# AN ATTEMPT TOWARDS DYNAMIC MODELING
# OF THE EARTH'S CLIMATE SYSTEM

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**ABSTRACT.** In this paper we attempt to develop a broader dynamic (mathematical) model for the climate system of the planet earth. This is based on two major components namely the atmosphere around the planet and the oceans all subject to the solar radiation, lunar gravity and their impact on land, sea and the atmosphere. It is assumed that the atmosphere-ocean interaction is the fundamental source of the global climate variability. Based on this fact we develop a mathematical model that takes into account all the possible major interactions. This model is further extended to a stochastic dynamic system in order to include uncertainties in many of the natural forces. The authors believe that this model will allow for numerical evaluation of many physical variables of interest possibly leading to a better understanding of the climate variability of the earth as a whole.

**AMS (MOS) Subject Classification.** 35K51,35K55,76N99, 93C20.

## 1. INTRODUCTION

The physics of the earth's climate system is complex depending on several interacting factors such as the Ocean, the land, the atmosphere surrounding the planet, the gravitational forces of the moon, the solar radiation etc. The dynamics of evolution of the earth's climate consists of a coupled system of partial differential equations and a set of nonhomogeneous boundary conditions some of which are variable and possibly stochastic in nature. Some of these boundary conditions provide coupling between some of the partial differential equations as seen later. This coupling poses a conundrum for the numerical simulists, geophysicists and mathematicians alike to quantify the comparative significance of the atmosphere, the oceans, the land and the natural interaction of the sun and the moon in working out the natural climate variability as we experience here on earth. Therefore construction of any mathematical model expected to predict the immediate future evolution given the present requires reasonable validation. To this end various studies have been undertaken by many investigators while deducing the complexities in different models. In the following paragraphs of this section we present a brief description of the ocean-atmosphere

models developed by few selected researchers in the investigation of the evolution and the dynamics of the global climate system.

In 1969, W. D. Sellers [1] presented a simple numerical model of the earth-atmosphere energy balance by using a single unknown i.e., the annual sea level temperature in $10^0$ latitude belts. The author assumes this dependent variable as a function of a constant solar radiation, atmospheric transparency, the albedo and the turbulent exchange coefficient for the atmosphere and oceans. He found that a change in the climate of one part of the earth must eventually influence the entire global climate before another steady-state condition can be achieved. A few years later the author [2] presented a modified global climate model by extending the annual variation in temperatures to include seasonal changes and taking into account the time-averaged thermodynamic energy equations for land and water. However the author cautioned against applying conditions which differ from those existing that day. Bryan [3], Manabe [4] and Manabe *et al.* [5] in 1969 developed a joint ocean-atmosphere model by simplifying a limited computational resolution in the context of a global domain. This investigation lacks a realistic topographic approach. However, the construction of this model was aimed at obtaining a quasi-steady state solution in terms of state variables varying around some fixed values. In 1975 Manabe *et al.* presented an improved version of the joint ocean-atmosphere model by extending their computational resolution to the global domain and this time focussing on a realistic topography. Also they enhance a factor of the difference between the time frame of the atmosphere and the ocean models from 100 (in their previous investigation) to 320 (in their improved analysis). The authors split their investigation into two parts. In the first part [6] they express the state of the atmosphere in terms of the primitive equations of motion using the so called sigma coordinate in a spherical coordinate system in which a normalized pressure is taken as the vertical coordinate. Using the initial conditions of an isothermal and dry air at rest the authors focus on identifying the effects of oceanic currents in the evolution of the climate. In the second part [7] of their investigation the authors present some results from a numerical experiment of the joint global ocean-atmosphere model.

John A. T. Bye [8] in 1985 studied the exchange of momentum between the atmosphere and the ocean. The author pointed out that this exchange is a two-way process and suggested that the fundamental dissipation processes for the abyssal ocean take place mostly in the air-sea boundary and less near the ocean bottom. He found that the ratio of the dissipation rates for the ocean surface and the atmosphere was approximately equal to 1/30 for both the surface shearing stress and oceanic meridional heat transport. A. E. Gill [9] presented a simple coupled model featuring the essential characteristics of the anomalies produced as a result of the interaction between the ocean and the atmosphere observed during 1982–83 in the

tropical region. However the proposed model lacked essential physical characteristics viz. density-driven currents and advection of the interacting fluids. It also ignored the transport phenomenon as the model equations were considered linear. The long-wave approximation were taken into account while ignoring the north-south variations in the currents. In the same year Benoit Cushman-Roisin [10] analysed the combined effect of hydrodynamic and thermodynamic processes during frontogenesis at the sea surface. These processes (wind-driven and buoyancy-induced currents) intertwine in a nonlinear fashion. Focusing his attention on the North Pacific ocean the author considered a linearized set of model equations. However these time-independent governing equations did not include transport phenomena, diffusivity of the currents and viscosity effects.

In 1988, Paul S. Schopf *et al.* [11] presented a coupled ocean-atmosphere model consisting of a two-level nonlinear primitive system of equations. The coupling was made with the help of a linear relationship between sea surface temperature (SST) and heating. In the atmospheric model the authors considered the general circulation driven by time independent mean atmospheric temperatures at the two levels in the direction of a zonally symmetric state with sufficiently large pole-to-equator temperature difference. However the authors did not take into account the moisture content in the atmospheric model. For the ocean model the authors considered the deep water (abyssal) layer with constant pressure and overlying layers connected through diffusion. The abyssal layer was considered to be at rest in the absence of horizontal pressure gradients. In the same year Manabe *et al.* [12] discovered two stable equilibria using time integrations of a global coupled ocean-atmosphere model. These equilibria originated from different initial conditions but under identical boundary conditions. This investigation discusses the mechanism which maintains the bistable equilibria. Julian P. McCreary, Jr. [13] in 1983 studied ocean-atmosphere interaction in the tropics. The coupled model is devised to analyse the interaction between the ocean and atmosphere. The ocean model is composed of a single baro-clinic mode of a two-layer ocean whereas the model atmosphere exists within two wind states. He finds in this investigation that solutions oscillate for suitable choices of parameters with various periods ranging from two to nine years. Furthermore, he points out that a Rossby wave in the subtropic regions is generated by wind curl during Hadley circulation. In 1988, Rosati *et al.* [14] developed a general circulation model (GCM) for the upper ocean simulation. In this 12 level model, the authors used data collected by the National Meteorological Centre for winds, temperature and humidity for the period 1982-83 and applied as surface boundary conditions by first finding the heat flux and wind stress. For the initial conditions they set the climatological temperature, salinity and currents equal to zero. However they neglected density differences in their model equations keeping in view the Boussinesq approximation.

The authors also used the hydrostatic rather than hydrodynamic pressure condition which is not realistic. Miller *et al.* [15] in 1990 studied the atmospheric wintertime circulation influenced by evolving mid-latitude SST anomalies in the Northern Hemisphere using a simplified model. However it was not clear from this investigation if the observed mid-latitude SST anomalies influence the extended-range hind-casts of the atmosphere. It is because of this reason that the atmospheric variability behaves as a dominant force on the mid-latitude SST anomalies.

In 1992, Arthur J. Miller [16] studied ocean-atmosphere interactions focusing on mid-latitude SST anomalies using a simplified coupled model. The results of this analysis reveal the effects of atmosphere on the mid-latitude ocean although various numerical studies have predicted the presence of counter influence, that is the ocean influencing the atmosphere. He pointed out that the response of the atmosphere to midlatitude SST anomalies is not considerably large as compared to its own intrinsic variability. This is the reason why one cannot solely rely on statistical observations. Thomas F. Stocker *et al.* [17], in the same year, investigated the ocean's thermohaline circulation in the Pacific, Atlantic and Indian oceans in their zonally averaged coupled ocean-atmosphere model. In the atmospheric part of the coupled model the investigators focussed their attention on the time-dependent energy balance model. For the oceanic part they considered an individualized zonally averaged domain for each major ocean basin. However, they used several approximations in their analysis viz., neglecting time-dependency and the nonlinear terms in the momentum equations. The authors also neglected meridional heat and salt transport due to gyre circulation. They focussed their attention on long time scales only in the climatic evolution. However, the authors did not take into account various processes like vegetation, snow and high latitude sea-ice. In 1997, Cubasch *et al.* [18] studied the effect of solar irradiance on climate system. In this analysis the authors focussed their investigation on the twentieth century global warming trend by determining the temperature change, its magnitude and implication. The investigators suggested that the existence of solar variability could have considerably contributed to the climate change but they also pointed out the fact that this solar forcing change cannot be the sole factor responsible for the global warming during the twentieth century. One can find many useful reading materials on geophysical fluid dynamics and related topics in [21–29] .

Our interest in this paper is to present a dynamic model of the earth's climate system as a whole by taking into account all major constituents. However, the authors believe that some important factors may have been omitted. Although these factors (or processes) may be important, their inclusion will increase further complexities in the model. Therefore, we focus our attention on a moderately complex model so that one can possibly attempt a numerical simulation. It is expected that this may still

pose some challenges in developing numerical codes for such a large system containing stochastic terms. Owing to these complexities many investigators have confined their domain of interest either to a regional context or to models with reduced number of state variables. Here we attempt to include all major factors defining the state of the earths climate. To the best of our knowledge it appears that no such attempt has been made in the literature. It is expected that this model will contribute in this direction.

The paper is organised as follows. In section 2 we present the system dynamics in terms of partial differential equations for the atmosphere and the oceans. In section 3, we discuss the boundary conditions including the conditions on the sea-air interface. In section 3, solvability of the system equations is discussed by presenting the system equations in the compact state-space form. In order to supplement the system equations the dynamics of greenhouse gas content in the atmosphere is given in section 5. In section 6, we present arguments to justify the existence of uncertainty in the natural inputs as well as the boundary data.

## 2. SYSTEM DYNAMICS

We consider the global climate system consisting of a volume of fluid in contact with a rotating oblate (nearly spherical) solid body. The body of fluid consist of two layers of different densities with randomly distributed topography moving through space. The coupled atmosphere-ocean system is driven mainly by the solar radiation, gravity (due to the sun and the moon) and inertia. The whole system is bounded by a domain $\Sigma \subset R^3$ consisting of two major components, the atmospheric domain $\Sigma_a \subset \Sigma$ and the oceanic domain $\Sigma_o \subset \Sigma$. Almost all the climatic events that we observe take place in these two domains. Henceforth, the subscripts "$a$" and "$o$" will refer to the atmosphere and the oceans respectively with domain $\Sigma = \Sigma_a \cup \Sigma_o$. Note that $\partial \Sigma_l$ represents a part of the atmospheric boundary which includes the earth's rigid surface consisting of biosphere, cryosphere and lithosphere.

2.1. **Atmospheric Dynamics.** The set of partial differential equations governing the dynamics of the atmospheric system confined in the domain $\Sigma_a$ over any time interval $I \equiv [0, T]$ (for any $T$ finite) are the equations of continuity (or mass conservation of water vapor), momentum and energy expressed as follows:

(2.1)
$$\frac{\partial}{\partial t}(\rho_a) = -div\,(\rho_a \mathbf{u}_a) + R_a,$$

$$\frac{\partial}{\partial t}(\rho_a \mathbf{u}_a) = -div\,[p_a \mathbb{I} + \rho_a \mathbf{u}_a \otimes \mathbf{u}_a] + div\,[\mu_a \{(\nabla \mathbf{u}_a) + (\nabla \mathbf{u}_a)'\}]$$

(2.2)
$$+ div\left[\left(\eta_a - \frac{2}{3}\mu_a\right)(div\mathbf{u}_a)\,\mathbb{I}\right] - 2\rho_a\,(\mathbf{\Omega} \times \mathbf{u}_a) + \rho_a \mathbf{f}_a,$$

$$(2.3) \qquad \frac{\partial}{\partial t}(C_a^v \rho_a T_a) = -(\mathbf{u}_a \cdot \nabla)(C_a^v \rho_a T_a) + div\,(k_a \nabla T_a) - \mathbf{u}_a \cdot \nabla p_a + Q_a.$$

Equation (2.1) describes the mass balance also known as the continuity equation. Here $\rho_a$ represents the density of moist air (including aerosol) and $\mathbf{u}_a = (u_x, u_y, u_z)$ denotes the velocity field and $R_a$ contains all possible sources or sinks. including flux density and possible phase transition. Equation (2.2) describes the conservation of momentum. Here $p_a$ stands for the hydrodynamic pressure, $\mathbb{I}$ the identity matrix, $\mathbf{\Omega}$ refers to the earth's rotation (spin) vector with $|\mathbf{\Omega}| = 7.292 \times 10^{-5}(\mathrm{rad})s^{-1}$, $\mu_a$ the dynamic viscosity and $\rho_a \mathbf{f}_a$ the body force including the centripetal force and possibly other random perturbations of the climate. The second coefficient of viscosity $\eta_a$ known as the bulk viscosity [21] requires that $\eta_a, \mu_a \geq 0$. The operator $\nabla$ represents the gradient in Cartesian co-ordinate system and $(\nabla \mathbf{u}_a)'$ is transpose of $(\nabla \mathbf{u}_a)$. Equation (2.3) describes the heat balance or the energy distribution (in the atmosphere). Here $T_a$ stands for the absolute temperature, $C_a^v$ is the heat capacity at constant volume, $k_a$ is the thermal conductivity of the moist air and $Q_a$ represents the radiation source. The first term on the right hand side of equation (2.3) is the convective or transport term, the second is the diffusion term (due to conduction) and the third term represents heat loss (or gain) due to hydrodynamic pressure and the last term $Q_a$ stands for all possible heat sources including the solar radiation and volcanic eruptions etc. The pressure $p_a$ is given by a suitable function of density and temperature called the constitutive law: $p_a = \psi_a(\rho_a, T_a)$. Clearly, this law provides a coupling among all the three equations given above.

2.2. **Oceanic Dynamics.** In a similar fashion one can develop the dynamics of the ocean (nearly incompressible fluid) evolving in the domain $\Sigma_o$ over any time interval $I$. The variables of interest are sea water density, the velocity vector and the temperature. They are given by the following set of equations:

$$(2.4) \qquad \frac{\partial}{\partial t}(\rho_o) = -div\,(\rho_o \mathbf{u}_o) + R_o,$$

$$\frac{\partial}{\partial t}(\rho_o \mathbf{u}_o) = -div\,[p_o \mathbb{I} + \rho_o \mathbf{u}_o \otimes \mathbf{u}_o] + div\,[\mu_o \{(\nabla \mathbf{u}_o) + (\nabla \mathbf{u}_o)'\}]$$

$$(2.5) \qquad + div\left[\left(\eta_o - \frac{2}{3}\mu_o\right)(div\,\mathbf{u}_o)\,\mathbb{I}\right] - 2\rho_o\,(\mathbf{\Omega} \times \mathbf{u}_o) + \rho_o \mathbf{f}_o,$$

$$(2.6) \qquad C_o^v \frac{\partial}{\partial t}(\rho_o T_o) = -C_o^v\,(\mathbf{u}_o \cdot \nabla)(\rho_o T_o) + div\,(k_o \nabla T_o) - \mathbf{u}_o \cdot \nabla p_o + Q_o.$$

The variable $\rho_o$ denotes the density of sea water which differs from that of the fresh water by a ratio called salinity $S$. That is $\rho_o = S\rho_f$ where $\rho_f$ denotes the density of fresh water. Equation (2.4) describes the mass balance (conservation of mass) with $R_o$ denoting the source (or sink). This is one of the sources of interaction of the sea with the atmosphere above it. Equation (2.5) describes the conservation of momentum

similar to equation (2.2) with the ocean variables denoted by the subscript $o$. Like the equation (2.3) of the atmospheric dynamics, equation (2.6) describes the energy balance of the sea with $T_o$ denoting the sea water temperature. Again, $Q_o$ denotes the source (or sink) of heat energy major part of which comes from solar radiation and possibly under water volcanic eruptions etc. Here also the pressure $p_o$ is given by another suitable function of density and temperature: $p_o = \psi_o(\rho_o, T_o)$ providing a similar coupling.

## 3. BOUNDARY CONDITIONS

The primary interaction between the land, sea and the atmosphere above them are through the boundary. The boundary of the atmosphere $\partial\Sigma_a$ consists of interior boundary and outer boundary given by $\partial\Sigma_a = \partial\Sigma_{ai} \cup \partial\Sigma_{ao}$. The interior boundary consists of the ocean surface (including rivers, dams, lakes, etc.), ice (glaciers, snow, ice-sheets etc.) and land surface (including deserts, mountains, vegetation, etc.) giving $\partial\Sigma_{ai} \equiv \partial\Sigma_{os} \cup \partial\Sigma_i \cup \partial\Sigma_l$. The exterior boundary $\partial\Sigma_{ao}$ interacts with the outer space. Similarly the oceanic boundary $\partial\Sigma_o(\equiv \partial\Sigma_{os} \cup \partial\Sigma_{ob})$ consists of its free surface $\partial\Sigma_{os}$ and its base including its basin $\partial\Sigma_{ob}$. Thus the complete boundary of interest $\partial\Sigma$ is given by

$$\partial\Sigma \equiv \partial\Sigma_{ao} \cup \partial\Sigma_{ai} \cup \partial\Sigma_{ob}.$$

3.1. **Atmospheric Boundary Conditions.** The atmospheric boundary conditions are given as follows:

$$(3.1) \qquad \rho_a = \begin{cases} 0 & \text{on } \partial\Sigma_l \\ 0 & \text{on } \partial\Sigma_i \\ 0 & \text{on } \partial\Sigma_{ao} \end{cases} \qquad \mathbf{u}_a \cdot \mathbf{n} = \begin{cases} 0 & \text{on } \partial\Sigma_l \\ 0 & \text{on } \partial\Sigma_i \\ 0 & \text{on } \partial\Sigma_{ao} \end{cases} \quad T_a = \begin{cases} T_1 & \text{on } \partial\Sigma_l \\ T_2 & \text{on } \partial\Sigma_i \\ 0 & \text{on } \partial\Sigma_{ao} \end{cases}$$

3.2. **Oceanic Boundary Conditions.** The boundary conditions for the oceans are given as follows:

$$(3.2) \qquad\qquad \rho_o = 0, \ \mathbf{u}_o \cdot \mathbf{n} = 0, \ T_o = T_3 \text{ on } \partial\Sigma_{ob}$$

3.3. **Boundary Conditions at the Ocean-Air Interface.** It is important to note that the ocean surface is dynamic in nature and it is given by a 2-D manifold described by $z = h(t, x, y)$. Thus the dynamic boundary of the interface is given by the manifold $\partial\Sigma_{os} = \{(x, y, z) : z = h(t, x, y)\}$. Therefore the boundary conditions at the interface of ocean and atmosphere are given as:

$$(3.3) \qquad\qquad \rho_a = \rho_o = 0, \ \mathbf{u}_a = \mathbf{u}_o, \ T_a = T_o = T_4 \text{ on } \partial\Sigma_{os}.$$

For details on free surface boundary condition see the reference [30].

**Comments:** As indicated above, the boundary conditions are of Dirichlet type. However, strictly speaking, some of these boundary conditions should be of Neumann type. For example, in the case of land-air interface, there is transmission of heat energy from the land to the air and this is given by an expression of the form

$$\beta_1 T_a + \beta_2 \frac{\partial T_a}{\partial \nu} = T_1 \text{ on } \partial\Sigma_\ell$$

where $\beta_1$ and $\beta_2$ are constants.

## 4. DISCUSSION ON SOLVABILITY OF SYSTEM EQUATIONS

First we note that the system of equations $(2.1) - (2.3)$ and $(2.4) - (2.6)$ are nonlinear partial differential equations defined on the domain $\Sigma_a \times \Sigma_o \subset R^3 \times R^3$. Denoting the state variable by $X \equiv (\rho_a, \mathbf{u_a}, T_a, \rho_o, \mathbf{u_o}, T_o)'$ and the natural input by $V \equiv (R_w, \mathbf{f}_a, Q_a, R_o, \mathbf{f}_o, Q_o)'$ we can write these equations as a system in compact form as follows

(4.1) $$\frac{\partial}{\partial t}\mathcal{N}(X) = \mathcal{A}(X) + \mathcal{C}(X, V),$$

(4.2) $$\mathcal{B}(X) = g,$$

where the operator $\mathcal{A}$ is a linear second order differential operator given by

$$\mathcal{A}(X) = \begin{pmatrix} 0 \\ div\left[\mu_a\{(\nabla\mathbf{u}_a) + (\nabla\mathbf{u}_a)'\}\right] + div\left[(\eta_a - 2/3\mu_a)(div\ \mathbf{u}_a)\mathbb{I}\right] \\ div(k_a\nabla T_a) \\ 0 \\ div\left[\mu_o\{(\nabla\mathbf{u}_o) + (\nabla\mathbf{u}_o)'\}\right] + div\left[(\eta_o - 2/3\mu_o)(div\mathbf{u}_o)\mathbb{I}\right] \\ div(k_o\nabla T_o) \end{pmatrix},$$

and the operators $\mathcal{N}$ and $\mathcal{C}$ are nonlinear given by

$$\mathcal{N}(X) = \begin{pmatrix} \rho_a \\ \rho_a\mathbf{u}_a \\ T_a \\ \rho_o \\ \rho_o\mathbf{u_o} \\ T_o \end{pmatrix},$$

and

$$\mathcal{C}(X,V) = \begin{pmatrix} -div(\rho_a \mathbf{u}_a) + R_w \\ -div\left[p_a \mathbb{I} + \rho_a \mathbf{u}_a \otimes \mathbf{u}_a\right] - 2\rho_a(\boldsymbol{\Omega} \times \mathbf{u_a}) + \rho_\mathbf{a}\mathbf{f_a} \\ -(\mathbf{u}_a \cdot \nabla)\left(C_a^v \rho_a T_a\right) - \mathbf{u}_a \cdot \nabla p_a + Q_a \\ -div(\rho_o \mathbf{u}_o) + R_s \\ -div\left[p_o \mathbb{I} + \rho_o \mathbf{u}_o \otimes \mathbf{u}_o\right] - 2\rho_o(\boldsymbol{\Omega} \times \mathbf{u_o}) + \rho_\mathbf{o}\mathbf{f_o} \\ -(\mathbf{u}_o \cdot \nabla)\left(C_o^v \rho_o T_o\right) - \mathbf{u}_o \cdot \nabla p_o + Q_o \end{pmatrix},$$

respectively. The boundary operator $\mathcal{B}$ is given by

$$\mathcal{B}(X) = \begin{pmatrix} \rho_a \\ \mathbf{u}_a \cdot \mathbf{n} \\ T_a \\ \rho_o \\ \mathbf{u}_o \cdot \mathbf{n} \\ T_o \end{pmatrix}.$$

In order to solve these equations one must specify the initial state $X_0$, the boundary data $g$ and the natural input (forces) $V$. Note that the initial state is a function of the spatial variable $\xi \equiv (\xi_1, \xi_2) \in \Sigma_a \times \Sigma_o$ given by

$$X_0(\xi) \equiv (\rho_a(t_0, \xi_1), \mathbf{u_a}(t_0, \xi_1), T_a(t_0, \xi_1), \rho_o(t_0, \xi_2), \mathbf{u_o}(t_0, \xi_2), T_o(t_0, \xi_2))'$$

for $\xi_1 \in \Sigma_a$ and $\xi_2 \in \Sigma_o$. The boundary data $g = g(t, \kappa), t \geq 0, \kappa \in \partial\Sigma$ is given by the functions shown on the righthand side of the equations (3.1)–(3.3). The natural input $V$ is also a function of time and space. Certainly, it is a formidable task to measure and collect this massive physical data without which it is impossible to solve the system equation (4.1). Thus we propose here an alternative approach. This is based on optimization theory applied to an inverse problem [20]. Consider any time interval $I \equiv [t_0, t_1]$ and suppose during this time period it is possible to measure and collect data from $n$ different regions of the atmosphere $\mathcal{R}_{a,i}, (i = 1, 2, \ldots, n)$ with $\bigcup_{i=1}^n \mathcal{R}_{a,i} \subset \Sigma_a$; and $m$ different regions of the Oceans $\mathcal{R}_{o,i}, (i = 1, 2, \ldots, m)$ with $\bigcup_{i=1}^m \mathcal{R}_{o,i} \subset \Sigma_o$. Let this observed data be denoted by $X^{ob}(t, \xi), \xi \in (\bigcup_{i=1}^n \mathcal{R}_{a,i} \subset \Sigma_a) \cup (\bigcup_{i=1}^m \mathcal{R}_{o,i} \subset \Sigma_o)$. Further, suppose the natural input $V$ during this period is recorded and so given. Now let us choose, to start with, any initial data $\zeta$ from $L_2(\Sigma_a \times \Sigma_o, R^5 \times R^5)$ and boundary data $\gamma$ from $L_2(I \times \partial\Sigma, R^5 \times R^5)$ and let $X(t, \xi; \zeta, \gamma)$ denote the corresponding weak solution of our system equation (4.1). Let $X_a$ and $X_o$ denote the atmospheric and oceanic components of $X$. We assume that the

(weak) solution of the system (4.1) is continuously dependent on the initial and the boundary data. Consider the mean square error (mismatch between the observed data and the computed model data) restricted to the domain of observation as the objective functional

(4.3)
$$J(\zeta, \gamma) \equiv \sum_{i=1}^{n_1} \int_{I \times \mathcal{R}_{a,i}} |X_a^{ob}(t, \xi) - X_a(t, \xi; \zeta, \gamma)|^2 d\xi dt$$
$$+ \sum_{i=1}^{n_2} \int_{I \times \mathcal{R}_{o,i}} |X_o^{ob}(t, \xi) - X_o(t, \xi; \zeta, \gamma)|^2 d\xi dt.$$

Our objective is to minimize this mismatch. For convenience of notation let us denote by

$$\mathcal{D} \equiv L_2(\Sigma_a \times \Sigma_o, R^5 \times R^5) \times L_2(I \times \partial\Sigma, R^5 \times R^5)$$

the set of admissible initial and boundary data. The problem now is to find a pair $(\zeta^*, \gamma^*) \in \mathcal{D}$ so that

$$J(\zeta^*, \gamma^*) \leq J(\zeta, \gamma) \; \forall \; (\zeta, \gamma) \in \mathcal{D}.$$

It is expected that with the increase of the number of observations $n$ and $m$, the estimate of the initial and boundary data is likely to converge towards the true data given that the weak solution is unique. Once the initial and boundary data is determined the solution of equation (4.1) can be computed for any period of time provided the natural input data $V$ is given for the same time period.

In case both the boundary data and the natural input data are given as functions of time and space, the objective functional can be taken as

(4.4)
$$J(\zeta) \equiv \sum_{i=1}^{n_1} \int_{I \times \mathcal{R}_{a,i}} |X_a^{ob}(t, \xi) - X_a(t, \xi; \zeta)|^2 d\xi dt$$
$$+ \sum_{i=1}^{n_2} \int_{I \times \mathcal{R}_{o,i}} |X_o^{ob}(t, \xi) - X_o(t, \xi; \zeta)|^2 d\xi dt,$$

where $X(t, \xi; \zeta)$ is the solution of equation (4.1) corresponding to the chosen initial data $\zeta \in L_2(\Sigma_a \times \Sigma_o, R^5 \times R^5)$ and the known boundary data and the natural input over the time period $I$.

In case the initial data $\zeta$, the boundary data $\gamma$ and the natural input process $\upsilon$ are all unknown, the objective functional is a function of all these arguments $J = J(\zeta, \gamma, \upsilon)$ and it is minimized on the space

$$\mathcal{D} \equiv H \times C(I, E) \times L_2(I, H)$$

where $H \equiv L_2(\Sigma_a \times \Sigma_o, R^5 \times R^5)$ is the state space (Hilbert space with standard topology) and $E \equiv L_2(\partial\Sigma, R^5 \times R^5)$ is the Hilbert space of boundary data and $L_2(I, H)$ is the space of natural (forces) input data.

In view of the above discussion, it is clear that, given the measured data over a large set of regions, it is possible to estimate the missing data including the initial condition by solving the inverse problem as discussed above. For general theory of inverse or equivalently identification problems the reader is referred to the book [19]. Moreover, a computer algorithm for inverse problems sequentially determining the unknown data is given in the reference [20].

## 5. COMPLETENESS OF THE MODEL

The system model presented above is not truly complete. There are other variables of interest, not included in the model, that affect the climate system. Many of these are due to human interference. For example, excessive use of fossil fuel for generation of electric power produces greenhouse gas emissions. One primary component of this gas is $CO_2$ which impacts the climate. The $CO_2$ gas gets absorbed in the ocean thereby increasing the acidity of the water and affecting marine life. Also $CO_2$ forms a blanket around the planet earth thereby trapping heat radiated from the ground and causing global temperature rise. The dynamics of this greenhouse gas content (including aerosols) in the atmosphere, denoted by $G$, can be described by diffusion convection equation as follows

$$(5.1) \qquad \frac{\partial G}{\partial t} - div(k_G \nabla G) + \mathbf{u}_a \cdot \nabla G = q, \ (t, \xi) \in I \times \Sigma_a$$

where $k_G$ is the diffusivity coefficient of the atmosphere (which may also depend on $G$ itself). The function $q = q(t, \xi)$ is the greenhouse gas emission rate as a function of time and space. Note the coupling of this equation with the atmospheric dynamics $(2.1) - (2.3)$ through the velocity vector $\mathbf{u}_a$ . Further, the heat source $Q_a$ in equation $(2.3)$ also depends on the variable $G$. This is due to the blockage of heat radiated from the ground due to formation of a blanket of greenhouse gas, in particular $CO_2$, around the globe. There are other variables of interest such as population of marine life which is affected by the increase of toxicity and temperature of water. To complete the model any number of such factors can be included in the abstract formulation of the state equation (4.1) and the boundary data (4.2). However there is no significant impact of marine population on the dynamics of the climate and therefore it is not necessary to include this.

## 6. RANDOMNESS IN THE MODEL

One of the prominent source of randomness is the ocean surface appearing in the boundary condition $(3.1) - (3.2)$. The other sources of noise (uncertainty) come from the input vector $V \equiv (R_a, f_a, Q_a, R_o, f_o, Q_o)'$. Each of the elements of this vector of natural forces has a deterministic component say $V_0$ and a stochastic component say $\tilde{V}$ which is difficult to explain through the reasoning of deterministic physics. However,

by use of the theory of stochastic processes such as Brownian motion one can develop a rigorous mathematical model which can be adjusted to predict (or estimate) the evolution of the state process probabilistically. To consider the stochastic system we need to introduce the probability space with filtration $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$ where $\Omega$ is the sample space and $\mathcal{F}$ is the sigma algebra of Borel subsets of $\Omega$ and $\mathcal{F}_{t\geq 0}$ is a nondecreasing family of subsigma algebras of the sigma algebra $\mathcal{F}$ and $P$ is the probability law. Since two of the components of the state process $(\rho_a, \rho_o)$ appear multiplicatively with the natural input $V$, we can replace $C(X, V)$ by $C(X, V_0) + \Lambda(X)\dot{W}$ where the last term is designed to model the multiplicative uncertainty. Here $\dot{W}$ denotes the space-time white noise which is the generalized derivative (in the distribution sense) of Brownian motion $W \equiv \{W(t), t \geq 0\}$ adapted to the sigma algebra $\mathcal{F}_{t\geq 0}$. Thus we can rewrite the model as

$$(6.1) \qquad \frac{\partial}{\partial t}\mathcal{N}(X) = \mathcal{A}X + \mathcal{C}(X, V_0) + \Lambda(X)\dot{W},$$

$$(6.2) \qquad \mathcal{B}(X) = g,$$

where the second equation describes the boundary condition. Here $\Lambda$ is a suitable deterministic function of the state. More precisely $\Lambda : H \longrightarrow \mathcal{L}(H)$ given that $W \equiv \{W(t), t \geq 0\}$ is an $H$ valued Brownian motion. Note that $\mathcal{L}(H)$ denotes the space of bounded linear operators in $H$. Since Brownian motion is not differentiable in the classical sense the model equation (6.1) is written rigorously as an Itô stochastic differential equation as follows:

$$(6.3) \qquad d\mathcal{N}(X) = \mathcal{A}(X)dt + \mathcal{C}(X, V_0)dt + \Lambda(X)dW,$$

subject to the boundary condition (6.2). By use of the theory of semigroups this equation can be converted into a stochastic integral equation and the question of existence of its (weak or mild) solution can be studied. We do not go into this subject here because this will be a substantial digression from our main objective here which is to present only a modest but reasonable mathematical model of the climate. We leave the question of existence and regularity properties of solution as an open problem. We wish to emphasize that any climate model must contain stochastic components since all the factors that determine the climate dynamics are not completely understood or known. In any case using such stochastic mathematical models, we can compute at any time $t \geq 0$ the probability of events such as development of a full blown cyclone. Let $\Gamma \subset H$ denote the set of states that characterize a cyclone and $Z \in H$ the state characterizing the preconditions for formation of a cyclone. For any given region $\mathcal{R}$ let $X_{\mathcal{R}}(\cdot)$ denote the state $X(\cdot)$ restricted to the spatial region $\mathcal{R}$. Then one may compute $P\{X_{\mathcal{R}}(t) \in \Gamma | X_{\mathcal{R}}(s) = Z\}$, that is, the probability of occurrence of a cyclone in the region $\mathcal{R}$ at time $t$ given that at an earlier time $s$, $X(s) = Z$. It is well known that the preconditions for a cyclone may involve the sea surface temperature

$T_o$ exceeding a certain critical value (say $T_c \equiv 26^oC$), the moisture content in the air above the see surface related to $\rho_a$, the vertical wind shear of the air mass above the see surface related to $\mathbf{u}_a$ etc.

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