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SOME NEW DYNAMIC INEQUALITIES ON DISCRETE TIME SCALES

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ABSTRACT. In this paper we prove some new dynamic inequalities on discrete time scales. These new inequalities contain some generalizations of the discrete inequalities due to Hardy, Copson, Leindler and Walsh. The main results will be proved using a general algebraic inequality and Keller's chain rule on time scales.

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1. Introduction

The classical Hardy inequality states that for $f \geq 0$ integrable over any finite interval $(0, x)$ and f^k integrable and convergent over $(0, \infty)$ and $k > 1$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^k dx \leq \left(\frac{k}{k-1}\right)^k \int_0^\infty f^k(x)dx. \tag{1.1}$$

The constant $(k/(k-1))^k$ is the best possible. This inequality is the continuous version of Hardy's discrete inequality

$$\sum_{n=1}^\infty \left(\frac{1}{n}\sum_{i=1}^n a_i\right)^k \leq \left(\frac{k}{k-1}\right)^k \sum_{n=1}^\infty a_n^k, \quad k > 1. \tag{1.2}$$

These Hardy type inequalities have been extensively studied in literature; we refer the reader to the books [17, 18, 22] and the papers [7, 14, 15, 16, 20, 21, 24]. Hardy in [10] proved that if $c > 1$, $\lambda(n) > 0$, $a(n) > 0$, and $\Lambda(n)$ and $A(n)$ are defined by

$$\Lambda(n) = \sum_{i=1}^n \lambda(i), \text{ and } A(n) = \sum_{i=1}^n a(i)\lambda(i), \tag{1.3}$$

then

$$\sum_{n=1}^\infty \lambda(n)\left(\frac{A(n)}{\Lambda(n)}\right)^c \leq \left(\frac{c}{c-1}\right)^c \sum_{n=1}^\infty \lambda(n)a^c(n). \tag{1.4}$$

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Copson in [5, Theorems 1.1, 1.2] proved that if $k > 1$, $\lambda(n) > 0$ and $c > 1$, then

$$\sum_{n=1}^{\infty} \lambda(n) \frac{(A(n))^k}{\Lambda^c(n)} \leq \left(\frac{k}{c-1}\right)^k \sum_{n=1}^{\infty} \lambda(n) \Lambda^{k-c}(n) a^k(n), \tag{1.5}$$

and if $0 < k < 1$ and $c < 1$, then

$$\sum_{n=1}^{\infty} \lambda(n) \Lambda^{k-c}(n) a^k(n) \leq \left(\frac{N}{k}\right)^k \sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda^c(n)} \left(\sum_{i=1}^{n} a(i)\lambda(i)\right)^k, \tag{1.6}$$

where N is 1 or $1-c$ according as $c \geq 0$ or $c < 0$. Copson proved his results by using the algebraic inequalities

$$x^{k-1}y - y^k \leq (k-1)x^{k-1}(x-y) \text{ when } k > 1, \tag{1.7}$$

and

$$(k-1)x^{k-1}y \geq (k-1)x^k + y^k \text{ when } 0 < k < 1,$$

where x and y are positive. Using the inequality (1.7), Elliott [8] proved that

$$\sum_{n=1}^{\infty} \lambda(n) \left(\frac{A(n)}{\Lambda(n)}\right)^k \leq \left(\frac{k}{k-1}\right)^k \sum_{n=1}^{\infty} \lambda(n) a^k(n) \text{ when } k < 0. \tag{1.8}$$

In [19] Leindler extended the results by Copson and proved that if $\sum_{i=n}^{\infty} \lambda(i) < \infty$, $k > 1$ and $0 \leq c < 1$, then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{(\Lambda^*(n))^c} (A(n))^k \leq \left(\frac{k}{1-c}\right)^k \sum_{n=1}^{\infty} \lambda(n) (\Lambda^*(n))^{k-c} a^k(n), \tag{1.9}$$

where $\Lambda^*(n) = \sum_{i=n}^{\infty} \lambda(i)$ and $A(n)$ is defined as in (1.3). Walsh [28] extended the results obtained by Copson and Elliott and proved that if $b \geq 1$, $c \geq 1$, and for all n, $a(n) > 0$, $\lambda(n) > 0$, $p(n) > 0$, and

$$p(n)\Lambda(n)/P(n) \geq p(n+1)\Lambda(n+1)/P(n+1) \tag{1.10}$$

where

$$\Lambda(n) = \sum_{s=a}^{n} \lambda(s), \; A(n) = \sum_{s=a}^{n} \lambda(s) a(s), \text{ and } P(n) = \sum_{s=a}^{n} p(s)\lambda(s),$$

then

$$\begin{aligned} &\sum_{n=a}^{\infty} \frac{\lambda(n) (A(n))^c p^b(n) (\Lambda(n))^{b-1}}{(P(n))^{b+c-1}} \\ &\leq \left(\frac{c}{c-1}\right)^b \sum_{n=a}^{\infty} \lambda(n) (P(n))^{1-c} (\Lambda(n))^{b-1} a^b(n) (A(n))^{c-b}. \end{aligned} \tag{1.11}$$

The inequality (1.11) was proved using the general algebraic inequality

$$c(1-x)^b \geq b\alpha^{b-1}(1-\beta^{1-c}x^c) + b(c-1)\alpha^{b-1}(1-\beta) + c(1-b)\alpha^b, \tag{1.12}$$

where $b \geq 1$, $c \geq 1$, $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ and x is assumed to lie in the range $(0,1)$. One can easily see that inequality (1.11) contains the inequalities due to Hardy and Copson.

The aim in this paper is to prove some new dynamic inequalities on discrete time scales, i.e., where the domain of the unknown function is a so-called discrete time scale $\mathbb{T}$ (i.e. $\mu(t) \neq 0$). The inequalities, as special cases, contain the above discrete inequalities when $\mathbb{T} = \mathbb{N}$, and can be used to derive new discrete inequalities when $\mathbb{T} = h\mathbb{N}$ and other different types of discrete (i.e. $\mu(t) \neq 0$) time scales. For dynamic inequalities of Hardy's type on time scales, we refer the reader to the papers [23, 25, 26, 27] and the references cited therein.

2. **Main Results**

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by: $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, where $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$, is said to be left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, is right-dense if $\sigma(t) = t$, is left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided g is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$.

The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t \geq 0$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. In this paper we present results on discrete (i.e. $\mu(t) \neq 0$) time scales (time scales with isolated points). Fix $t \in \mathbb{T}$ and let $x : \mathbb{T} \to \mathbb{R}$. Define $x^{\Delta}(t)$ to be the number (if it exists) with the property that given any $\epsilon > 0$ there is a neighborhood U of t with $|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|$, for all $s \in U$. In this case, we say $x^{\Delta}(t)$ is the (delta) derivative of x at t and that x is (delta) differentiable at t. We will frequently use the following results due to Hilger [13]. Throughout the paper we will assume that $g : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$.

(i) If g is differentiable at t, then g is continuous at t.

(ii) If g is continuous at t and t is right-scattered, then g is differentiable at t with $g^{\Delta}(t) = \frac{g(\sigma(t)) - g(t)}{\mu(t)}$.

(iii) If g is differentiable at t, then $g(\sigma(t)) = g(t) + \mu(t)g^{\Delta}(t)$.

Without loss of generality, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. For more details of time scale analysis we refer the reader to the two books by Bohner and Peterson [2], [3] which summarize and organize much of the time scale calculus. The cases when the time scale is equal to the the integers represent the classical theories of discrete inequalities. In this paper, we will refer to the (delta) integral which we can define as follows. If $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(s)\Delta s := G(t) - G(a)$. It can be shown (see [2]) that if $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}$. An infinite integral is defined as $\int_a^{\infty} f(t)\Delta t = \lim_{b\to\infty} \int_a^b f(t)\Delta t$. We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$, here $g^{\sigma} = g \circ \sigma$) of two differentiable function f and g

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}, \text{ and } \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}. \tag{2.1}$$

The following simple consequence of Keller's chain rule [2, Theorem 1.90] and the integration by parts formula on time scales is needed in the proof of the main results

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_0^1 [hx^{\sigma} + (1-h)x]^{\gamma-1}\, dh x^{\Delta}(t), \tag{2.2}$$

and

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{2.3}$$

Throughout this section, we will assume that the functions in the statements of the theorems are nonnegative and rd-continuous functions and the integrals considered are assumed to exist.

In the following we prove the time scale version of the Walsh type inequality on discrete (i.e. $\mu(t) \neq 0$) time scales. The idea is to use Walsh's inequality (1.12) and the chain rule (2.2).

Theorem 2.1. *Let $\mathbb{T}$ be a discrete time scale and $b > 1$, $c > 1$ and define*

$$\begin{cases} \Lambda(t) := \int_a^t \lambda(s)\Delta s, \\ A(t) := \int_a^t a(s)\lambda(s)\Delta s, \qquad \text{for any } t \in [a,\infty)_{\mathbb{T}}. \\ P(t) := \int_a^t p(s)\lambda(s)\Delta s, \end{cases} \tag{2.4}$$

If

$$(P^{\sigma}(t))^{1-c}(A^{\sigma}(t))^{c} \geq (P(t))^{1-c}(A(t))^{c}, \tag{2.5}$$

then

$$\int_a^{\infty} \frac{\lambda(t)p^b(t)\,(A^{\sigma}(t))^c\,(\Lambda^{\sigma}(t))^{b-1}}{(P^{\sigma}(t))^{b+c-1}}\Delta t \leq \left(\frac{c}{c-1}\right)^b \int_a^{\infty} \frac{\lambda(t)a^b(t)\,(A^{\sigma}(t))^{c-b}}{(P^{\sigma}(t))^{c-1}\,(\Lambda^{\sigma}(t))^{1-b}}\Delta t. \tag{2.6}$$

Proof. Fix $t \in (a, \infty)_{\mathbb{T}}$. Let

$$x := \frac{A(t)}{A^\sigma(t)} < 1, \qquad \beta := \frac{P(t)}{P^\sigma(t)} < 1, \text{ and } \quad \alpha := \eta \frac{\lambda(t)}{\Lambda^\sigma(t)}, \tag{2.7}$$

where $\eta\,(= \eta(t))$ is positive and will be determined below such that $\alpha < 1$. As a result of these substitutions in (1.12), we get that

$$\begin{aligned} c\left(1 - \frac{A(t)}{A^\sigma(t)}\right)^b &= c\left(\frac{\mu(t)A^\Delta(t)}{A^\sigma(t)}\right)^b = c\left(\frac{\mu(t)a(t)\lambda(t)}{A^\sigma(t)}\right)^b \\ &\geq b\left(\eta\frac{\lambda(t)}{\Lambda^\sigma(t)}\right)^{b-1}\left(1 - \left(\frac{P(t)}{P^\sigma(t)}\right)^{1-c}\left(\frac{A(t)}{A^\sigma(t)}\right)^c\right) \\ &\quad + b(c-1)\frac{\lambda^{b-1}(t)\mu(t)p(t)\lambda(t)}{(\Lambda^\sigma(t))^{b-1}P^\sigma(t)}\eta^{b-1} + \frac{c(1-b)\lambda^b(t)}{(\Lambda^\sigma(t))^b}\eta^b. \end{aligned} \tag{2.8}$$

Multiplying both sides by

$$\lambda^{1-b}(t)\,(A^\sigma(t))^c\,(\Lambda^\sigma(t))^{b-1}\,(P^\sigma(t))^{1-c}, \tag{2.9}$$

gives

$$\begin{aligned} &\frac{c\lambda(t)(a(t)\mu(t))^b\,(A^\sigma(t))^{c-b}}{(P^\sigma(t))^{c-1}\,(\Lambda^\sigma(t))^{1-b}} \\ &\geq b\eta^{b-1}\left[(P^\sigma(t))^{1-c}\,(A^\sigma(t))^c - (P(t))^{1-c}\,(A(t))^c\right] \\ &\quad + \lambda(t)\,(A^\sigma(t))^c\left[\frac{b(c-1)\mu(t)p(t)}{(P^\sigma(t))^c}\eta^{b-1} + \frac{c(1-b)\eta^b}{(P^\sigma(t))^{c-1}\Lambda^\sigma(t)}\right]. \end{aligned} \tag{2.10}$$

Now, we consider the last term on the right hand side, namely

$$f(\eta) := M\eta^b + K\eta^{b-1}, \tag{2.11}$$

as a function of η, where

$$K := \frac{b(c-1)\mu(t)p(t)}{(P^\sigma(t))^c}, \quad \text{and } M := \frac{c(1-b)\,(P^\sigma(t))^{1-c}}{\Lambda^\sigma(t)}.$$

By differentiation, we see that the function $f(\eta)$ has a maximum value at

$$\eta = \frac{c-1}{c}\frac{p(t)\mu(t)\Lambda^\sigma(t)}{P^\sigma(t)} > 0, \tag{2.12}$$

and the maximum value of $f(\eta)$ is given by

$$\max_{\eta>0} f(\eta) = \frac{(c-1)^b}{c^{b-1}}\frac{p^b(t)\mu^b(t)\,(\Lambda^\sigma(t))^{b-1}}{(P^\sigma(t))^{b+c-1}}. \tag{2.13}$$

From (2.7) and (2.12), we see that (with the above η),

$$\begin{aligned} \alpha &= \eta\frac{\lambda(t)}{\Lambda^\sigma(t)} = \frac{c-1}{c}\frac{p(t)\mu(t)\lambda(t)}{P^\sigma(t)} = \frac{c-1}{c}\frac{\mu(t)P^\Delta(t)}{P^\sigma(t)} \\ &= \frac{c-1}{c}\frac{P^\sigma(t) - P(t)}{P^\sigma(t)}, \end{aligned}$$

and note $0<\alpha<1$. Then (2.10) and (2.13) imply that

$$\frac{c\lambda(t)(a(t)\mu(t))^b\left(A^\sigma(t)\right)^{c-b}}{\left(P^\sigma(t)\right)^{c-1}\left(\Lambda^\sigma(t)\right)^{1-b}} \geq b\eta^{b-1}\left[\left(P^\sigma(t)\right)^{1-c}\left(A^\sigma(t)\right)^c-\left(P(t)\right)^{1-c}\left(A(t)\right)^c\right] + \frac{(c-1)^b}{c^{b-1}}\frac{\lambda(t)\mu^b(t)p^b(t)\left(A^\sigma(t)\right)^c\left(\Lambda^\sigma(t)\right)^{b-1}}{(P^\sigma(t))^{b+c-1}}.$$

Using the condition (2.5), we have that

$$\frac{c\lambda(t)a^b(t)\left(A^\sigma(t)\right)^{c-b}}{\left(P^\sigma(t)\right)^{c-1}\left(\Lambda^\sigma(t)\right)^{1-b}} \geq \frac{(c-1)^b}{c^{b-1}}\frac{\lambda(t)\left(A^\sigma(t)\right)^c\left(\Lambda^\sigma(t)\right)^{b-1}}{p^{-b}(t)(P^\sigma(t))^{b+c-1}}. \tag{2.14}$$

Integration from a to ∞ yields

$$c\int_a^\infty \frac{\lambda(t)a^b(t)\left(A^\sigma(t)\right)^{c-b}}{\left(P^\sigma(t)\right)^{c-1}\left(\Lambda^\sigma(t)\right)^{1-b}}\Delta t \geq \frac{(c-1)^b}{c^{b-1}}\int_a^\infty \frac{\lambda(t)\left(A^\sigma(t)\right)^c\left(\Lambda^\sigma(t)\right)^{b-1}}{p^{-b}(t)(P^\sigma(t))^{b+c-1}}\Delta t. \tag{2.15}$$

Thus

$$\int_a^\infty \frac{\lambda(t)\left(A^\sigma(t)\right)^c\left(\Lambda^\sigma(t)\right)^{b-1}p^b(t)}{(P^\sigma(t))^{b+c-1}}\Delta t \leq \left(\frac{c}{c-1}\right)^b\int_a^\infty \frac{\lambda(t)a^b(t)\left(A^\sigma(t)\right)^{c-b}}{\left(P^\sigma(t)\right)^{c-1}\left(\Lambda^\sigma(t)\right)^{1-b}}\Delta t,$$

which is the desired inequality (2.6). The proof is complete. □

When $c>b$, we have the following result.

Corollary 2.2. *Let $\mathbb{T}$ be a discrete time scale and $c>b>1$ and $\Lambda(t)$, $A(t)$ and $P(t)$ are defined as in Theorem* 2.1. *If* (2.5) *holds, then*

$$\int_a^\infty \frac{\lambda(t)\left(A^\sigma(t)\right)^c p^b(t)\left(\Lambda^\sigma(t)\right)^{b-1}}{(P^\sigma(t))^{b+c-1}}\Delta t \leq \left(\frac{c}{b-1}\right)^c\int_a^\infty \frac{\lambda(t)a^b(t)\left(A^\sigma(t)\right)^{c-b}}{\left(P^\sigma(t)\right)^{c-1}\left(\Lambda^\sigma(t)\right)^{1-b}}\Delta t. \tag{2.16}$$

As a special case when $p(t)=1$ for all t, we see that $P^\sigma(t)=\Lambda^\sigma(t)$ and then (2.16) reduces to

$$\int_a^\infty \frac{\lambda(t)\left(A^\sigma(t)\right)^c}{(\Lambda^\sigma(t))^b}\Delta t \leq \left(\frac{c}{b-1}\right)^c\int_a^\infty \lambda(t)(A^\sigma(t))^{c-b}a^b(t)\Delta t. \tag{2.17}$$

As a special case when $b=c$ and $p(t)=1$ for all t, we see that $P(t)=\Lambda(t)$ and then (2.17) reduces to

$$\int_a^\infty \frac{\lambda(t)\left(A^\sigma(t)\right)^c}{(\Lambda^\sigma(t))^c}\Delta t \leq \left(\frac{c}{c-1}\right)^c\int_a^\infty \lambda(t)a^c(t)\Delta t. \tag{2.18}$$

As a special case when $\lambda(t)=1$, we have the inequality

$$\int_a^\infty \left(\frac{1}{\sigma(t)-a}\int_a^{\sigma(t)} a(s)\Delta s\right)^c\Delta t \leq \left(\frac{c}{c-1}\right)^c\int_a^\infty a^c(t)\Delta t. \tag{2.19}$$

As a special case when $\mathbb{T}=\mathbb{N}$, we have that $\mu(t)=1$ and then the inequality (2.17) becomes the Copson inequality

$$\sum_{n=a}^\infty \frac{\lambda(n)A^c(n)}{\Lambda^b(n)} \leq \left(\frac{c}{b-1}\right)^c\sum_{n=a}^\infty \lambda(n)(A(n))^{c-b}a^c(n), \tag{2.20}$$

where $c > 1$ and $b > 1$, and for all n, $a(n) > 0$, $\lambda(n) > 0$ and

$$A(n) = \sum_{s=a}^{n} \lambda(s)a(s), \text{ and } \Lambda(n) = \sum_{s=a}^{n} \lambda(s). \tag{2.21}$$

As a special case when $\mathbb{T} = \mathbb{N}$, we have that $\mu(t) = 1$ and then the inequality (2.18) becomes the following Copson discrete inequality

$$\sum_{n=a}^{\infty} \lambda(n) \left(\frac{A(n)}{\Lambda(n)}\right)^{c} \leq \left(\frac{c}{c-1}\right)^{c} \sum_{n=a}^{\infty} \lambda(n)a^{c}(n), \tag{2.22}$$

where $c > 1$ and $A(n)$ and $\Lambda(n)$ are defined as in (2.21). As a special case when $\mathbb{T} = \mathbb{N}$ and $a = 1$, we have that $\mu(t) = 1$ and then the inequality (2.18) becomes the following Hardy discrete inequality

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{s=1}^{n} a(s)\right)^{c} \leq \left(\frac{c}{c-1}\right)^{c} \sum_{n=1}^{\infty} a^{c}(n). \tag{2.23}$$

Theorem 2.3. *Let $\mathbb{T}$ be a discrete time scale and $c > 1$, $b > 1$ and $\Lambda(t)$, $A(t)$ and $P(t)$ are defined as in Theorem 2.1. If*

$$\frac{p^{\sigma}(t)\Lambda^{\sigma^2}(t)}{P^{\sigma^2}(t)} \leq \frac{p(t)\Lambda^{\sigma}(t)}{P^{\sigma}(t)}, \tag{2.24}$$

then

$$\int_{a}^{\infty} \frac{\lambda(t)\mu^{b-1}(t) \left(A^{\sigma}(t)\right)^{c} p^{b}(t)}{\left(P^{\sigma}(t)\right)^{b+c-1} \left(\Lambda^{\sigma}(t)\right)^{1-b}} \Delta t \leq \left(\frac{c}{b-1}\right)^{c} \int_{a}^{\infty} \frac{\lambda(t)\mu^{b-1}(t)a^{b}(t) \left(A^{\sigma}(t)\right)^{c-b}}{\left(P^{\sigma}(t)\right)^{c-1} \left(\Lambda^{\sigma}(t)\right)^{1-b}} \Delta t. \tag{2.25}$$

Proof. Fix $t \in (a, \infty)_{\mathbb{T}}$. We proceed as in the proof of Theorem 2.1, to get

$$\begin{aligned} & c\frac{\left(\mu(t)a(t)\right)^{b} \lambda^{b}(t)}{\left(A^{\sigma}(t)\right)^{b}} \\ & \geq b\left(\eta\right)^{b-1} \frac{\lambda^{b-1}(t) \left(\left(P^{\sigma}(t)\right)^{1-c} \left(A^{\sigma}(t)\right)^{c} - \left(P(t)\right)^{1-c} \left(A(t)\right)^{c}\right)}{\left(P^{\sigma}(t)\right)^{1-c} \left(A^{\sigma}(t)\right)^{c} \left(\Lambda^{\sigma}(t)\right)^{b-1}} \\ & \quad + b(c-1)\frac{\lambda^{b-1}(t)\mu(t)p(t)\lambda(t)}{\left(\Lambda^{\sigma}(t)\right)^{b-1} P^{\sigma}(t)}\eta^{b-1} + \frac{c(1-b)\lambda^{b}(t)}{\left(\Lambda^{\sigma}(t)\right)^{b}}\eta^{b}. \end{aligned} \tag{2.26}$$

Multiplying both sides of (2.26) by

$$\frac{\lambda^{1-b}(t) \left(A^{\sigma}(t)\right)^{c} \left(\Lambda^{\sigma}(t)\right)^{b-1} \left(P^{\sigma}(t)\right)^{1-c}}{\mu(t)},$$

we get that

$$\begin{aligned}
&\frac{c\lambda(t)(a(t)\mu(t))^b\left(A^\sigma(t)\right)^{c-b}}{\mu(t)\left(P^\sigma(t)\right)^{c-1}\left(\Lambda^\sigma(t)\right)^{1-b}} \\
&\quad\geq b\left(\eta\right)^{b-1}\frac{\left[\left(P^\sigma(t)\right)^{1-c}\left(A^\sigma(t)\right)^c-\left(P(t)\right)^{1-c}\left(A(t)\right)^c\right]}{\mu(t)} \\
&\quad\quad+\frac{\lambda(t)\left(A^\sigma(t)\right)^c}{\mu(t)}\left[\frac{b(c-1)\mu(t)p(t)}{\left(P^\sigma(t)\right)^c}\eta^{b-1}+\frac{c(1-b)}{\left(P^\sigma(t)\right)^{c-1}\left(\Lambda^\sigma(t)\right)}\eta^b\right]. \qquad (2.27)
\end{aligned}$$

Now, we consider the last term on the right hand side, namely

$$f_1(\eta):=M_1\eta^b+K_1\eta^{b-1},$$

as a function of η, where

$$K_1:=\frac{b(c-1)\mu(t)p(t)}{\left(P^\sigma(t)\right)^c}, \quad \text{and } M:=\frac{c(1-b)\left(P^\sigma(t)\right)^{1-c}}{\Lambda^\sigma(t)}.$$

By differentiation, we see that the function $f_1(\eta)$ has a maximum value at

$$\eta=\frac{c-1}{c}\frac{p(t)\mu(t)\Lambda^\sigma(t)}{P^\sigma(t)}>0,$$

and the maximum value of $f_1(\eta)$ is given by

$$\max_{\eta>0}f_1(\eta)=\frac{(c-1)^b}{c^{b-1}}\frac{p^b(t)\mu^b(t)\left(\Lambda^\sigma(t)\right)^{b-1}}{(P^\sigma(t))^{b+c-1}}.$$

Fix $T\in(a,\infty)_{\mathbb{T}}$. Integrate from a to T to obtain (the $\eta(t)$ is as above)

$$\begin{aligned}
&c\int_a^T\frac{\lambda(t)(a(t)\mu(t))^b\left(A^\sigma(t)\right)^{c-b}}{\mu(t)\left(P^\sigma(t)\right)^{c-1}\left(\Lambda^\sigma(t)\right)^{1-b}}\Delta t \\
&\quad\geq b\int_a^T\left(\eta(t)\right)^{b-1}\frac{\left[\left(P^\sigma(t)\right)^{1-c}\left(A^\sigma(t)\right)^c-\left(P(t)\right)^{1-c}\left(A(t)\right)^c\right]}{\mu(t)}\Delta t \\
&\quad\quad+\frac{(c-1)^b}{c^{b-1}}\int_a^T\frac{\lambda(t)\mu^{b-1}(t)\left(A^\sigma(t)\right)^c\left(\Lambda^\sigma(t)\right)^{b-1}}{p^{-b}(t)(P^\sigma(t))^{b+c-1}}\Delta t. \qquad (2.28)
\end{aligned}$$

Using the product rule and the integration by parts formula, we see that

$$\begin{aligned}
&\int_a^T\left(\eta(t)\right)^{b-1}\frac{\left[\left(P^\sigma(t)\right)^{1-c}\left(A^\sigma(t)\right)^c-\left(P(t)\right)^{1-c}\left(A(t)\right)^c\right]}{\mu(t)}\Delta t \\
&=\int_a^T\left(\eta(t)\right)^{b-1}\left[\left(P(t)\right)^{1-c}\left(A(t)\right)^c\right]^\Delta\Delta t \\
&=\left(\eta(T)\right)^{b-1}\left[\left(P(T)\right)^{1-c}\left(A(T)\right)^c\right]-\int_a^T\left(\left(\eta(t)\right)^{b-1}\right)^\Delta\left(P^\sigma(t)\right)^{1-c}\left(A^\sigma(t)\right)^c\Delta t,
\end{aligned}$$

since $A(a) = P(a) = 0$. From the chain rule and condition (2.24), we see that

$$\begin{aligned} \left((\eta(t))^{b-1}\right)^{\Delta} &= (b-1)(\eta(t))^{\Delta}\int_0^1 [h\eta^{\sigma}+(1-h)\eta]^{b-2}\,dh \\ &= \frac{(b-1)}{\mu(t)}\left(\frac{p^{\sigma}(t)\Lambda^{\sigma^2}(t)}{P^{\sigma^2}(t)}-\frac{p(t)\Lambda^{\sigma}(t)}{P^{\sigma}(t)}\right) \\ &\quad\times\int_0^1 [h\eta^{\sigma}+(1-h)\eta]^{b-2}\,dh \leq 0. \end{aligned}$$

This implies that

$$\int_a^T (\eta(t))^{b-1}\frac{\left[(P^{\sigma}(t))^{1-c}(A^{\sigma}(t))^{c}-(P(t))^{1-c}(A(t))^{c}\right]}{\mu(t)}\Delta t \geq 0.$$

Using this in (2.28) and we obtain

$$\int_a^T \frac{c\lambda(t)a^{b}(t)\mu^{b-1}(t)\,(A^{\sigma}(t))^{c-b}}{(P^{\sigma}(t))^{c-1}(\Lambda^{\sigma}(t))^{1-b}}\Delta t \geq \frac{(c-1)^{b}}{c^{b-1}}\int_a^T \frac{\lambda(t)\mu^{b-1}(t)\,(A^{\sigma}(t))^{c}(\Lambda^{\sigma}(t))^{b-1}}{p^{-b}(t)(P^{\sigma}(t))^{b+c-1}}\Delta t.$$

Let $T \to \infty$ and we obtain (2.25). The proof is complete. □

Remark 2.4. As a special case when $\mathbb{T} = \mathbb{N}$, we see that $\sigma(t) = t+1$ and then condition (2.24) becomes

$$\frac{p(n)\Lambda(n+1)}{P(n+1)} \geq \frac{p(n+1)\Lambda(n+2)}{P(n+2)}, \tag{2.29}$$

where $\Lambda(n+1) = \sum_{s=a}^{n}\lambda(n)$ and $P(n+1) = \sum_{s=a}^{n}\lambda(n)p(n)$. One can easily see that condition (2.29) is the same condition (1.10) imposed by Walsh.

Theorem 2.5. *Let $\mathbb{T}$ be a discrete time scale and $b > 1$, $c > 1$ and define*

$$\begin{cases} \Lambda(t) = \int_a^t \lambda(s)\Delta s, & \\ \Gamma(t) := \int_t^{\infty} a(s)\lambda(s)\Delta s, & \text{for } t \in [a,\infty)_{\mathbb{T}}. \\ F(t) := \int_t^{\infty} p(s)\lambda(s)\Delta s, & \end{cases} \tag{2.30}$$

If

$$(F(t))^{1-c}(\Gamma(t))^{c} \geq (F^{\sigma}(t))^{1-c}(\Gamma^{\sigma}(t))^{c}, \tag{2.31}$$

then

(2.32)
$$\int_a^{\infty}\frac{\lambda(t)\,(\Gamma(t))^{c}(p(t))^{b}(\Lambda^{\sigma}(t))^{b-1}}{(F(t))^{b+c-1}}\Delta t \leq \left(\frac{c}{c-1}\right)^{b}\int_a^{\infty}\frac{\lambda(t)(a(t))^{b}(\Gamma(t))^{c-b}}{(F(t))^{c-1}(\Lambda^{\sigma}(t))^{1-b}}\Delta t.$$

Proof. Fix $t \in (a,\infty)_{\mathbb{T}}$. Let

$$x := \frac{\Gamma^\sigma(t)}{\Gamma(t)} < 1, \qquad \beta := \frac{F^\sigma(t)}{F(t)} < 1, \text{ and } \quad \alpha := \eta \frac{\lambda(t)}{\Lambda^\sigma(t)}, \tag{2.33}$$

where $\eta\,(=\eta(t))$ is positive and will be determined below such that $\alpha < 1$. As a result of these substitutions in (1.12), we get that

$$\begin{aligned} c\left(1-\frac{\Gamma^\sigma(t)}{\Gamma(t)}\right)^b &= c\left(\frac{-\mu(t)\Gamma^\Delta(t)}{\Gamma(t)}\right)^b = c\left(\frac{\mu(t)a(t)\lambda(t)}{\Gamma(t)}\right)^b \\ &\geq b\left(\eta\frac{\lambda(t)}{\Lambda^\sigma(t)}\right)^{b-1}\left(1-\left(\frac{F^\sigma(t)}{F(t)}\right)^{1-c}\left(\frac{\Gamma^\sigma(t)}{\Gamma(t)}\right)^c\right) \\ &\quad +b(c-1)\eta^{b-1}\frac{\lambda^{b-1}(t)\mu(t)p(t)\lambda(t)}{(\Lambda^\sigma(t))^{b-1}F(t)} + \frac{c(1-b)\lambda^b(t)}{(\Lambda^\sigma(t))^b}\eta^b. \end{aligned} \tag{2.34}$$

Multiplying both sides by

$$\lambda^{1-b}(t)\,(\Gamma(t))^c\,(\Lambda^\sigma(t))^{b-1}\,(F(t))^{1-c}, \tag{2.35}$$

gives

$$\begin{aligned} &\frac{c\lambda(t)(a(t)\mu(t))^b\,(\Gamma(t))^{c-b}}{(F(t))^{c-1}\,(\Lambda^\sigma(t))^{1-b}} \\ &\geq b\,(\eta(t))^{b-1}\left[(F(t))^{1-c}\,(\Gamma(t))^c-(F^\sigma(t))^{1-c}\,(\Gamma^\sigma(t))^c\right] \\ &\quad +\lambda(t)\,(\Gamma(t))^c\left[\frac{b(c-1)p(t)\mu(t)}{(F(t))^c}\eta^{b-1}+\frac{c(1-b)}{(F(t))^{c-1}\,(\Lambda^\sigma(t))}\eta^b\right]. \end{aligned} \tag{2.36}$$

Now, we consider the last term on the right hand side, namely $g(\eta) := M_1\eta^b + K_1\eta^{b-1}$, as a function of η, where

$$K_1 := \frac{b(c-1)p(t)\mu(t)}{F^c(t)} \text{ and } M_1 := \frac{c(1-b)}{(F(t))^{c-1}\,\Lambda^\sigma(t)}.$$

The function $g(\eta)$ has a maximum value at

$$\eta := \frac{c-1}{c}\frac{p(t)\mu(t)\Lambda^\sigma(t)}{F(t)} > 0, \tag{2.37}$$

and the maximum value is given by

$$\max_{\eta\geq 0} g(\eta) = \frac{(c-1)^b}{c^{b-1}}\frac{(p(t)\mu(t))^b\,(\Lambda^\sigma(t))^{b-1}}{(F(t))^{b+c-1}}. \tag{2.38}$$

From (2.33) and (2.37), we see that (with the above η)

$$\begin{aligned} \alpha &= \eta\frac{\lambda(t)}{\Lambda^\sigma(t)} = \frac{c-1}{c}\frac{p(t)\mu(t)\lambda(t)}{F(t)} = \frac{c-1}{c}\frac{-\mu(t)F^\Delta(t)}{F(t)} \\ &= \frac{c-1}{c}\frac{-(F^\sigma(t)-F(t))}{F(t)} = \frac{c-1}{c}\frac{F(t)-F^\sigma(t)}{F(t)}, \end{aligned}$$

and note $0<\alpha<1$. Then (2.31), (2.36) and (2.38) imply that

$$\begin{aligned}\frac{c\lambda(t)(a(t)\mu(t))^{b}\left(\Gamma(t)\right)^{c-b}}{\left(F(t)\right)^{c-1}\left(\Lambda^{\sigma}(t)\right)^{1-b}} &\geq b\left(\eta(t)\right)^{b-1}\left[\left(F(t)\right)^{1-c}\left(\Gamma(t)\right)^{c}-\left(F^{\sigma}(t)\right)^{1-c}\left(\Gamma^{\sigma}(t)\right)^{c}\right]\\ &\quad+\frac{(c-1)^{b}}{c^{b-1}}\frac{\lambda(t)\left(p(t)\mu(t)\right)^{b}\left(\Gamma(t)\right)^{c}\left(\Lambda^{\sigma}(t)\right)^{b-1}}{\left(F(t)\right)^{b+c-1}}\\ &\geq \frac{(c-1)^{b}}{c^{b-1}}\frac{\lambda(t)\left(p(t)\mu(t)\right)^{b}\left(\Gamma(t)\right)^{c}\left(\Lambda^{\sigma}(t)\right)^{b-1}}{\left(F(t)\right)^{b+c-1}}.\end{aligned}$$

Integration from a to ∞ yields

$$c\int_{a}^{\infty}\frac{\lambda(t)(a(t))^{b}\left(\Gamma(t)\right)^{c-b}}{\left(F(t)\right)^{c-1}\left(\Lambda^{\sigma}(t)\right)^{1-b}}\Delta t\geq\frac{(c-1)^{b}}{c^{b-1}}\int_{a}^{\infty}\frac{\lambda(t)\left(\Gamma(t)\right)^{c}p^{b}(t)\left(\Lambda^{\sigma}(t)\right)^{b-1}}{\left(F(t)\right)^{b+c-1}}\Delta t.$$

Thus

$$\int_{a}^{\infty}\frac{\lambda(t)\left(\Gamma(t)\right)^{c}p^{b}(t)\left(\Lambda^{\sigma}(t)\right)^{b-1}}{\left(F(t)\right)^{b+c-1}}\Delta t\leq\left(\frac{c}{c-1}\right)^{b}\int_{a}^{\infty}\frac{\lambda(t)a^{b}(t)\left(\Gamma(t)\right)^{c-b}}{\left(F(t)\right)^{c-1}\left(\Lambda^{\sigma}(t)\right)^{1-b}}\Delta t,$$

which is the desired inequality (2.30). The proof is complete. □

Remark 2.6. As a special case of Theorem 2.5 when $\mathbb{T}=\mathbb{N}$, we see that $\mu(t)=1$ and

$$\Lambda(n):=\sum_{s=a}^{n}\lambda(s)\Delta s,\quad\Gamma(n)=\sum_{s=n}^{\infty}a(s)\lambda(s),\ \text{ and } F(n)=\sum_{s=n}^{\infty}p(s)\lambda(s),$$

and the condition (2.31) becomes

$$\left(F(n)\right)^{1-c}\left(\Gamma(n)\right)^{c}\geq\left(F(n+1)\right)^{1-c}\left(\Gamma(n+1)\right)^{c}.$$

In this time scale inequality (2.32) reduces to

$$\sum_{n=a}^{\infty}\frac{\lambda(n)\left(\Gamma(n)\right)^{c}p^{b}(n)\left(\Lambda_{1}(n)\right)^{b-1}}{\left(F(n)\right)^{b+c-1}}\leq\left(\frac{c}{c-1}\right)^{b}\sum_{n=a}^{\infty}\frac{\lambda(n)a^{b}(n)\left(\Gamma(n)\right)^{c-b}}{\left(F(n)\right)^{c-1}\left(\Lambda(n)\right)^{1-b}}.\tag{2.39}$$

Theorem 2.7. *Let $\mathbb{T}$ be a discrete time scale and $b>1$, $c>1$ and define*

$$\begin{cases}\Lambda(t)=\int_{a}^{t}\lambda(s)\Delta s,\\ A(t)=\int_{a}^{t}a(s)\lambda(s)\Delta s, & \text{for } t\in[a,\infty)_{\mathbb{T}}.\\ F(t)=\int_{t}^{\infty}p(s)\lambda(s)\Delta s,\end{cases}\tag{2.40}$$

If

$$\left(F(t)\right)^{1-c}\left(A^{\sigma}(t)\right)^{c}\geq\left(F^{\sigma}(t)\right)^{1-c}\left(A(t)\right)^{c},\tag{2.41}$$

then

(2.42)
$$\int_{a}^{\infty}\frac{\lambda(t)\left(A^{\sigma}(t)\right)^{c}p^{b}(t)\left(\Lambda^{\sigma}(t)\right)^{b-1}}{\left(F(t)\right)^{b+c-1}}\Delta t\leq\left(\frac{c}{c-1}\right)^{b}\int_{a}^{\infty}\frac{\lambda(t)a^{b}(t)\left(A^{\sigma}(t)\right)^{c-b}}{\left(F(t)\right)^{c-1}\left(\Lambda^{\sigma}(t)\right)^{1-b}}\Delta t.$$

Proof. Fix $t \in (a, \infty)_{\mathbb{T}}$. Let

$$x := \frac{A(t)}{A^{\sigma}(t)} < 1, \quad \beta := \frac{F^{\sigma}(t)}{F(t)} < 1, \quad \text{and} \quad \alpha := \eta \frac{\lambda(t)}{\Lambda^{\sigma}(t)}, \tag{2.43}$$

where $\eta\,(= \eta(t))$ is positive and will be determined below such that $\alpha < 1$. As a result of these substitutions in (1.12), we get that

$$\begin{aligned} c\left(1-\frac{A(t)}{A^{\sigma}(t)}\right)^{b} &= c\left(\frac{\mu(t)A^{\Delta}(t)}{A^{\sigma}(t)}\right)^{b} = c\left(\frac{\mu(t)a(t)\lambda(t)}{A^{\sigma}(t)}\right)^{b} \\ &\geq b\left(\eta\frac{\lambda(t)}{\Lambda^{\sigma}(t)}\right)^{b-1}\left(1-\left(\frac{F^{\sigma}(t)}{F(t)}\right)^{1-c}\left(\frac{A(t)}{A^{\sigma}(t)}\right)^{c}\right) \\ &\quad + b(c-1)\eta^{b-1}\frac{\lambda^{b-1}(t)p(t)\mu(t)\lambda(t)}{\left(\Lambda^{\sigma}(t)\right)^{b-1}F(t)} + \alpha\frac{c(1-b)\eta^{b}\lambda^{b}(t)}{\left(\Lambda^{\sigma}(t)\right)^{b}}. \end{aligned}$$

Multiplying both sides by $\lambda^{1-b}(t)\left(A^{\sigma}(t)\right)^{c}\left(\Lambda^{\sigma}(t)\right)^{b-1}\left(F(t)\right)^{1-c}$, we have that

$$\begin{aligned} &\frac{c\lambda(t)(a(t)\mu(t))^{b}\left(A^{\sigma}(t)\right)^{c-b}}{\left(F(t)\right)^{c-1}\left(\Lambda^{\sigma}(t)\right)^{1-b}} \\ &\geq b\left(\eta(t)\right)^{b-1}\left[\left(F(t)\right)^{1-c}\left(A^{\sigma}(t)\right)^{c}-\left(F^{\sigma}(t)\right)^{1-c}\left(A(t)\right)^{c}\right] \\ &\quad + \lambda(t)\left(A^{\sigma}(t)\right)^{c}\left[\frac{b(c-1)p(t)\mu(t)}{\left(F(t)\right)^{c}}\eta^{b-1} + \frac{c(1-b)}{\left(F(t)\right)^{c-1}\left(\Lambda^{\sigma}(t)\right)}\eta^{b}\right]. \end{aligned}$$

The rest of the proof is similar to the proof of Theorem 2.1. The proof is complete. ☐

Remark 2.8. As a special case in Theorem 2.7, when $\mathbb{T} = \mathbb{N}$ *and* $b > 1$, $c > 1$, we see that

$$\Lambda(n) := \sum_{s=1}^{n-1}\lambda(s), \; A(n) = \sum_{s=1}^{n-1}a(s)\lambda(s), \text{ and } F(n) = \sum_{s=n}^{\infty}p(s)\lambda(s), \; \textit{for } n \geq 1,$$

and condition (2.41) becomes

$$\left(F(n)\right)^{1-c}\left(A(n+1)\right)^{c} \geq \left(F(n+1)\right)^{1-c}\left(A(n)\right)^{c}.$$

Then, we have the following discrete inequality

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{\lambda(n)\left(\Lambda(n+1)\right)^{b-1}p^{b}(n)}{(F(n))^{b+c-1}}\left(\sum_{i=1}^{n}\lambda(i)a(i)\right)^{c} \\ &\quad \leq \left(\frac{c}{c-1}\right)^{b}\sum_{n=1}^{\infty}\frac{\lambda(n)(a(n))^{b}}{\left(F(n)\right)^{c-1}\left(\Lambda(n+1)\right)^{1-b}}\left(\sum_{i=1}^{n}\lambda(i)a(i)\right)^{c-b}. \end{aligned}$$

If $c = b$ and $p(n) = 1$ for all n, then we have the following inequality

$$\sum_{n=1}^{\infty}\frac{\lambda(n)\left(\Lambda(n+1)\right)^{c-1}}{(F(n))^{2c-1}}\left(\sum_{i=1}^{n}\lambda(i)a(i)\right)^{c} \leq \left(\frac{c}{c-1}\right)^{c}\sum_{n=1}^{\infty}\frac{\lambda(n)(a(n))^{c}}{\left(F(n)\right)^{c-1}\left(\Lambda(n+1)\right)^{1-c}}.$$

In the following, we will prove a new inequality which as a special case contains the inequality (1.9) proved by Leindler.

Theorem 2.9. *Let $\mathbb{T}$ be a discrete time scale and $b > 1$, $c > 1$ and define*

$$\begin{cases} \Omega(t) := \int_t^\infty \lambda(s)\Delta s, \\ A(t) = \int_a^t a(s)\lambda(s)\Delta s, \qquad \text{for } t \in [a,\infty)_{\mathbb{T}}. \\ F(t) = \int_t^\infty p(s)\lambda(s)\Delta s, \end{cases} \tag{2.44}$$

If

$$(A^\sigma(t))^c F^{1-c}(t) \geq (F^\sigma(t))^{1-c}(A(t))^c, \tag{2.45}$$

then

(2.46)

$$\int_a^\infty \frac{\lambda(t)(A^\sigma(t))^b p^b(t)(\Omega(t))^{b-1}}{(F(t))^{b+c-1}}\Delta t \leq \left(\frac{c}{c-1}\right)^b \int_a^\infty \frac{\lambda(t)a^b(t)(A^\sigma(t))^{c-b}(\Omega(t))^{b-1}}{(F(t))^{c-1}}\Delta t.$$

Proof. Fix $t \in (a,\infty)_{\mathbb{T}}$. Let

$$x := \frac{A(t)}{A^\sigma(t)} < 1, \qquad \beta := \frac{F^\sigma(t)}{F(t)} < 1, \text{ and } \quad \alpha := \eta\frac{\lambda(t)}{\Omega(t)}, \tag{2.47}$$

where $\eta\,(=\eta(t))$ is positive and will be determined below such that $\alpha < 1$. As a result of these substitutions in (1.12), we get that

$$\begin{aligned} &c\left(\frac{\mu(t)a(t)\lambda(t)}{A^\sigma(t)}\right)^b \\ &\geq b\left(\eta\frac{\lambda(t)}{\Omega(t)}\right)^{b-1}\left(1-\left(\frac{F^\sigma(t)}{F(t)}\right)^{1-c}\left(\frac{A(t)}{A^\sigma(t)}\right)^c\right) \\ &+b(c-1)\eta^{b-1}\frac{\lambda^{b-1}(t)p(t)\mu(t)\lambda(t)}{(\Omega(t))^{b-1}F(t)} + \frac{c(1-b)\eta^b\lambda^b(t)}{(\Omega(t))^b}. \end{aligned} \tag{2.48}$$

Multiplying both sides by $\lambda^{1-b}(t)(A^\sigma(t))^c(\Omega(t))^{b-1}(F(t))^{1-c}$, we have that

$$\begin{aligned} &\frac{c\lambda(t)(a(t)\mu(t))^b(A^\sigma(t))^{c-b}}{(F(t))^{c-1}(\Omega(t))^{1-b}} \\ &\geq b\eta^{b-1}\frac{\lambda^{b-1}(t)\left[(A^\sigma(t))^{b+c}F^{1-c}(t)-(A^\sigma(t))^b(F^\sigma(t))^{1-c}(A(t))^c\right]}{\Omega^{b-1}(t)F^{1-c}(t)(A^\sigma(t))^c} \\ &+\lambda(t)(A^\sigma(t))^b\left[\frac{b(c-1)p(t)\mu(t)}{F^c(t)}\eta^{b-1}+\frac{c(1-b)}{(F(t))^{c-1}\Omega(t)}\eta^b\right]. \end{aligned} \tag{2.49}$$

Now, we consider the last term on the right hand side, namely

$$F(\eta) := M\eta^b + K\eta^{b-1}, \tag{2.50}$$

as a function of η, where

$$K := \frac{b(c-1)p(t)\mu(t)}{(F(t))^c}, \quad \text{and } M := \frac{c(1-b)}{\Omega(t)(F(t))^{c-1}}.$$

The function $F(\eta)$ has a maximum value at

$$\eta=\frac{c-1}{c}\frac{p(t)\mu(t)\Omega(t)}{\digamma(t)}>0, \tag{2.51}$$

and the maximum value of $F(\eta)$ is given by

$$\max_{\eta>0}F(\eta)=\frac{(c-1)^{b}}{c^{b-1}}\frac{p^{b}(t)\mu^{b}(t)\left(\Omega(t)\right)^{b-1}}{(\digamma(t))^{b+c-1}}. \tag{2.52}$$

From (2.47) and (2.51), we see that (with the above η)

$$\alpha=\eta\frac{\lambda(t)}{\Omega(t)}=\frac{c-1}{c}\frac{p(t)\mu(t)\lambda(t)}{\digamma(t)},$$

and note $0<\alpha<1$. Then (2.49) and (2.52) imply that

$$\begin{aligned}&\frac{c\lambda(t)(a(t)\mu(t))^{b}\left(A^{\sigma}(t)\right)^{c-b}\left(\Omega(t)\right)^{b-1}}{\left(\digamma(t)\right)^{c-1}}\\&\geq b\left(\eta\frac{\lambda(t)}{\Omega(t)}\right)^{b-1}\frac{\left(A^{\sigma}(t)\right)^{b+c}\digamma^{1-c}(t)-\left(A^{\sigma}(t)\right)^{b}\left(\digamma^{\sigma}(t)\right)^{1-c}\left(A(t)\right)^{c}}{\digamma^{1-c}(t)\left(A^{\sigma}(t)\right)^{c}}\\&\quad+\frac{(c-1)^{b}}{c^{b-1}}\frac{\lambda(t)(\mu(t))^{b}\left(A^{\sigma}(t)\right)^{b}\left(\Omega(t)\right)^{b-1}}{p^{-b}(t)(\digamma(t))^{b+c-1}}.\end{aligned}$$

Using the condition (2.45), we have that

$$\frac{c\lambda(t)(a(t)\mu(t))^{b}\left(A^{\sigma}(t)\right)^{c-b}}{\left(\digamma(t)\right)^{c-1}\left(\Omega(t)\right)^{1-b}}\geq\frac{(c-1)^{b}}{c^{b-1}}\frac{\lambda(t)(\mu(t))^{b}\left(A^{\sigma}(t)\right)^{b}\left(\Omega(t)\right)^{b-1}}{p^{-b}(t)(\digamma(t))^{b+c-1}}.$$

The result follows now. □

Remark 2.10. As a special case in Theorem 2.9, when $\mathbb{T}=\mathbb{N}$ *and* $b>1$, $c>1$, we get that

$$\Omega(n):=\sum_{s=n}^{\infty}\lambda(s),\quad A(n)=\sum_{s=1}^{n-1}a(s)\lambda(s),\text{ and }\ \digamma(n)=\sum_{s=n}^{\infty}\digamma(s)\lambda(s),\ \text{ for } n\geq 1, \tag{2.53}$$

and the condition (2.45) becomes

$$\left(\digamma(n)\right)^{1-c}\left(A(n+1)\right)^{c}\geq\left(\digamma(n+1)\right)^{1-c}\left(A(n)\right)^{c},$$

and the inequality (2.46) reduces to the following discrete inequality

$$\sum_{n=1}^{\infty}\frac{\lambda(n)\left(\Omega(n)\right)^{b-1}\digamma^{b}(n)}{(\digamma(n))^{b+c-1}}\left(\sum_{i=1}^{n}\lambda(i)a(i)\right)^{b}\leq\left(\frac{c}{c-1}\right)^{b}\sum_{n=1}^{\infty}\frac{\lambda(n)(a(n))^{b}}{\left(\digamma(n)\right)^{c-1}\left(\Omega(n)\right)^{1-b}}.$$

If $\digamma(n)=1$, we have the following inequality of Leindler's type

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{(\Omega(n))^{c}}\left(\sum_{i=1}^{n}\lambda(i)a(i)\right)^{b}\leq\left(\frac{c}{c-1}\right)^{b}\sum_{n=1}^{\infty}\lambda(n)(a(n))^{b}\left(\Omega(n)\right)^{b-c}.$$

Remark 2.11. If $c = b$, then we have the following inequality of Hardy's type

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{(\Omega(n))^c} \left(\sum_{i=1}^{n} \lambda(i) a(i) \right)^b \leq \left(\frac{c}{c-1} \right)^c \sum_{n=1}^{\infty} \lambda(n) (a(n))^c.$$

REFERENCES

[1] R. P. Agarwal, M. Bohner, D. O'Regan and S. H. Saker, Some Wirtinger-type inequalities on time scales and their applications, Pacific J. Math. 252 (2011), 1–26.

[2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[3] M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[4] E. T. Copson, Note on series of positive terms, J. London. Math. Soc. 2 (1927), 9–12.

[5] E. T. Copson, Note on series of positive terms, J. London. Math. Soc. 3 (1928), 49–51.

[6] E. T. Copson, Some integral inequalities, Prof. Roy. Soc. Edinburg. Sect. A 75 (1975/1976), 157–164.

[7] E. B. Elliott, A simple expansion of some recently proved facts as to convergency, J. London Math. Soc. 1 (1926), 93–96.

[8] E. B. Elliott, A Further note on sums of positive terms, J. London Math. Soc., s 1–4 (1929), 21–23.

[9] G. H. Hardy, Notes on a theorem of Hilbert, Math. Z. 6 (1920), 314–317.

[10] G. H. Hardy, Notes on some points in the integral calculus, Messenger Math. 54 (1925), 150–156.

[11] G. H. Hardy, Notes on some points in the integral calculus, Messenger Math. 57 (1928), 12–16.

[12] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities,* 2nd Ed. Cambridge Univ. Press 1952.

[13] S. Hilger, Analysis on measure chains —a unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18–56.

[14] M. Izumi and S. Izumi, On some inequalities for Fourier Series, J. D'Analyse Math. 21 (1968), 277–291.

[15] S. Kaijser, L. Nikolova, L. E. Persson and A. Wedestig, Hardy-type inequalities via convexity, Math. Ineq. Appl. 8 (2005), 403–417.

[16] S. Kaijser, L. E. Persson and A. Öberg, On Carleman and Knopp's inequalities, J. Approx. Theor. 117 (2002), 140–151.

[17] A. Kufner and Lars-Erik Persson, *Weighted Inequalities of Hardy Type*, World Scientific Publishing (2003).

[18] A. Kufner, L. Maligranda and L. Persson, *The Hardy inequalities: About its History and Some Related Results*, Pilsen (2007).

[19] L. Leindler, Generalization of inequalities of Hardy and Littlewood, Acta Sci. Math. (Szeged), 31 (1970), 297–285.

[20] N. Levinson, Generalizations of an inequality of hardy, Duke Math. J. 31 (1964), 389–394.

[21] J. A. Oguntuase and C. O. Imoru, New generalizations of Hardy's integral inequalities, J. Math. Anal. Appl. 241 (2000), 73–82.

[22] B. Opic and A. Kufner, *Hardy-type inequalities*, Longman Scientific& Technical, Harlow, UK, (1989).

[23] U. M. Ozkan and H. Yildirim, Hardy-Knopp-type inequalities on time scales, Dyn. Sys. Appl. 17 (2008), 477–486.

[24] B. G. Pachpatte, A note on certain inequalities related to Hardy's inequality, Indian J. Pure Appl. Math 23 (1992), 773–776.

[25] S. H. Saker and Donal O'Regan, Extensions of dynamic inequalities of Hardy's type on time scales, Mathematica Slovaca (accepted).

[26] M. R. Sidi, A. F. M. Torres, Hölder's and Hardy's two dimensional diamond-alpha inequalities on time scales, Annals of Univ. Craiva, Math. Comp. Series 37 (2010), 1–11.

[27] A. Tuna and S. Kutukcu, Some integrals inequalities on time scales, Appl. Math. Mech. Eng. Ed. 29 (2008), 23–29

[28] C. E. Walsh, Inequalities for positive series, Edinburgh Mathematical Notes 32 (1940), xxx – xxxii.