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OPTIMAL PROPORTIONAL REINSURANCE AND INVESTMENT IN JUMP DIFFUSION MARKETS WITH NO SHORT-SELLING AND NO BORROWING

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ABSTRACT. The optimal reinsurance and investment problem for insurance has attracted a lot of attention of researchers in the field of stochastic control for a long time. Along this line we discuss this problem in the case of jump diffusion markets when neither short-selling nor borrowing is allowed. Here, we specifically assume that the risk process of the insurance company is a diffusion process. The insurance company can transfer its risk by reinsurance and also invest its surplus in the financial market, where we model the price of the risky asset by a geometric Lévy process. To maximize the CARA (Constant Absolute Risk Aversion) utility of terminal wealth, the HJB equation with no short-selling constraint has been considered, and we obtain the closed form of the value function by a standard method. However, only a handful of people have discussed this problem under both constraints, (*i.e.* no short-selling and no borrowing). This is because the problem is much more general in this context, and becomes so complex that analytical solution could hardly be obtained. Therefore, we provide, under the no short-selling and no borrowing constraints, a numerical solution via Markov chain approximation, which proves to be effective and amenable.

Key words: Optimal reinsurance, Jump-diffusion markets, HJB equation, Markov chain approximation, No short-selling and no borrowing, CARA utility function.

1. Introduction

In recent years, optimal control in insurance has been widely discussed. The topics can mainly be divided into three categories: optimal investment and reinsurance, optimal dividend payments, and pension plans. Three kinds of objective functions are treated: the expected utility, the expected discounted value, and the ruin probability. Minimization of the ruin probability attracted much interest in the earlier period, but in recent days that interest gradually faded from people's attention since it is a little far from the practice. The other two cases continue to attract further research.

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For an insurance company, insurance services and investment activities are two channels to make profits. The reduction of the underwritten risk and the choice of the right portfolio policy are major factors deeply related to its further development. The company buys reinsurance treaties to transfer part of its underwritten risk to another insurance company in case that the risk of claim is too intensive, which inversely make its earnings increase. Meanwhile, it has to invest the surplus asset in the financial market to earn more. However, only when the company manages its portfolio well that it will be competitive. In other words, a weak portfolio will pull down its business performance.

To get acquainted with the insurance company's action, mathematical models have been built to simulate the dynamics of the associated economy. Based on the characterization of the surplus process, recent results can be mainly divided into three categories. Hipp and Plum (2000), Schmidli (2002) considered this kind of risk process, and solved the problem of minimizing the ruin probability. However, they did not take into account the borrowing constraints. Azcue and Muler (2009) reconsidered the problem under borrowing constraints. In the second type, the surplus process is approximated by a diffusion process. More papers are based on this model because the diffusion processes are well studied and convenient for numerical analysis. Luo and Taksar (2011) represented the surplus process by a pure diffusion. They assumed the company invested its surplus into a Black-Scholes risky asset and a risk free asset, aiming at minimizing the probability of absolute ruin. Precisely, they assumed that only a limited amount was invested in the risky asset and that no short-selling was allowed. But they allowed the company to borrow to continue financing. Furthermore, Luo, Taksar and Tsoi (2008) discussed similar problems under presence or absence of short-selling and/or borrowing. On the other aspect, *i.e.* aiming at maximizing the expected exponential utility of terminal wealth, Bai and Guo (2008) considered the model with no-shorting constraint,(they also considered the object of minimizing the probability of ruin). Cao and Wan (2009) completed the problem. They discussed the company's optimal policy in all cases. Zhang et al. (2009) added transaction costs to the risky asset, with CVaR controlling the risk, and found the optimal value function and corresponding strategies. In the third type, the surplus process is represented by a jump diffusion process. Yang and Zhang (2005) assumed the risk process to be a compound Poisson process perturbed by a standard Brownian motion. And they derived the optimal investment policies when the insurer can invest in the money market and in a risky asset. Wang (2007) went further: the claim process is only supposed to be a pure jump process. But they all ignored the reinsurance and the corresponding restrictions. Liang et al. (2011) went even further; they added reinsurance to the model both in the compound Poisson risk model and the Brownian motion model. However, they didn't impose restrictions on the control variables, i.e.

they didn't consider no borrowing constraint. Zhang et al. (2011) discussed a similar situation under the classical risk model with no short-selling and no borrowing.

In the aforementioned works, the dynamics of the risky assets all follow a geometric Brownian motion. But as we all know, in fact, the price of the risky asset doesn't always move like this. More practical types have to be taken into consideration. The first one should be a geometric Lévy process, *i.e.* we can assume the financial market to be a Lévy type Black-Scholes market. Framstad et al. (1998) first discussed this. But their problem is to decide the optimal consumption when the portfolio is constructed in a jump diffusion market. Kostadinova (2007) considered the optimal investment for an insurance company under a risk constraint on the Value-at-Risk. Lin and Yang (2011) considered the optimal investment and reinsurance in a jump diffusion market. They assumed the surplus of the insurance company and the return of the risky asset both to be a jump diffusion risk process under no short-selling. However, they didn't consider no borrowing constraint and their assumption was too strong to get a unique root $\hat{a}$ with $0 < \hat{a} < 1$.

In our paper, we consider the problem of optimal reinsurance and investment by an insurance company. The company (1) purchases a proportional reinsurance to transfer its risk and (2) invests its surplus to a jump diffusion financial market consisting of a bank account (a risk-free asset) and a stock (a risky asset), while the price of the stock is governed by a geometric Lévy process. We adopt the diffusion approximation to represent the surplus process. We do not allow short-selling and/or borrowing. In the case of no short-selling constraint, we solve the problem analytically. The method to derive the explicit solution is standard, which is the same as that in Browne (1995), Yang and Zhang (2005), Liang, Bai, and Guo (2011), and Zhang, Liu, and Kannan (2011). Next, we use the Markov chain approximation method to present the numerical results for the more general case, (*i.e.* with both constraints). It was first used in Zhang, Liu, and Kannan (2011), and we discussed it more carefully here. Our goal is to find the optimal reinsurance and investment policy which maximizes the CARA utility of the terminal wealth.

The rest of the paper is organized as follows. Section 2 is devoted to build up our model, present the necessary assumptions, and state the optimization problem. We present there the HJB equation and the verification theorem. Section 3 solves the corresponding HJB equation and finds the closed form expression for the solution under the constraint of no short-selling. In section 4, we solve the general problem, (*i.e.* with no short-selling and no borrowing), numerically through one example. Section 5 provides a discussion of this work and also of ongoing work.

2. Problem Formulation

2.1. **The Model.** We consider the problem of optimal reinsurance and investment of an insurance company that purchases proportional reinsurance to transfer its risk and invests its surplus to a jump diffusion market consisting of a bank account and a stock. Here the price of the stock is modeled by a geometric Lévy process.

In the classical risk model, the surplus of a collective contract or a large portfolio is modeled as

$$R_t = x + ct - \sum_{i=1}^{N_t} Y_i.$$

Here, x is the initial capital, N is a Poisson process with rate λ, and $\{Y_i\}$ forms a sequence of strictly positive iid random variables representing the payouts, which are independent of N, and having the first moment κ and the second moment s^2, and $c = (1+\gamma_0)\lambda\kappa > 0$ is the premium rate, in which γ_0 is a relative safety loading.

As discussed by Taksar and Markussen (2003), the surplus process can be approximated by a diffusion process

$$dR_t = \mu dt + \sigma_1 \, dw_t^1,$$

where $\mu = c - \lambda\kappa$ represents the constant drift, $\sigma_1 = \sqrt{s^2 + \kappa^2}$ is the diffusion coefficient that implies the volatility of the surplus, and w_t^1 is a standard Brownian motion.

In Bäuerle (2005) and Chen *et al.* (2010), they assumed that at each moment, the insurance company was allowed to reinsure part of its losses or to acquire new business (*e.g*, to be reinsurer for other insurance companies). They used the value of risk exposure $a_t \in [0, \infty)$ to indicate the reinsurance or the new business decision, that is,

$$a_t \in \begin{cases} [0,1], & \text{reinsurance;} \\ (1,\infty), & \text{new business.} \end{cases}$$

However, we do not take new business decision in our model, which limits a_t to take values in $[0, 1]$.

Combining this reinsurance in the above SDE for surplus, we get

$$dR_t = \big(\mu - (1 - a_t)\theta\big)dt + a_t\sigma_1 dw_t^1.$$

We note here that the assumption $\theta > \mu$ implies that the reinsurance is not cheap. Thus the reinsurance turns out to be cheap when $\theta = \mu$.

The insurance company has two kinds of investments in its portfolio; one is the non-risky bank (bond, money-market) account and the other is an investment in the risky stock market. Assuming that the bank account offers a fixed interest rate $r_0 > 0$, the price $P_1(t)$ of the bond is given by

$$dP_1(t) = r_0 P_1(t)dt.$$

The initial price of the bond is $P_1(0) = p_1 > 0$.

The price $P_2(t)$ of the stock is a càdlàg process satisfying the following stochastic differential equation

$$dP_2(t) = r_1 P_2(t)\ dt + \sigma_2 P_2(t)\ dw_t^2 + P_2(t-)\int_{-1}^{\infty} \eta \tilde{N}(dt, d\eta) \tag{2.1}$$

The initial price of the stock is $P_2(0-) = p_2 > 0$. Here, $r_1 > r_0$ is the expected yield rate of the stock, which is assumed to be a positive constant and should be higher than that of the bank account, σ_2 is a positive constant, and w_t^2 is a standard Brownian motion independent of w_t^1. Furthermore,

$$\tilde{N}(t, z) = N(t, z) - tq(z), t \geq 0, z \in \mathcal{B}(-1, \infty)$$

is the compensator of a homogeneous Possion random measure $N(t, z)$ on $\mathbb{R}^+ \times \mathcal{B}(-1, \infty)$ with intensity measure $E[N(t, z)] = tq(z)$, where $dq(\eta)$ is the Lévy measure associated to N. We only allow jump sizes $\eta \in (-1, \infty)$ so that the process $P_2(t)$ will remain positive for all $t \geq 0$ a.s, see Framstad, Øksendal and Sulem (1998).

As pointed out above, the insurance company invests its surplus in two assets, with a fraction $b_t \in [0, 1]$ to purchase in the risky asset, and the balance $1 - b_t$ in the bond market. Note that when there is no short-selling, $b_t \geq 0$. Furthermore, if there is no borrowing, then $b_t \leq 1$.

Therefore, our model is built as follows: we start with a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, and two independent standard Brownian motions w_t^1 and w_t^2 adapted to $\{\mathcal{F}_t\}$. Once the portfolio policy $(a(\cdot), b(\cdot))$ is chosen, the dynamics of the surplus process $R_t^{a,b}$ is given by

$$\begin{aligned} dR_t &= [\mu - (1 - a_t)\theta + r_0(1 - b_t)R_t + r_1 b_t R_t]dt + a_t\sigma_1 dw_t^1 \\ &\quad + b_t\sigma_2 R_t dw_t^2 + b_t R_{t-}\int_{-1}^{\infty} \eta\tilde{N}(dt, d\eta), \end{aligned} \tag{2.2}$$

with $R_0 = x$, where θ is the reinsurance premium rate. Under mild conditions (see e.g. Øksendal (2003), Theorem 11.2.3), it suffices to consider *Markov controls*, *i.e.* controls of the form

$$(a_{t-}, b_{t-}) = (a(R_{t-}), b(R_{t-})).$$

With a slight abuse of notation, we continue to write (a_t, b_t) in place of $(a(R_{t-}), b(R_{t-}))$.

Next, we give the definition of admissible Markovian control in our framework.

Definition 2.1. Considering (a_t, b_t) as our Markov control, we call it ADMISSIBLE and write $(a_t, b_t) \in \Pi$ if:

- the processes a_t and b_t are predictable;

- the processes a_t and b_t satisfy the integrability condition
$$\int_0^t (a_s^2 + b_s^2) ds < \infty, \text{a.e.}, \text{forall } t \geq 0;$$
- the SDE(2.2) has a unique solution corresponding to (a_t, b_t);
- if the initial endowment $x \in \mathcal{S}$, then $R_t \in \mathcal{S}$ for all $t > 0$, where $\mathcal{S}$ is the SOLVENCY REGION:
$$\mathcal{S} = \{x \in \mathbb{R}; x > 0\}.$$

Now the problem can be stated as follows: We consider the CARA (Constant Absolute Risk Aversion) utility function, (which is also called the exponential utility function), given by

$$u(x) = c_0 - \frac{\delta}{\gamma} e^{-\gamma x}, \tag{2.3}$$

where c_0, δ and γ are all positive constants. The value function $V(t,x)$ is given by

$$V(t,x) = \sup_{(a,b)\in\Pi} V^{a,b}(t,x) = \sup_{(a,b)\in\Pi} E[u(R_T^{a,b}) | R_t^{a,b} = x]. \tag{2.4}$$

The problem is that the insurer desires to maximize her expected terminal utility of wealth at a terminal time T.

2.2. **HJB equation and Verification Theorem.** We note, from Øksendal and Sulem (2007), that the generator $\mathcal{A}^{a,b}$ of the time-space process (dt, dR_t) is given by

$$\begin{aligned} \mathcal{A}^{a,b} V(t,x) &= V_t + [\mu - (1-a_t)\theta + r_0(1-b_t)x + r_1 b_t x] V_x + \frac{1}{2}(a_t^2\sigma_1^2 + b_t^2\sigma_2^2 x^2) V_{xx} \\ &\quad + \int_{-1}^{\infty} \big[V(t, x + bx\eta) - V(t,x) - V_x(t,x) bx\eta\big] dq(\eta) \end{aligned} \tag{2.5}$$

Using the standard proof it is not difficult to obtain the associated Hamilton-Jacobi-Bellman (HJB) equation.

Theorem 2.2. *Assume that the value function V defined by* (2.4) *is in* $\mathcal{C}^{1,2}([0,T] \times \mathbb{R}_+)$. *Then V satisfies the following Hamilton-Jacobi-Bellman equation:*

$$\sup_{(a,b)\in\Pi} \mathcal{A}^{a,b} V(t,x) = 0, \tag{2.6}$$

with the boundary condition

$$V(T,x) = u(x), \tag{2.7}$$

where, the utility function $u(x)$ is given in (2.3).

Similar to Theorem 3.1 in Øksendal and Sulem (2007), we get the verification theorem for our case. The proof is standard and we omit it.

Theorem 2.3 (Verification Theorem). *a) Suppose that* $\phi \in \mathcal{C}^{1,2}([0,T)\times\mathbb{R}_+)\cap\mathcal{C}^0([0,T]\times\mathbb{R}_+)$ *satisfies the following conditions*

(i)

$$\sup_{(a,b)\in\Pi}\mathcal{A}^{(a,b)}\phi(t,x)\le 0,\quad (t,x)\in[0,T)\times\mathbb{R}_+\ , \tag{2.8}$$

(ii)

$$\phi(T,x)\ge u(x),\quad x\in\mathbb{R}_+\ , \tag{2.9}$$

(iii) and for all $\pi\in\Pi$,

$$E^x\Bigg[|\phi(T,R^\pi_T)| + \int_0^T\Bigg\{|\mathcal{A}^{(a,b)}\phi(T,R^\pi_T)| + [a^2\sigma_1^2+b^2\sigma_2^2(R^\pi_T)^2]\phi_x^2(t,R^\pi_t) + \int_{-1}^{\infty}|\phi(t,R^\pi_t+bR^\pi_t\eta)-\phi(t,R^\pi_t)|^2q(d\eta)\Bigg\}\Bigg]dt<\infty.$$

Then, $\phi\ge V$ *on* $[0,T]\times\mathbb{R}_+$.

b) Suppose further that, for each $x\in\mathcal{S}$, *there exists an admissible control* $(a^*,b^*)\in\Pi$ *such that*

(iv)

$$\mathcal{A}^{a^*,b^*}\phi(t,x)=0, \tag{2.10}$$

(v) and $\{\phi(\tau,R^{a^*,b^*}_\tau)\}_{\tau\le T}$ *is uniformly integrable.*

Then,(a^*_t,b^*_t) *is an optimal control and*

$$\phi=V\ \ on\ \ [0,T]\times\mathbb{R}_+\ .$$

3. **The Solution of HJB Equation under no short-selling**

In this section, we solve the HJB Equation (2.6) under the assumption of no short-selling in the investment. This implies that $b_t\ge 0$. For technical consideration, set $\tilde{b}_t=b_tx$ though we continue to write b_t here instead of $\tilde{b}_t$.

To solve the HJB equation, we shall fit a solution of the form

$$V(t,x)=c_0-\frac{\delta}{\gamma}\exp\left\{-\gamma xe^{r_0(T-t)}+h(T-t)\right\}, \tag{3.1}$$

where $h(\cdot)$ is a suitable function such that (3.1) is a solution to (2.6), with the boundary condition $h(0)=0$.

From (3.1), we note

$$\begin{aligned}
V_t &= (V-c_0)\left(\gamma xr_0e^{r_0(T-t)}-h'(T-t)\right)\\
V_x &= (V-c_0)\left(-\gamma e^{r_0(T-t)}\right)>0\\
V_{xx} &= (V-c_0)\left(\gamma^2e^{2r_0(T-t)}\right)<0\\
V(t,x+b_t\eta)-V(t,x)-V_x(t,x)b_t\eta &= (V-c_0)\left(\exp\left\{-\gamma b_t\eta e^{r_0(T-t)}\right\}-1+\gamma b_t\eta e^{r_0(T-t)}\right)
\end{aligned}$$

Substituting these into (2.5), differentiating the generator (2.5) with respect to a_t, and setting the derivative equal to zero, we get

$$\hat{a}_t = -\frac{\theta}{\sigma_1^2} \cdot \frac{V_x}{V_{xx}} = \frac{\theta}{\gamma\sigma_1^2} e^{-r_0(T-t)} > 0. \tag{3.2}$$

Recall that $a_t \in [0,1]$. Based on the relationship between $\hat{a}_t$ and the interval $[0,1]$, in order to get the optimal policy a_t^* we have to discuss the optimal value of a_t in three cases:

- $\theta \leq \gamma\sigma_1^2$;
- $\gamma\sigma_1^2 < \theta < \gamma\sigma_1^2\ e^{r_0 T}$;
- $\theta \geq \gamma\sigma_1^2\ e^{r_0 T}$.

CASE I: Let $\gamma\sigma_1^2 < \theta < \gamma\sigma_1^2\ e^{r_0 T}$.

Since $a_t \in [0,1]$, we can conclude that the optimal policy a_t^* should be

$$a_t^* = \begin{cases} \hat{a}_t, & \hat{a}_t \leq 1; \\ 1, & \hat{a}_t > 1. \end{cases}$$

When $0 \leq t \leq T + \dfrac{1}{r_0} \ln \dfrac{\gamma\sigma_1^2}{\theta}$, we have $\hat{a}_t \leq 1$, and $a_t^* = \hat{a}_t = \dfrac{\theta}{\gamma\sigma_1^2}\ e^{-r_0(T-t)}$. The HJB equation (2.6) becomes:

$$\begin{aligned} 0 = \inf_{b_t} G(b_t) \quad &= \quad -h'(T-t) - (\mu-\theta)\gamma\ e^{r_0(T-t)} - \frac{\theta^2}{2\sigma_1^2} \\ &\quad + \inf_{b_t} \Big\{ -\gamma(r_1-r_0)b_t\ e^{r_0(T-t)} + \frac{1}{2} b_t^2 \sigma_2^2 \gamma^2\ e^{2r_0(T-t)} \\ &\quad + \int_{-1}^{\infty} \left(\exp\left\{-\gamma b_t \eta\ e^{r_0(T-t)}\right\} - 1 + \gamma b_t \eta\ e^{r_0(T-t)}\right)\ dq(\eta) \Big\} \end{aligned} \tag{3.3}$$

Differentiating $G(b_t)$ with respect to b_t and setting the derivative equal to zero, we get

$$g(b_t) = b_t \sigma_2^2 \gamma^2 e^{r_0(T-t)} - \gamma(r_1-r_0) + \int_{-1}^{\infty} \gamma\eta \left(1 - \exp\{-\gamma b_t \eta\ e^{r_0(T-t)}\}\right)\ dq(\eta) = 0. \tag{3.4}$$

Lemma 3.1. *Equation* (3.4) *has a unique finite positive root* b_t^*.

Proof. Since

$$g'(b_t) = \sigma_2^2 \gamma^2 e^{r_0(T-t)} + \int_{-1}^{\infty} \gamma^2 \eta^2 \exp\left\{-\gamma b_t \eta\ e^{r_0(T-t)}\right\} dq(\eta) \cdot e^{r_0(T-t)} > 0,$$

$g(b_t)$ is a strictly increasing function of b_t, we have

$$g(0) = -\gamma(r_1-r_0) < 0, \quad \text{and} \quad \mathrm{g}(\infty) = \infty.$$

Thus, $g(b_t) = 0$ has a unique positive root b_t^*. $\square$

Remark 3.2. 1. From Equation (3.4), we see that b_t^* is dependent on σ_2, r_0, r_1, $T-t$, γ, and the distribution of the jump sizes, which implies the risk tolerance of the insurer.

2. Note that b_t^* is an increasing function with respect to $r_1 - r_0$ and γ, and a decreasing function respect to σ_2 and t.
3. Larger the positive jumps are, the bigger b_t^* is, and *vice versa*.

Returning back to our study in Case I, let us substitute b_t^* into (3.3) so that

$$\begin{aligned} h'(T-t) &= -(\mu-\theta)\gamma\ e^{r_0(T-t)} - \frac{\theta^2}{2\sigma_1^2} - \gamma(r_1-r_0)b_t^*\ e^{r_0(T-t)} + \frac{1}{2}b_t^{*2}\sigma_2^2\gamma^2 e^{2r_0(T-t)} \\ &\quad + \int_{-1}^{\infty}\left(\exp\left\{-\gamma b_t^*\eta\ e^{r_0(T-t)}\right\} - 1 + \gamma b_t^*\eta\ e^{r_0(T-t)}\right)\ dq(\eta) \end{aligned}$$

Integrating this over $[0,t]$ we obtain

$$\begin{aligned} h(T-t) &= \frac{\gamma}{r_0}(\mu-\theta)(e^{r_0T} - e^{r_0(T-t)}) + \gamma(r_1-r_0)\int_0^t b_s^*\ e^{r_0(T-s)}ds \\ &\quad + \frac{\theta^2}{2\sigma_1^2}t - \frac{1}{2}\sigma_2^2\gamma^2\int_0^t b_s^{*2}\ e^{2r_0(T-s)}ds \\ &\quad - \int_0^t\int_{-1}^{\infty}\left(\exp\left\{-\gamma b_s^*\eta\ e^{r_0(T-s)}\right\} - 1 + \gamma b_s^*\eta\ e^{r_0(T-s)}\right)\ dq(\eta)ds + h(T) \end{aligned} \tag{3.5}$$

where $h(T)$ will be deduced later.

When $T + \frac{1}{r_0}\ln\frac{\gamma\sigma_1^2}{\theta} \leq t \leq T$, we have $\hat{a}_t > 1$. Then, $a_t^* = 1$. The HJB equation (2.6) now takes the form:

$$\begin{aligned} 0 &= \inf_{b_t}\tilde{G}(b_t) \\ &= -h'(T-t) - \gamma\mu e^{r_0(T-t)} + \frac{1}{2}\sigma_1^2\gamma^2 e^{2r_0(T-t)} \\ &\quad + \inf_{b_t}\Bigg\{ -\gamma(r_1-r_0)b_t\ e^{r_0(T-t)} + \frac{1}{2}b_t^2\sigma_2^2\gamma^2 e^{2r_0(T-t)} \\ &\quad + \int_{-1}^{\infty}\left(\exp\left\{-\gamma b_t\eta\ e^{r_0(T-t)}\right\} - 1 + \gamma b_t\eta\ e^{r_0(T-t)}\right)\ dq(\eta)\Bigg\} \end{aligned} \tag{3.6}$$

Differentiating $\tilde{G}(b_t)$ with respect to b_t and setting the derivative equal to zero, we get the same equation as (3.4). It follows now from Lemma3.1 that, if the jump part is explicitly defined, we can derive the root b_t^* or get the numeric solution, after all, the root does exist and is unique. Therefore,

$$\begin{aligned} h'(T-t) &= -\gamma\mu\ e^{r_0(T-t)} + \frac{1}{2}\sigma_1^2\gamma^2\ e^{2r_0(T-t)} - \gamma(r_1-r_0)b_t^*\ e^{r_0(T-t)} \\ &\quad + \frac{1}{2}b_t^{*2}\sigma_2^2\gamma^2 e^{2r_0(T-t)} \\ &\quad + \int_{-1}^{\infty}\left(\exp\left\{-\gamma b_t^*\eta\ e^{r_0(T-t)}\right\} - 1 + \gamma b_t^*\eta\ e^{r_0(T-t)}\right)\ dq(\eta). \end{aligned}$$

Integration of this from t to T yields

$$\begin{aligned} h(T-t) &= \frac{\gamma}{r_0}\mu(1-e^{r_0(T-t)}) - \gamma(r_1-r_0)\int_t^T b_s^* \ e^{r_0(T-s)}ds \\ &\quad -\frac{\gamma^2}{4r_0}\sigma_1^2(1-e^{2r_0(T-t)}) + \frac{1}{2}\sigma_2^2\gamma^2\int_t^T b_s^{*2}e^{2r_0(T-s)}ds \\ &\quad +\int_t^T\int_{-1}^{\infty}\left(\exp\left\{-\gamma b_s^* \ \eta e^{r_0(T-s)}\right\} - 1 + \gamma b_s^* \ \eta e^{r_0(T-t)}\right) \ dq(\eta)ds \end{aligned} \tag{3.7}$$

By the continuity in t of the value function at $t = t_0 = T + \frac{1}{r_0}\ln\frac{\gamma\sigma_1^2}{\theta}$, it is easy to show that

$$\begin{aligned} h(T) &= \frac{\gamma}{r_0}\mu(1-e^{r_0T}) - \gamma(r_1-r_0)\int_0^T b_s^* \ e^{r_0(T-s)} \ ds + \frac{\gamma}{r_0}\theta(e^{r_0T}-e^{r_0(T-t_0)}) \\ &\quad -\frac{\theta}{2\sigma_1^2}t_0 - \frac{\gamma^2}{4r_0}\sigma_1^2(1-e^{2r_0(T-t_0)}) + \frac{1}{2}\sigma_2^2\gamma^2\int_0^T b_s^{*2} \ e^{2r_0(T-s)}ds \\ &\quad +\int_0^T\int_{-1}^{\infty}\left(\exp\left\{-\gamma b_s^* \ \eta e^{r_0(T-s)}\right\} - 1 + \gamma b_s^*\eta \ e^{r_0(T-s)}\right) \ dq(\eta)ds. \end{aligned} \tag{3.8}$$

By the Verification Theorem (2.3) *supra*, we can confirm that the solution of the HJB equation is the value function that we are looking for.

Theorem 3.3. *Let $\gamma\sigma_1^2 < \theta < \gamma\sigma_1^2 e^{r_0T}$.*

The value function is given by

$$V(t,x) = c_0 - \frac{\delta}{\gamma}\exp\{-\gamma x e^{r_0(T-t)} + h(T-t)\} \tag{3.9}$$

Set $t_0 = T + \frac{1}{r_0}\ln\frac{\gamma\sigma_1^2}{\theta}$.

1. *When $0 \le t \le t_0$, we have*

$$\begin{aligned} h(T-t) &= \frac{\gamma}{r_0}(\mu-\theta)(e^{r_0T}-e^{r_0(T-t)}) + \gamma(r_1-r_0)\int_0^t b_s^* \ e^{r_0(T-s)} \ ds \\ &\quad +\frac{\theta^2}{2\sigma_1^2}t - \frac{1}{2}\sigma_2^2\gamma^2\int_0^t b_s^{*2} \ e^{2r_0(T-s)} \ ds \\ &\quad -\int_0^t\int_{-1}^{\infty}\exp\left\{-\gamma b_s^*\eta \ e^{r_0(T-s)}\right\} - 1 + \gamma b_s^*\eta \ e^{r_0(T-s)} \ dq(\eta)ds + h(T), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} h(T) &= \frac{\gamma}{r_0}\mu(1-e^{r_0T}) - \gamma(r_1-r_0)\int_0^T b_s^* \ e^{r_0(T-s)}ds + \frac{\gamma}{r_0}\theta\left(e^{r_0T}-e^{r_0(T-t_0)}\right) \\ &\quad -\frac{\theta}{2\sigma_1^2}t_0 - \frac{\gamma^2}{4r_0}\sigma_1^2\left(1-e^{2r_0(T-t_0)}\right) + \frac{1}{2}\sigma_2^2\gamma^2\int_0^T b_s^{*2} \ e^{2r_0(T-s)} \ ds \\ &\quad +\int_0^T\int_{-1}^{\infty}\exp\left\{-\gamma b_s^*\eta \ e^{r_0(T-s)}\right\} - 1 + \gamma b_s^*\eta \ e^{r_0(T-s)} \ dq(\eta)ds. \end{aligned}$$

The optimal retention level is

$$a_t^* = \frac{\theta}{\gamma\sigma_1^2} e^{-r_0(T-t)}$$

.

2. *Next when $t_0 \le t \le T$, we have*

$$\begin{aligned} h(T-t) &= \frac{\gamma}{r_0}\mu(1-e^{r_0(T-t)}) - \gamma(r_1-r_0)\int_t^T b_s^* \; e^{r_0(T-s)} \; ds \\ &\quad -\frac{\gamma^2}{4r_0}\sigma_1^2\left(1-e^{2r_0(T-t)}\right) + \frac{1}{2}\sigma_2^2\gamma^2 \int_t^T b_s^{*2} \; e^{2r_0(T-s)} \; ds \\ &\quad + \int_t^T \int_{-1}^{\infty} \exp\left\{-\gamma b_s^* \eta e^{r_0(T-s)}\right\} - 1 + \gamma b_s^* \eta \; e^{r_0(T-t)} \; dq(\eta) ds \end{aligned} \tag{3.11}$$

And the optimal retention level is $a_t^* = 1$.

Moreover, the optimal fund invested in the risky asset, b_t^*, is the unique positive root of the Equation (3.4).

We shall be very concise in the next two cases.

CASE II: Let $\theta \le \gamma\sigma_1^2$.

In this case, we have $\hat{a}_t \le 1$, and therefore $a_t^* = \hat{a}_t = \dfrac{\theta}{\gamma\sigma_1^2} \; e^{-r_0(T-t)}$, for $0 \le t \le T$. The expression of $h(T-t)$ is the same as (3.5). For $t = T$, that expression gives us

$$\begin{aligned} h(T) &= \frac{\gamma}{r_0}(\mu-\theta)(1-e^{r_0 T}) - \gamma(r_1-r_0)\int_0^T b_s^* \; e^{r_0(T-s)} \; ds \\ &\quad -\frac{\theta^2}{2\sigma_1^2}T + \frac{1}{2}\sigma_2^2\gamma^2 \int_0^T b_s^{*2} \; e^{2r_0(T-s)} ds \\ &\quad + \int_0^T \int_{-1}^{\infty} \exp\left\{-\gamma b_s^* \; \eta e^{r_0(T-s)}\right\} - 1 + \gamma b_s^* \; \eta e^{r_0(T-s)} \; dq(\eta) ds \end{aligned} \tag{3.12}$$

We shall proceed now to summarize the results for the Case II.

Theorem 3.4. *Assume that $\theta \le \gamma\sigma_1^2$. For all $0 \le t \le T$, we have*

$$V(t,x) = c_0 - \frac{\delta}{\gamma} \; \exp\left\{-\gamma x e^{r_0(T-t)} + \tilde{h}(T-t)\right\}, \tag{3.13}$$

where $\tilde{h}(T-t)$ has the same expression as $h(T-t)$ in (3.10) *with*

$$\begin{aligned} h(T) &= \frac{\gamma}{r_0}(\mu-\theta)(1-e^{r_0 T}) - \gamma(r_1-r_0)\int_0^T b_s^* \; e^{r_0(T-s)} ds - \frac{\theta^2}{2\sigma_1^2}T \\ &\quad + \frac{1}{2}\sigma_2^2\gamma^2 \int_0^T b_s^{*2} \; e^{2r_0(T-s)} \; ds \\ &\quad + \int_0^T \int_{-1}^{\infty} \exp\left\{-\gamma b_s^* \eta \; e^{r_0(T-s)}\right\} - 1 + \gamma b_s^* \eta \; e^{r_0(T-s)} \; dq(\eta) ds \end{aligned}$$

Moreover, the optimal policy is given by $a_t^* = \dfrac{\theta}{\gamma\sigma_1^2}\, e^{-r_0(T-t)}$*, and* b_t^* *is the unique positive root of the Equation* (3.4)*.*

CASE III: Let $\theta \geq \gamma\sigma_1^2 e^{r_0 T}$.

Here, we get $\hat{a}_t \geq 1$, so that $a_t^* = 1$, for $0 \leq t \leq T$. Now $h(T-t)$ is the same as the expression in (3.7).

Summary of the results for the final case.

Theorem 3.5. *Assume that* $\theta \geq \gamma\sigma_1^2 e^{r_0 T}$*.*

For all $0 \leq t \leq T$*, we have*

$$V(t,x) = c_0 - \frac{\delta}{\gamma}\ \exp\left\{-\gamma x\ e^{r_0(T-t)} + \bar{h}(T-t)\right\}, \tag{3.14}$$

where $\bar{h}(T-t)$ *has the same expression as* $h(T-t)$ *in* (3.11)*.*

Moreover, $a_t^* = 1$*, and* b_t^* *is the same as the above.*

Remark 3.6. The problem is categorized into three cases, based on the relationships among θ, γ, σ_1, r_0 and T, which mainly manifests the risk tolerance of the insurer versus the cost of reinsurance.
(a) When θ is relatively small (CASE II), the insurer is willing to buy reinsurance, as can be seen from Theorem 3.4.
(b) As θ takes moderate values,(CASE I), the insurer buys reinsurance at the early stages (time) of the contract. However, as the initial/contract time is close to the maturity time, the insurer gives up reinsurance to keep more surplus as benefits.
(c) As θ increases, the insurer is unwilling to buy any reinsurance, (CASE III, $a_t^* = 1$ means non-reinsurance).

Remark 3.7. When the insurer is willing to purchase reinsurance, the optimal retention level is $a_t^* = \frac{\theta}{\gamma\sigma_1^2} e^{-r_0(T-t)}$, (in fact, $1 - a_t^*$ is the optimal reinsurance proportion). Therefore, larger the σ_1 is, the smaller a_t^* is. Therefore, the more uncertain the claims are, the more the risk is transferred to the reinsurer in case of bankruptcy.

When t is larger, i.e. the present time is nearer to the terminal time, a_t^* is larger, too. The reason is that when the time interval of contract period becomes smaller, it is less likely that much payout will be claimed. So, the insurer is more willing to keep the surplus to herself rather than buying reinsurance.

Remark 3.8. We also note that a_t^* and b_t^* do not affect one another. This is because we assumed independence between the compensation process and the dynamics of the risky asset. In fact, if the two processes are related, a more complex model could be built, such as plugging in copula or just correlation coefficient. We will discuss it in a future work.

4. Markov chain approximation method under both constraints

In this section, we take a specific case as an example. The parameters are listed below:

- the minimal and maximal surplus of the company are respectively $m = 0$ and $M = 1$.
- $\mu = 0.01$, $\theta = 0.012 > \mu$ which means it is not a cheap reinsurance.
- $r_0 = 0.05$, $r_1 = 0.2$, $\sigma_1 = 0.1$, $\sigma_2 = 0.5$.
- $c_0 = 5$, $\delta = 0.5$, $\gamma = 2$.
- $\lambda = 2$, $T = 5$.

We use the Markov chain approximation method to tackle this problem. The approximation scheme is similar to that in Section 4 of Zhang, Liu, Kannan (2011):
(1) Firstly, we discretize the problem, cutting the x-interval into $\frac{M-m}{h}$ pieces with $h = 0.05$, and slicing the t-interval into $\dfrac{T-0}{\tau}$ pieces with $\tau = 0.1$.
(2) Next, we construct the Markov chain to approximate the initial process, write down the matrix of transition probability, including the diffusion part and the jump part. As for the jump part, we assume the jump sizes have the exponential distribution on $[-1, \infty)$ with the intensity $\lambda = 2$.
(3) Finally, we calculate the value function according to the dynamic programming equation:

$$V^{h,\tau}(x, n\tau) = \sup_{(a,b)\in[0,1]\times[0,1]} \left\{ \sum_y p^{(JD)}(x, y|a, b, \lambda) V^{h,\tau}(y, n\tau + \tau) \right\}.$$

This can be solved by a simple backward iteration procedure. Here, $p^{(JD)}(x, y|a, b, \lambda)$ denotes the transition probability of the Markov chain from state x to state y, (with the given variables a, b and λ). We use the function **fmincon** in MATLAB to solve this problem.

After establishing the algorithm, the value function and the optimal policy could be obtained. The results are presented in Figures 1, 2, and 3.

Figure 1 depicts the value function related to time and wealth. It's concave and smooth.

Figure 2 depicts the optimal retention level. Affected by the boundary for wealth x and the jump-diffusion process, it fluctuates and depends some what to the wealth x. Fortunately, we can still observe the broad outline and the tendency.

When the wealth is small, the insurer does not possess enough fund to buy reinsurance, so it is prone to accept the risk of claims herself.

As the wealth grows, the insurer is more willing to buy reinsurance to transfer risk to some extent. Figure **3** shows the optimal fraction invested on the risky asset.

It is interesting that the more wealth the insurer has, the less likely that she will buy the risky asset. Of course, why would anyone want to take the risk of loss as the one already owns nearly the expected wealth?

Remark 4.1. As can be seen from the figures, except for the value function, the retention level and the proportion of stock have some volatility and are not that smooth. There are at least two reasons for this.

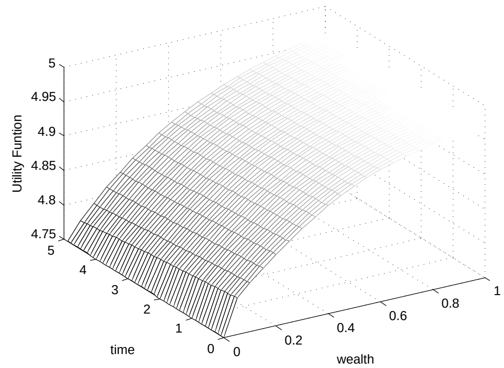

FIGURE 1. Value function as a function of time and initial wealth at time t

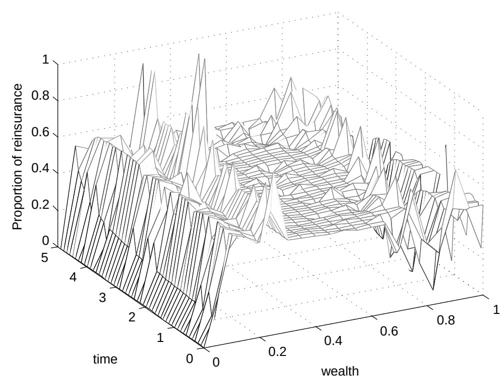

FIGURE 2. Optimal retention level as a function of time and initial wealth at time t

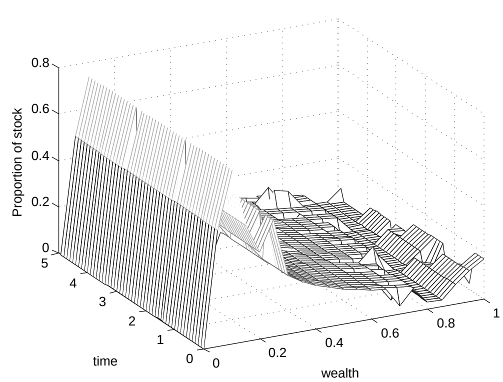

FIGURE 3. Optimal investment fraction as a function of time and initial wealth at time t

(1) Firstly, there exists some errors when we discretize the model, especially the jump part.

(2) Secondly, we cannot make h quite small, or else the transition probabilities would not exist, which will lead to the failure of this numerical method. It means, we have to limit the upper bound of the term $\frac{\tau}{h^2}$.

Remark 4.2. When we do the backward iteration procedure, we use the function **fmincon** in Matlab. If the constraints become more complex, e.g. when the explicit expression couldnot be derived, we will not be able to use the existing function, and have to look for other algorithms. We are still working on it, and we have found the alternative optimization method to solve similar stochastic control problems. Results will appear in another article.

5. **Concluding Remarks**

In this paper, we discussed the optimal proportional reinsurance and investment in jump diffusion markets with the constraints of no short-selling and no borrowing. Closed form of the value function was derived under the no short-selling condition. If both no short-selling and no borrowing constraints are taken into consideration, it is difficult to separate t and x. However, we can use Markov chain approximation method to solve this problem numerically. It has the advantage that the business strategy can be dynamically optimized, even R_t exists in the HJB equation, when analytical solutions are hard to derive due to the inseparability of t and x. Strict conditions are imposed to make sure that the matrix of transition probabilities are well defined, which is also strongly related to the partition of t and x. Also the question of more effective algorithm might be raised, and we are still working on it.

In addition, more realistic factors can be brought in, such as the transaction costs, the related structure of the different dynamic processes, and the risk management. However, as more factors are taken into consideration, the more difficult the problem becomes, leading to tougher and more challenging work. Indeed, analytical solutions will be even harder to derive, so we have to apply numerical algorithms to solve those problems. Markov chain approximation method is effective in certain situations, and it still has a long way to go.

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