1 - \gamma)^{K/\mu}(c_2(\delta) + \mu q(c_2(\delta))) \\ &\leq (1 - \gamma)^{K/\mu}(c_1(H) + \mu q(c_1(H))) \\ &< c_1(\eta). \end{aligned}$$

The proof is complete. $\square$

Remark 3.2. Condition (ii) of Theorem 3.1 guarantees that the increase of V at $n+1$ is bounded by $q(V(n, x(n)))$ when $V(n, \bar{x}(n))$ exceeds some kind of upper bound measured by $p_1(V(n+s,\bar{x}(n+s))$ for all $s \in \mathbb{N}_{-r}$, otherwise $V(n+1, f(n,x_n))$ is bounded by $p_2(\max_{s\in\mathbb{N}_{-r}}\{V(n+s,\bar{x}(n+s))\})$.

Remark 3.3. V may increase in Theorem 3.1 at time $n+1$ when $V(n,\bar{x}(n))$ exceeds $p_1(V(n+s,\bar{x}(n+s))$ for all $s \in \mathbb{N}_{-r}$. Thus the system may be unstable without impulse.

Remark 3.4. Inequalities (3.1) and (3.2) in Theorem 3.1 guarantee that V decreases at a rate greater than $1-\gamma$ in adjacent interval while condition (iv) only guarantees that V is nonincreasing.

Theorem 3.5. *Assume $V : \mathbb{N} \times B_\rho \to \mathbb{R}^+$ is a positive definite and continuous in x, where $B_\rho = \{x \in \mathbb{R}^m : \|x\| < \rho\}$. There exist scalar nondecreasing functions $q(\cdot)$, $p_1(\cdot)$, $p_2(\cdot)$, $g(\cdot)$ with $q(w) > 0$, $0 < p_1(w) < w$, $0 < p_2(w) < w$, $g(w) > 0$ for any $w > 0$ such that*

(i) $c_1(\|x\|) \le V(n,x) \le c_2(\|x\|)$, $x \in B_\rho, c_1, c_2 \in \mathcal{K}$;

(ii) *for any $n \in \mathbb{N}$, if $V(n,\bar{x}(n)) \ge p_1(V(n+s,\bar{x}(n+s)))$ for all $s \in \mathbb{N}_{-r}$, then*

$$V(n+1, f(n,\bar{x}_n)) \le V(n,\bar{x}(n)) - q(V(n,\bar{x}(n))),$$

otherwise

$$V(n+1, f(n,\bar{x}_n)) \le p_2(\max_{s\in\mathbb{N}_{-r}}\{V(n+s,\bar{x}(n+s))\});$$

(iii) $V(N_k,\bar{x}(N_k)) \le g(V(N_k, x(N_K)))$, *for all $k \in \mathbb{N}$;*

(iv) $g(p_2(w)) \le w$, $w - (\tau - r)q(p_2(w))) \le p_2(w)$ *and $\tau > r$ for any $w > 0$ with* $\tau = \min_{k\in\mathbb{N}}\{N_{k+1} - N_k\}$.

Then, the trivial solution of system (2.1) is US. It is UAS if

$$g(p_2(w)) \le (1-\gamma)w \tag{3.9}$$

for some $\gamma \in (0,1)$.

Proof. Denote $U(n) = \max_{\theta\in\mathbb{N}_{-r}}\{V(n+\theta,\bar{x}(n+\theta))\}$.

For any $n_0 \in \mathbb{N}$, and any $\epsilon > 0$, let $\delta = \delta(\epsilon) > 0$ with $0 < \delta < \epsilon$ and $g(c_2(\delta)) < c_1(\epsilon)$, if $\|\phi\|_r \le \delta$, we claim that

$$V(n,\bar{x}(n)) < c_1(\epsilon), \text{ for any } n \ge n_0, n \in \mathbb{N}.$$

If this is not true, since $V(n,\bar{x}(n)) < c_1(\epsilon)$ for all $n \le n_0$, then there exists $n^* \in [N_k, N_{k+1})$ such that

$$V(n^*,\bar{x}(n^*)) \ge c_1(\epsilon) \text{ and } V(n,\bar{x}(n)) < c_1(\epsilon) \text{ for all } n_0 - r \le n < n^*. \tag{3.10}$$

If $n^* \ne N_k$, we have

$$V(n+1, x(n+1)) \;\le\; \max\{V(n,\bar{x}(n)) - q(V(n,\bar{x}(n))), p_2(U(n))\}$$

$$\begin{aligned} &\le \max\{U(n)-q(V(n,\bar{x}(n))),U(n)\} \\ &\le U(n)<c_1(\epsilon), \end{aligned}$$

which is contradict with assumption of n^*.

If $n^*=N_k$, first, We claim that there exists $\hat{n}\in(N_{k-1},N_{k-1}+\tau-r]$, such that $V(\hat{n},x(\hat{n}))\le p_2(U(N_k))$. If this is not true, then

$$V(n,x(n))>p_2(U(N_k)) \text{ for all } n\in(N_{k-1},N_{k-1}+\tau-r]. \tag{3.11}$$

From condition (ii), if $V(n,\bar{x}(n))\ge p_1(V(n+s,\bar{x}(n+s)))$ for all $n\in[N_{k-1},N_{k-1}+\tau-r-1]$ and all $s\in\mathbb{N}_{-r}$, we have

$$\begin{aligned} V(N_{k-1}+\tau-r,x(N_{k-1}+\tau-r)) &\le V(N_{k-1},\bar{x}(N_{k-1}))-\sum_{i=N_{k-1}}^{N_{k-1}+\tau-r-1}q(V(i,\bar{x}(i))) \\ &\le U(N_{k-1})-(\tau-r)q(p_2(U(N_{k-1}))) \\ &\le p_2(U(N_{k-1})), \end{aligned}$$

otherwise, there exist some $\hat{n}\in(N_{k-1},N_{k-1}+\tau-r-1]$ such that

$$V(\hat{n}+1,x(\hat{n}+1))\le p_2(U(\hat{n}))\le p_2(U(N_{k-1})),$$

which is contradict with (3.11). Thus our claim is true.

Now suppose for some $s\in[N_{k-1},N_k)$, $V(s,\bar{x}(s))\le p_2(U(N_{k-1}))$. Then

$$\begin{aligned} V(s+1,x(s+1)) &\le \max\{V(s,\bar{x}(s))-q(V(s,\bar{x}(s))),p_2(U(s))\} \\ &\le \max\{p_2(U(N_{k-1}))-q(V(s,\bar{x}(s))),p_2(U(N_{k-1}))\} \\ &\le p_2(U(N_{k-1})). \end{aligned}$$

Thus $V(n,x(n))\le p_2(U(N_{k-1}))$ for all $n\in[\hat{n},N_k]$ by mathematical induction.

Hence, we have

$$\begin{aligned} V(n^*,\bar{x}(n^*)) &\le g(V(N_k,x(N_k))) \\ &\le g(p_2(U(N_{k-1}))) \\ &\le U(N_{k-1}) \\ &< c_1(\epsilon), \end{aligned}$$

which contradict with assumption of n^*.

Therefore, we obtain that

$$c_1(\|\bar{x}(n)\|)<c_1(\epsilon).$$

Thus, for any $n\ge n_0$, $n\in\mathbb{N}$, $\|\bar{x}(n)\|\le\epsilon$. Hence, the trivial solution of system (2.1) is US.

In the following, we will prove that the trivial solution of system (2.1) is uniformly attractive if inequality (3.9) holds. Similar as the proof in Theorem 3.1, for any fixed positive number $H > 0, H < \rho_1$, we just prove that there exists $\delta(H) > 0$ and $\delta(H) < \rho_1$, for any positive real number η satisfying $0 \le \eta \le H$, for any $n_0 \in \mathbb{N}$, $\phi \in C$, there exists a positive integer $K = K(\eta)$ independent of n_0 and δ, such that (3.8) holds when $\|\phi\|_r \le \delta$ and $n \ge n_0 + K$.

Since

$$\begin{aligned} U(N_{k+1}) &\le \max\{V(N_{k+1}, \bar{x}(N_{k+1})), V(N_{k+1}-1, \bar{x}(N_{k+1}-1)), \dots, \\ &\qquad V(N_{k+1}-r, \bar{x}(N_{k+1}-r))\} \\ &\le \max\{g(V(N_{k+1}, x(N_{k+1}))), p_2(U(N_k))\} \le (1-\gamma)U(N_k). \end{aligned}$$

Thus, for all $N_k \le n < N_{k+1}$, we have

$$U(n) \le U(N_k) \le (1-\gamma)U(N_{k-1}) \le \cdots \le (1-\gamma)^{k-1}U(N_1) \le (1-\gamma)^k U(n_0).$$

For any $\eta : 0 < \eta < \rho_1$ and the above $\delta = \delta(H) > 0$, let $\mu = \max_{k\in\mathbb{N}}\{N_{k+1} - N_k\}$, there exists a $K = K(\eta) > 0$ such that $(1-\gamma)^{K/\mu} c_1(H) < c_1(\eta)$. For any $n_0 \in \mathbb{N}$ and $\|\phi\|_r \le \delta$, when $n \ge n_0 + K$, by condition (i), we have

$$V(n, \bar{x}(n)) \le U(n) \le (1-\gamma)^k U(n_0) \le (1-\gamma)^{K/\mu} c_2(\delta) \le (1-\gamma)^{K/\mu} c_1(H) \le c_1(\eta).$$

Hence (3.8) holds. The proof is complete. □

Remark 3.6. Condition (ii) in Theorem 3.5 guarantees that V decreases at time $n+1$ when $V(n, \bar{x}(n))$ exceeds some upper bound measured by $p_1(V(n+s, \bar{x}(n+s))$ for any $n \in \mathbb{N}$ and all $s \in \mathbb{N}_{-r}$, otherwise $V(n+1, f(n, x_n))$ is bounded by $p_2(\max_{s\in\mathbb{N}_{-r}}\{V(n+s, \bar{x}(n+s))\})$. Thus the system is stable without impulse.

Remark 3.7. Inequality (3.9) in Theorem 3.5 guarantees that V decreases at a rate greater than $1-\gamma$ in adjacent intervals while condition (iv) only guarantees that V is nonincreasing.

4. **Stability of linear impulsive discrete systems with time delay**

In this section, we shall obtain results on uniform asymptotical stability for linear impulsive discrete system with time delay.

Consider the linear impulsive discrete systems with time delay of the form

$$\begin{cases} x(n+1) = A\bar{x}(n) + \sum_{i=1}^{m_0} B_i \bar{x}(n - \tau_i(n)), & n_0 \in \mathbb{N}, n \ge n_0, \\ \bar{x}(n) = \begin{cases} x(n), & \\ x(N_k) + I_k(x(N_k)), & \end{cases} & \begin{matrix} n \ne N_k, \\ n = N_k, k \in \mathbb{N}, \end{matrix} \\ x_{n_0} = \phi, \end{cases} \tag{4.1}$$

where $x \in \mathbb{R}^m$, $A, B_1, \dots, B_{m_0} \in \mathbb{R}^{m\times m}$, $\phi \in C$ and $\tau_i : \mathbb{N} \to \mathbb{N}_{-r}$ with $r > 0$ and $r \in \mathbb{N}$ represents the time delay. We have the following theorems.

Theorem 4.1. *Suppose there exist a symmetric positive definite matrix P, let $\lambda_0 \geq \max\{\lambda(A^TPAP^{-1})\}$, $\lambda_j \geq \max\{\lambda(B_j^TPB_jP^{-1})\}$, $j = 1, 2, \ldots, m_0$ and $\alpha_2 = \max\{\lambda((E+I_k)^T(E+I_k))\}$ where $\lambda(S)$ denote all the eigenvalues of matrix S. And there exist $\alpha_1 : 0 < \alpha_1 < 1$, $\gamma : 0 < \gamma < 1$, $\eta > 0$, $\epsilon > 0$ such that the following inequalities hold:*

$$-\left(a + \frac{b}{\alpha_1}\right) > 0 \tag{4.2}$$

$$0 < (a\alpha_1 + b + \eta) < 1 \tag{4.3}$$

$$\alpha_2 \cdot (a\alpha_1 + b + \eta) \leq 1 - \gamma \tag{4.4}$$

$$1 + (\tau - r)(a + \frac{b}{\alpha_1})(a\alpha_1 + b + \eta) \leq a\alpha_1 + b + \eta \tag{4.5}$$

where $\tau = \min_{k\in\mathbb{N}}\{N_{k+1} - N_k\}$, $a = (1 + \frac{1}{\epsilon})\lambda_0 - 1$ and $b = (1+\epsilon)m_0\sum_{j=1}^{m_0}\lambda_j$. Then, the trivial solution of system (4.1) *is UAS.*

Proof. Denote $x_n = x(n)$, $\bar{x}_n = \bar{x}(n)$, $\lambda_{\min} = \min\{\lambda(P)\}$ and $\lambda_{\max} = \max\{\lambda(P)\}$. Let $V(n) = x(n)^TPx(n)$ and $V(\bar{n}) = \bar{x}(n)^TP\bar{x}(n)$. It can be seen that condition (i) of Theorem 3.5 holds with $c_1(w) = \lambda_{\min}w$ and $c_2(w) = \lambda_{\max}w$.

For any $n \in [N_k, N_{k+1})$, we have

$$\begin{aligned}
V(n+1) - V(\bar{n}) &= \left(A\bar{x}_n + \sum_{j=1}^{m_0} B_j\bar{x}_{n-j}\right)^T P \left(A\bar{x}_n + \sum_{j=1}^{m_0} B_j\bar{x}_{n-j}\right) - \bar{x}_n^TP\bar{x}_n \\
&= \bar{x}_n^TA^TPA\bar{x}_n + 2\bar{x}_n^TA^TP\sum_{j=1}^{m_0}(B_j\bar{x}_{n-j}) \\
&\quad + \sum_{j=1}^{m_0}(\bar{x}_{n-j}^TB_j^T)P\sum_{j=1}^{m_0}(B_j\bar{x}_{n-j}) - \bar{x}_n^TP\bar{x}_n \\
&\leq \bar{x}_n^T(A^TPA - P)\bar{x}_n + \frac{1}{\epsilon}\bar{x}_n^TA^TPA\bar{x}_n \\
&\quad + (1+\epsilon)\sum_{j=1}^{m_0}(\bar{x}_{n-j}^TB_j^T)P\sum_{j=1}^{m_0}(B_j\bar{x}_{n-j}) \\
&\leq \bar{x}_n^T\left(\left(1 + \frac{1}{\epsilon}\right)A^TPA - P\right)\bar{x}_n + (1+\epsilon)\sum_{j=1}^{m_0}(\bar{x}_{n-j}^TB_j^T)P\sum_{j=1}^{m_0}(B_j\bar{x}_{n-j}) \\
&\leq \bar{x}_n^T\left(\left(1 + \frac{1}{\epsilon}\right)A^TPA - P\right)\bar{x}_n + (1+\epsilon)\sum_{i=1}^{m_0}\sum_{j=1}^{m_0}(\bar{x}_{n-i}^TB_i^TPB_j\bar{x}_{n-j}) \\
&\leq \bar{x}_n^T\left(\left(1 + \frac{1}{\epsilon}\right)A^TPA - P\right)\bar{x}_n + (1+\epsilon)m_0\sum_{j=1}^{m_0}(\bar{x}_{n-j}^TB_j^TPB_j\bar{x}_{n-j})
\end{aligned}$$

$$\leq \bar{x}_n^T\left(\left(1+\frac{1}{\epsilon}\right)A^TPA-P\right)\bar{x}_n+(1+\epsilon)m_0\sum_{j=1}^{m_0}(\lambda_j\bar{x}_{n-j}^TP\bar{x}_{n-j})$$

$$\leq((1+\frac{1}{\epsilon})\lambda_0-1)V(\bar{n})+(1+\epsilon)m_0\sum_{j=1}^{m_0}(\lambda_jV(\overline{n-j})). \tag{4.6}$$

Denote $p_1(w)=\alpha_1 w$. If $V(n)\geq p_1(V(\overline{n+s}))=\alpha_1\cdot V(\overline{n+s})$ for any $s\in\mathbb{N}_{-m}$, we get

$$V(n+1)-V(\bar{n})\leq\left(a+\frac{b}{\alpha_1}\right)V(\bar{n})=-q(V(\bar{n})),$$

where $q(w)=-(a+\frac{b}{\alpha_1})w$. From inequality (4.2), condition (ii) of Theorem 3.5 holds.

If there exist some $s_0\in\mathbb{N}_{-m}$, such that $V(n)<p_1(V(\overline{n+s_0}))$, we get

$$\begin{aligned} V(n+1)-V(\bar{n}) &\leq \left(\left(1+\frac{1}{\epsilon}\right)\lambda_0-1\right)\alpha_1V(\overline{n+s_0})+2m_0\sum_{j=1}^{m_0}(\lambda_jV(\overline{n-j}))\\ &\leq \left(\left(1+\frac{1}{\epsilon}\right)\alpha_1\lambda_0-\alpha_1+(1+\epsilon)m_0\sum_{j=1}^{m_0}\lambda_j\right)\|V(\bar{n})\|_r\\ &\leq (a\alpha_1+b+\eta)\|V(\bar{n})\|_r\\ &= p_2(\|V(\bar{n})\|_r), \end{aligned}$$

where $p_2(w)=(a\alpha_1+b+\eta)w$. From inequality (4.3), condition (iii) of Theorem 3.5 holds.

Let $g(w)=\alpha_2 w$. For any $w>0$, from inequalities (4.4) and (4.5), we have

$$g(p_2(w))=\alpha_2\cdot(a\alpha_1+b+\eta)w\leq(1-\gamma)w,$$

$$\begin{aligned} w-(\tau-r)q(p_2(w)) &= w+(\tau-r)\left(a+\frac{b}{\alpha_1}\right)(a\alpha_1+b+\eta)w\\ &\leq (a\alpha_1+b+\eta)w\\ &= p_2(w). \end{aligned}$$

Thus, all the conditions of Theorem 3.5 hold. The proof is complete. □

Theorem 4.2. *Suppose there exist an symmetric positive definite matrix P, let $\lambda_0\geq\max\{\lambda(A^TPAP^{-1})\}$, $\lambda_j\geq\max\{\lambda(B_j^TPB_jP^{-1})\}$ and $\alpha_2=\max\{\lambda((E+I_k)^T(E+I_k))\}$ where $\lambda(S)$ denote all the eigenvalues of matrix S. And there exist $\alpha_1:0<\alpha_1<1$, $\gamma:0<\gamma<1$, $\eta>0$, $\epsilon>0$ such that the following inequalities hold:*

$$\left(a+\frac{b}{\alpha_1}\right)>0 \tag{4.7}$$

$$0<(a\alpha_1+b)<1 \tag{4.8}$$

$$\alpha_2\cdot\left(1+a+\frac{b}{\alpha_1}\right)\leq(1-\gamma)(a\alpha_1+b), \tag{4.9}$$

$$a\alpha_1 + b + \mu\left(a + \frac{b}{\alpha_1}\right) \le 1 - \gamma, \tag{4.10}$$

where $\mu = \max_{k\in\mathbb{N}}\{N_{k+1} - N_k\}$, $a = (1+\frac{1}{\epsilon})\lambda_0 - 1$ *and* $b = (1+\epsilon)m_0\sum_{j=1}^{m_0}\lambda_j$. *Then, the trivial solution of system* (4.1) *is UAS.*

Proof. Denote $x_n = x(n)$ and $\bar{x}_n = \bar{x}(n)$. Let $V(n) = x(n)^T Px(n)$ and $V(\bar{n}) = \bar{x}(n)^T P\bar{x}(n)$. From the proof of Theorem 4.1, condition (i) of Theorem 3.1 holds.

Let $p_1(w) = \alpha_1 w$. If $V(n) \ge p_1(V(\overline{n+s})) = \alpha_1 \cdot V(\overline{n+s})$ for any $s \in \mathbb{N}_{-m}$, by inequality (4.6), we get

$$V(n+1) - V(\bar{n}) \le \left(a + \frac{b}{\alpha_1}\right)V(\bar{n}) = q(V(\bar{n})),$$

where $q(w) = (a + \frac{b}{\alpha_1})w$. From inequality (4.7), condition (ii) of Theorem 3.1 holds.

If there exist some $s_0 \in \mathbb{N}_{-m}$, such that $V(n) < p_1(V(\overline{n+s_0}))$, we get

$$V(n+1) - V(\bar{n}) \le (a\alpha_1 + b)\|V(\bar{n})\|_r = p_2(\|V(\bar{n})\|_r),$$

where $p_2(w) = (a\alpha_1 + b)w$. From inequality (4.8), condition (iii) of Theorem 3.1 holds.

Let $g(w) = \alpha_2 w$. For any $w > 0$, from inequalities (4.9) and (4.10), we have

$$g(w + q(w)) = \alpha_2(1 + a + \frac{b}{\alpha_1})) \cdot w \le (1-\gamma)(a\alpha_1 + b)w,$$

$$p_2(w) + \mu q(w) = (a\alpha_1 + b + \mu(a + \frac{b}{\alpha_1}))w \le (1-\gamma)w.$$

Thus, all the conditions of Theorem 3.1 hold. The proof is complete. □

5. Examples

To illustrate our theorems obtained in the previous sections, we now consider some numerical examples.

Example 5.1. Consider the linear impulsive discrete system with time delay

$$\begin{cases} x(n) = A\bar{x}(n-1) + B\bar{x}(n-2), & n_0 \in \mathbb{N}, n \ge n_0, \\ \bar{x}(n) = \begin{cases} x(n), & n \ne N_k, \\ x(N_k) + I_k(x(N_k)), & n = N_k, k \in \mathbb{N}, \end{cases} \\ x_{n_0} = \phi, \end{cases} \tag{5.1}$$

where $A = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0.6 & 0 \\ 0 & -0.6 \end{bmatrix}$, $I_k = \alpha\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\phi \in \{\phi : \mathbb{N} \times \{-1, 0\} \to \mathbb{R}^2\}$.

Suppose we define the Lyapunov function $V(n) = \|x(n)\|$. For any $\phi : \mathbb{N} \times \{-1, 0\} \to \mathbb{R}^2$ with $\phi(0) \ne 0$, we have

$$\Delta V(n) \le -0.98V(n) + 0.72V(n-1).$$

We are interested in uniformly stabilizing system (5.1) through the use of impulses according to Theorem 3.5. For simplicity, let us assume that the impulse times are equally spaced. In other words, $N_k - N_{k-1} = \tau > 0$ for all k. Take $p_1(w) = p_2(w) = 0.8w$, $g(w) = 1.2w$, $\tau = 5$. If $V(n) \geq p_1(V(n+s)) = 0.8V(n+s)$ for all $s \in \mathbb{N}_{-1}$, we have

$$\begin{aligned} V(n+1) - V(n) &\leq -0.98V(n) + 0.72V(n+s) \\ &\leq -0.08V(n) \\ &= -q(V(n)). \end{aligned}$$

If for any n, there exist some $s \in \mathbb{N}_{-1}$ such that $V(n) < p_1(V(n+s)) = 0.8V(n+s)$, we have

$$\begin{aligned} V(n+1) &\leq V(n) - 0.98V(n) + 0.72V(n+s) \\ &\leq 0.736V(n+s) \\ &\leq 0.736U(n) \\ &\leq 0.8U(n). \end{aligned}$$

It can be seen that all the conditions of Theorem 3.5 are satisfied. Figure 1 shows that the system is uniform stable.

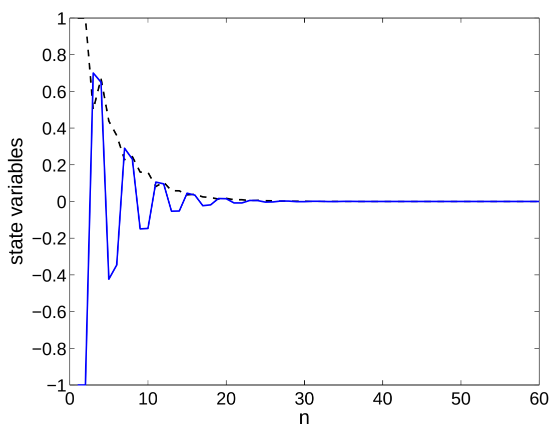

FIGURE 1. Simulation of state variables in example 5.1.

Example 5.2. Consider the same type of linear impulsive discrete system with time delay in Example 5.1 with $A = \begin{bmatrix} 0.01 & 1.01 \\ 0.5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$, $I_k = \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\phi \in \{\phi : \mathbb{N} \times \{-1, 0\} \to \mathbb{R}^2\}$.

Let $\lambda_0 = 1.05$, $\lambda_1 = 0.005$, $\alpha_1 = 0.88$, $\alpha_2 = 0.7$ $\gamma = 0.1$, $\epsilon = 20$ and $\mu = \max_{k=1,2,\ldots}\{N_k - N_{k-1}\} = 3$. Thus, all the conditions of Theorem 4.2 are satisfied

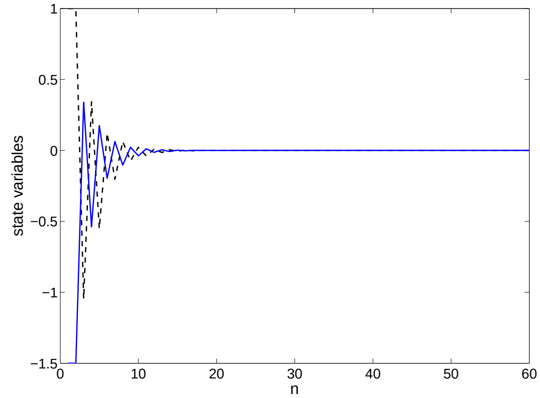

FIGURE 2. Simulation of state variables in example 5.2 .

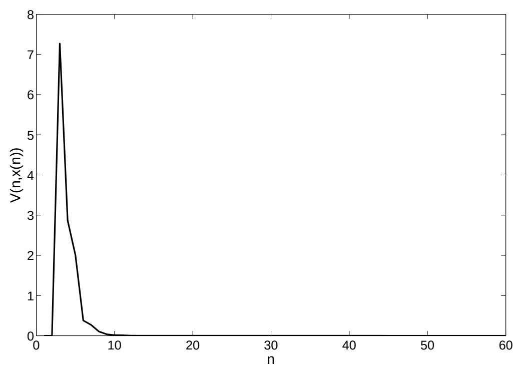

FIGURE 3. Simulation of $V(n, x(n))$ in example 5.2 .

with $P = \begin{bmatrix} 5.8922 & 0.0891 \\ 0.0891 & 7.6430 \end{bmatrix}$. The simulation results are shown in Figure 2 and Figure 3, which show that the system is uniformly asymptotically stable.

Remark 5.3. It can be seen that system (5.1) in Example 5.1 is still uniformly stable without impulse. But the impulse is necessary to achieve uniform stability in Example 5.2.

Example 5.4. Consider the nonlinear impulsive discrete system with time delay:

$$\begin{cases} x(n+1) = A\bar{x}(n) + F(n, \bar{x}(n), \bar{x}(n-h(n))), & n_0 \in \mathbb{N}, n \geq n_0, \\ \bar{x}(n) = \begin{cases} x(n), \\ x(N_k) + I_k(x(N_k)), \end{cases} & \begin{matrix} n \neq N_k, \\ n = N_k, k \in \mathbb{N}, \end{matrix} \\ x_{n_0} = \phi, \end{cases} \tag{5.2}$$

where $m = 2,\ h(n) = 1$ or $2,\ A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.1 & 0.1 \\ 0.1 & 0 & 0.5 \end{bmatrix}$ and $F(n, x(n), x(n-h(n))) = \frac{1}{4}((x_1(n-h(n))/(1+\sin^2 n+\|x(n)\|^2), x_2(n-h(n))\sin(x_3(n)), x_3(n-h(n))cos(x_3(n)))^T$.

We are interested in uniformly asymptotically stabilizing system (5.2) through the use of impulses according to Theorem 3.1. For simplicity, let us assume that the impulse times are equally spaced. In other words, $N_k - N_{k-1} = \tau > 0$ for all k. Let $V(n, x) = x^T x$. Take $p_1(w) = 0.9w,\ g(w) = 1.3w$.

If $V(n) \geq p_1(V(n+s) = 0.9V(n+s)$ for all $s \in \mathbb{N}_{-r}$, we have

$$V(n+1) - V(n) \quad \leq \quad -0.2408V(n).$$

If for any n and some $s \in \mathbb{N}_{-r}$, $V(n) < p_1(V(n+s)) = 0.9V(n+s)$, we have

$$V(n+1) \quad \leq \quad 0.6833V(n).$$

Choose $\gamma = 0.1$. It can be seen that all the conditions of Theorem 3.1 are satisfied with $\tau \geq 3.925$. Figure 4 shows that the system is uniformly asymptotically stable with $\tau = 4$.

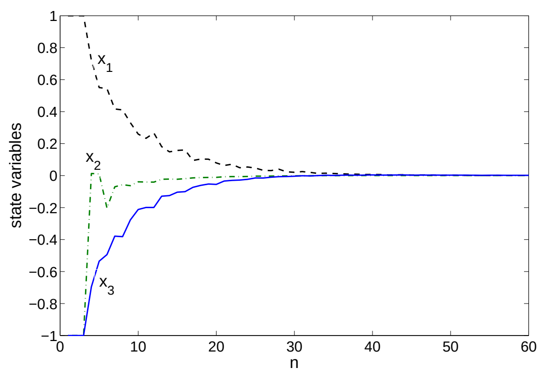

FIGURE 4. Simulation of state variables in example 5.4.

6. **Conclusion**

In this paper, we have studied impulsive discrete systems with time delay. By using Lyapunov function and the Razumikhin technique we have established several uniform stability and uniform asymptotic stability criteria for such systems. We have shown that impulses can be used to stabilize a unstable system. Numerical examples have been provided to demonstrate our theoretical results.

7. **Acknowledgement**

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REFERENCES

[1] J. Nilsson, B. Bernhardsson, and B. Wittenmark. Stochastic analysis and control of real-time systems with random time delays. *Automatica*, 34:57–64, 1998.

[2] E. Biberovic, A. Iftar, and H. Ozbay. A solution to the robust flow control problem for networks with multiple bottlenecks. In *IEEE Conference on Decision and Control*, 3:2303–2308, 2001.

[3] A. Leleve, P. Fraisse, and P. Dauchez. Telerobotics over IP networks: Towards a low-level real-timearchitecture. In *IEEE/RSJ International Conference on Intelligent Robots and Systems*, 2:643–648, 2001.

[4] D. Chatterjee and D. Liberzon. Stability analysis of deterministic and stochastic switched systems via a comparison principle and multiple Lyapunov functions. *SIAM Journal on Control and Optimization*, 45:174–206, 2006.

[5] J.K. Hale and S.M.V. Lunel. *Introduction to functional differential equations*. Springer Verlag, 1993.

[6] K. Gu, V. Kharitonov, and J. Chen. *Stability of time-delay systems*. Birkhauser, 2003.

[7] E. Fridman. New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems. *Systems & Control Letters*, 43:309–319, 2001.

[8] V.L. Kharitonov and A.P. Zhabko. Lyapunov–Krasovskii approach to the robust stability analysis of time-delay systems. *Automatica*, 39:15–20, 2003.

[9] C.R. Knospe and M. Roozbehani. Stability of linear systems with interval time-delay. In *American Control Conference*, 1458–1463, 2003.

[10] B. Liu and D.J. Hill. Comparison principle and stability of discrete-time impulsive hybrid systems. *IEEE Transactions on Circuits and Systems I : fundamental theory and applications*, 56:233–246, 2009.

[11] X. Liu. Stability of impulsive control systems with time delay. *Mathematical and Computer Modelling*, 39:511–519, 2004.

[12] X. Liao, L. Wang, and P. Yu. *Stability of dynamical systems*. Elsevier Science, 2007.

[13] X.Z. Liu. Impulsive stabilization of nonlinear systems. *IMA Journal of Mathematical Control and Information*, 10:11–19, 1993.

[14] G. Ballinger and X. Liu. Existence, uniqueness and boundedness results for impulsive delay differential equations. *Applicable Analysis*, 74:71–93, 2000.

[15] B. Liu, X. Liu, K.L. Teo, and Q. Wang. Razumikhin-type theorems on exponential stability of impulsive delay systems. *IMA Journal of Applied Mathematics*, 71:47–61, 2006.

[16] J.E. Prussing, L.J. Wellnitz, and W.G. Heckathorn. Optimal impulsive time-fixed direct-ascent interception. *Journal of Guidance Control Dynamics*, 12:487–494, 1989.

[17] X. Liu and A.R. Willms. Impulsive controllability of linear dynamical systems with applications to maneuvers of spacecraft. *Mathematical Problems in Engineering*, 2:277–299, 1996.

[18] X. Liu and K. Rohlf. Impulsive control of a Lotka-Volterra system. *IMA Journal of Mathematical Control and Information*, 15:269–284, 1998.

[19] X. Liu and G. Ballinger. Uniform asymptotic stability of impulsive delay differential equations. *Computers and Mathematics with Applications*, 41:903–915, 2001.

[20] S. Zhang and M.P. Chen. A new Razumikhin theorem for delay difference equations. *Computers and Mathematics with Applications*, 36:405–412, 1998.

[21] B. Liu and H.J. Marquez. Razumikhin-type stability theorems for discrete delay systems. *Automatica*, 43:1219–1225, 2007.