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FIRST-ORDER DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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ABSTRACT. We study a first-order boundary value problem subject to some boundary conditions given by Riemann-Stieltjes integrals. Using a monotone iterative method, we formulate sufficient conditions which guarantee the existence of extremal or quasi-solutions in the corresponding region bounded by upper and lower solutions of our problems. The case when a unique solution exists is also investigated. Some examples are given to illustrate our results.

Key words: Boundary value problems for differential equations, monotone iterative technique, lower and upper solutions, existence of solutions, extremal solutions, quasi-solutions.

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1. INTRODUCTION

In this paper, we investigate the first-order differential equation of the form:

$$x'(t) = f(t, x(t)) \equiv Fx(t), \quad t \in J = [0, T], \quad T < \infty, \tag{1.1}$$

subject to various nonlocal Boundary Conditions (BCs):

$$x(0) = \lambda[x] + d, \tag{1.2}$$

$$x(T) = \lambda[x] + d, \tag{1.3}$$

$$x(0) = -\lambda[x] + d, \tag{1.4}$$

$$x(T) = -\lambda[x] + d, \tag{1.5}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R})$, $d \in \mathbb{R}$. Here, λ denotes a linear functional on $C(J)$ given by Riemann-Stieltjes integral

$$\lambda[x] = \int_0^T x(t)dA(t)$$

with a suitable function A of bounded variation. The advantage is that the well-studied multipoint and integral BCs are both included as special cases.

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Note that, BCs in (1.2)–(1.5) cover some nonlocal BCs, for example

$$\begin{aligned}
\lambda[x] &= \beta x(T) + a,\\
\lambda[x] &= \sum_{i=1}^{m} \beta_i x(\xi_i) + b, \quad 0 < \xi_1 < \xi_2 < \cdots < \xi_m < T,\\
\lambda[x] &= \int_0^T x(t)g(t)dt + c, \ \ g \in C(J, \mathbb{R}_+).
\end{aligned}$$

If $a = 0$ and $\beta = 1$ or $\beta = -1$, then we have periodic or anti-periodic problem, see for example [1], [3], [11], [12], see also [7]; for multipoint problems, see [4], [6], and BC with $g(t) = k \in \mathbb{R}$, see [5]. Indeed, we have more papers in which boundary value problems have been discussed with BCs as in the above mentioned special cases for λ.

It is important to indicate that boundary conditions involving Stieltjes integrals appeared in some papers in which the problem of existence of positive solutions to differential equations have been discussed. When we apply fixed point theorems in cones, then we can obtain conditions which guarantee the existence of positive solutions in cones for boundary problems also for cases when the measure dA changes sign, see for example, [13], [14], [8], [9].

It is well known, that the monotone iterative technique offers an approach for obtaining approximate solutions to boundary value problems of differential equations, see for example [12], [2]. According to our knowledge, using the monotone iterative technique, the existence results are formulated only for special cases of functional λ. In this paper, we study problem (1.1) under quite general boundary conditions given by functional λ. To obtain the existence results we use the monotone iterative method based on inequalities and therefore we are not able to discuss our problems when the measure dA changes sign. Therefore, we discuss boundary value problems under the assumption that the measure dA in functional λ is non-negative. Looking on BCs (1.4), (1.5) we see that the measure $-dA$ is now non-positive to cover the case of anti-periodic solutions too. The monotone iterative method has also been discussed in paper [10] to second order differential equations with Stieltjes integrals.

We establish sufficient conditions under which boundary value problems have solutions: extremal, quasi or a unique solution too. Two examples are added to verify theoretical results. Let us introduce the following assumption:

H_1 : $f \in C(J \times \mathbb{R}, \mathbb{R})$, A is a function of bounded variation and the measure dA is non-negative.

2. LEMMAS

First, we consider two boundary value problems:

$$\left\{\begin{array}{ll} x'(t) & = -M(t)x(t) + h(t), \quad t \in J, \\ x(0) & = \lambda[x] + d, \quad d \in \mathbb{R}, \end{array}\right. \tag{2.1}$$

$$\left\{\begin{array}{ll} x'(t) & = M(t)x(t) - h(t), \quad t \in J, \\ x(T) & = \lambda[x] + d, \quad d \in \mathbb{R}. \end{array}\right. \tag{2.2}$$

Lemma 2.1. *Suppose that:*

$H_2:$ $M, h \in C(J, \mathbb{R})$, *and* A *is a function of bounded variation and moreover*

$$\int_0^T \exp\left(-\int_0^t M(s)ds\right) dA(t) \neq 1.$$

Then problem (2.1) *has a unique solution given by*

$$\begin{aligned} x(t) &= P(t)\left\{\left[1 - \int_0^T P(s)dA(s)\right]^{-1}\left[d + \int_0^T \left(P(s)\int_0^s P^{-1}(u)h(u)du\right)dA(s)\right]\right\} \\ &\quad + P(t)\int_0^t P^{-1}(s)h(s)ds \end{aligned}$$

with

$$P(t) = \exp\left(-\int_0^t M(\xi)d\xi\right).$$

Proof. Note that

$$x(t) = \exp\left(-\int_0^t M(s)ds\right)\left[x(0) + \int_0^t \exp\left(\int_0^s M(\tau)d\tau\right)h(s)ds\right].$$

Now, using the boundary condition $x(0) = \lambda[x] + d$ and Assumption H_2, we have the assertion of this lemma. □

Similarly as Lemma 2.1, we can prove the following result.

Lemma 2.2. *Suppose that:*

$H_3:$ $M, h \in C(J, \mathbb{R})$, *and* A *is a function of bounded variation and moreover*

$$\int_0^T \exp\left(-\int_t^T M(s)ds\right) dA(t) \neq 1$$

Then problem (2.2) *has a unique solution given by*

$$\begin{aligned} x(t) &= Q(t)\left\{\left[1 - \int_0^T Q(s)dA(s)\right]^{-1}\left[d + \int_0^T \left(Q(s)\int_s^T Q^{-1}(\tau)h(\tau)d\tau\right)dA(s)\right]\right\} \\ &\quad + Q(t)\int_t^T Q^{-1}(s)h(s)ds \end{aligned}$$

with

$$Q(t) = \exp\left(-\int_t^T M(\xi)d\xi\right).$$

Lemma 2.3. *Suppose that:*

H_4 : $M \in C(J,\mathbb{R})$, *A is a function of bounded variation, the measure dA is non-negative and moreover*

$$\int_0^T \exp\left(-\int_0^t M(s)ds\right) dA(t) < 1. \tag{2.3}$$

Let $p \in C^1(J,\mathbb{R})$ and

$$\begin{cases} p'(t) & \leq & -M(t)p(t), \quad t \in J, \\ p(0) & \leq & \lambda[p]. \end{cases}$$

Then $p(t) \leq 0$ on J.

Proof. Indeed,

$$p(t) \leq \exp\left(-\int_0^t M(s)ds\right) p(0).$$

Now, using the condition $p(0) \leq \lambda[p]$ and (2.3), we have the assertion. □

In a similar way, we can prove the following lemma.

Lemma 2.4. *Suppose that:*

H_5 : $M \in C(J,\mathbb{R})$, *A is a function of bounded variation, the measure dA is non-negative and moreover*

$$\int_0^T \exp\left(-\int_t^T M(s)ds\right) dA(t) < 1. \tag{2.4}$$

Let $p \in C^1(J,\mathbb{R})$ and

$$\begin{cases} p'(t) & \geq & M(t)p(t), \quad t \in J, \\ p(T) & \leq & \lambda[p]. \end{cases}$$

Then $p(t) \leq 0$ on J.

3. SOME COMMENTS TO SECTION 2

1. Let

$$\int_0^T x(t)dA(t) = \gamma x(\mu), \quad \gamma \geq 0.$$

Then conditions (2.3) and (2.4) take respectively the form

$$\gamma \exp\left(-\int_0^\mu M(s)ds\right) < 1, \quad \mu \in (0,T],$$

$$\gamma \exp\left(-\int_\mu^T M(s)ds\right) < 1, \quad \mu \in [0,T).$$

2. Let

$$\int_0^T x(t)dA(t) = \sum_{i=1}^m \gamma_i x(\mu_i), \quad \gamma_i > 0.$$

Then conditions (2.3) and (2.4) take respectively the form

$$\sum_{i=1}^{m} \gamma_i \exp\left(-\int_0^{\mu_i} M(s)ds\right) < 1, \quad 0 < \mu_1 < \mu_2 < \cdots < \mu_m \leq T,$$

$$\sum_{i=1}^{m} \gamma_i \exp\left(-\int_{\mu_i}^{T} M(s)ds\right) < 1, \quad 0 \leq \mu_1 < \mu_2 < \cdots < \mu_m < T.$$

3. Let

$$\int_0^T x(t)dA(t) = \int_0^T x(t)g(t)dt, \quad g \in C(J, \mathbb{R}_+).$$

Then conditions (2.3) and (2.4) take respectively the form

$$\int_0^T \exp\left(-\int_0^t M(s)ds\right) g(t)dt < 1,$$

$$\int_0^T \exp\left(-\int_t^T M(s)ds\right) g(t)dt < 1.$$

4. Let

$$\int_0^T x(t)dA(t) = \sum_{i=1}^{m} \int_{\mu_i}^{\gamma_i} x(s)g(s)ds,$$

where $g \in C(J, \mathbb{R}_+)$, $\quad 0 \leq \mu_1 < \gamma_1 < \mu_2 < \gamma_2 < \cdots < \mu_m < \gamma_m \leq T$.

Then conditions (2.3) and (2.4) take respectively the form

$$\sum_{i=1}^{m} \int_{\mu_i}^{\gamma_i} \exp\left(-\int_0^t M(s)ds\right) g(t)dt < 1,$$

$$\sum_{i=1}^{m} \int_{\mu_i}^{\gamma_i} \exp\left(-\int_t^T M(s)ds\right) g(t)dt < 1.$$

5. Also we can consider the case when the above points **2** and **4** are combined.

4. EXISTENCE OF SOLUTIONS TO PROBLEMS (1.1), (1.2) and (1.1), (1.3)

Now, we derive a fixed point result for nondecreasing mappings in ordered spaces which play a central role in our investigations. We say that $Q : [a, b] \to [a, b]$ is nondecreasing if $Qx \leq Qy$ for $x, y \in [a, b]$ and $x \leq y$. We say that $x \in [a, b]$ is the least fixed point of Q in $[a, b]$ if $x = Qx$ and if $x \leq y$ whenever $y \in [a, b]$ and $y = Qy$. The greatest fixed point of Q in $[a, b]$ is defined similarly, by reversing the inequality. If both least and greatest fixed point of Q in $[a, b]$ exist, we call them extremal fixed points of Q in $[a.b]$.

Theorem 4.1 (see [2]). *Let $[a, b]$ be an ordered interval in a subset Y of an ordered Banach space X and let $Q : [a, b] \to [a, b]$ be a nondecreasing mapping. If each sequence $\{Qx_n\} \subset Q([a, b])$ converges, whenever $\{x_n\}$ is a monotone sequence in $[a, b]$, then the sequence of Q-iteration of a converges to the least fixed point x_* of*

Q and the sequence of Q-iteration of b converges to the greatest fixed point x^* of Q. Moreover,

$$x_* = \min\{y \in [a,b] : \ y \geq Qy\}, \quad \textit{and} \ \ x^* = \max\{y \in [a,b] : \ y \leq Qy\}.$$

Let us introduce the following definition.

We say that $u \in C^1(J,\mathbb{R})$ is a lower solution of (1.1), (1.2) if

$$u'(t) \leq Fu(t), \ t \in J, \quad u(0) \leq \lambda[u] + d,$$

and it is an upper solution of (1.1), (1.2) if the above inequalities are reversed.

Theorem 4.2. *Assume that Assumption H_1 holds. Let $y_0, z_0 \in C^1(J,\mathbb{R})$ be lower and upper solutions of problem* (1.1), (1.2) *respectively and $y_0(t) \leq z_0(t)$, $t \in J$. In addition, we assume that:*

H_6 : *there exists a function $M \in C(J,\mathbb{R})$ such that condition* (2.3) *holds and*

$$f(t,u_1) - f(t,v_1) \leq M(t)[v_1 - u_1]$$

if $y_0(t) \leq u_1 \leq v_1 \leq z_0(t)$.

Then problem (1.1), (1.2) *has, in the sector $[y_0,z_0]$, extremal solutions, where*

$$[y_0,z_0] = \{w \in C^1(J,\mathbb{R}) : y_0(t) \leq w(t) \leq z_0(t), \ \ t \in J\}.$$

Proof. For each $h \in C(J,\mathbb{R})$, problem (2.1) has a unique solution x given in Lemma 2.1. Indeed, x is also a unique fixed point of operator S_h, so $x = S_h x$. Choose $h_1, h_2 \in C(J,\mathbb{R})$ such that $h_1(t) \leq h_2(t)$ on J. Let x_1, x_2 denote the solutions of problem (2.1) with h_1, h_2 instead of h, respectively. Put $p = x_1 - x_2$. Then,

$$\begin{cases} p'(t) &= -M(t)p(t) + h_1(t) - h_2(t) \leq -M(t)p(t), \ \ t \in J, \\ p(0) &= \lambda[p]. \end{cases}$$

In view of Lemma 2.3, we see that $x_1(t) \leq x_2(t)$ on J; so the operator S_h is nondecreasing. It is also continuous.

For $u \in [y_0,z_0]$, we put

$$\mathcal{F}u(t) = Fu(t) + M(t)u(t),$$

where the operator F is defined as in problem (1.1). We define the operator $S = S_{\mathcal{F}}$. Let $x_1 = Sy_0$, $x_2 = Sz_0$, so

$$\begin{cases} x_1'(t) &= -M(t)x_1(t) + \mathcal{F}y_0(t), \\ x_1(0) &= \lambda[x_1] + d, \end{cases}$$

and

$$\begin{cases} x_2'(t) &= -M(t)x_2(t) + \mathcal{F}z_0(t), \\ x_2(0) &= \lambda[x_2] + d. \end{cases}$$

Now, apply Lemma 2.3 with $p = y_0 - x_1$; so it is easy to show, using the definition of the lower solution y_0, that $y_0 \le x_1 = Sy_0$. Similarly, we can show $Sz_0 = x_2 \le z_0$. Put $x = x_1 - x_2$. Then

$$\begin{cases} x'(t) &= -M(t)x(t) + Fy_0(t) - Fz_0(t) + M(t)[y_0(t) - z_0(t)] \le -M(t)x(t), \\ x(0) &= \lambda[x]. \end{cases}$$

Using again Lemma 2.3, we see that $x_1 \le x_2$; so the operator S is nondecreasing. It means that $y_0 \le Su \le z_0$ for $u \in [y_0, z_0]$. Hence $S : [y_0, z_0] \to [y_0, z_0]$ and operator S is bounded because $\|Su\| \le \max(\|y_0\|, \|z_0\|) = B$.

Let $\{y_n\}$ be a monotone sequence in $[y_0, z_0]$; so $y_0 \le Sy_n \le z_0$. Hence $\|Sy_n\| \le B$. It is easy to show that $\{Sy_n\}$ is equicontinuous. By Arzeli-Ascoli theorem, $\{Ay_n\}$ is compact. It proves that $\{Sy_n\}$ converges in $S([y_0, z_0])$. Finally, operator S has a least and a greatest fixed point in $[y_0, z_0]$, by Theorem 4.1. It results that problem (1.1), (1.2) has minimal and maximal solutions in $[y_0, z_0]$. This ends the proof. □

Theorem 4.3. *Assume that all assumptions of Theorem* 4.2 *hold. In addition, we assume that the following assumption H_7 holds with:*

H_7 : *there exists a function $L \in C(J, \mathbb{R})$ such that $M(t) + L(t) \ge 0$, $t \in J$, condition* (2.3) *holds with L instead of $-M$ and*

$$f(t, v_1) - f(t, u_1) \le L(t)[v_1 - u_1]$$

if $y_0(t) \le u_1 \le v_1 \le z_0(t)$.

Then problem (1.1), (1.2) *has, in the sector $[y_0, z_0]$, a unique solution.*

Proof. In view of Theorem 4.2, $y_0 \le y \le z \le z_0$, where y, z are corresponding minimal and maximal solutions of problem (1.1), (1.2) in $[y_0, z_0]$. Put $p = z - y$. Hence,

$$\begin{cases} p'(t) &= Fz(t) - Fy(t) \le L(t)p(t), \\ p(0) &= \lambda[p]. \end{cases}$$

By Lemma 2.3, $z \le y$, so the assertion holds. □

Now, we will discuss problem (1.1), (1.3). We say that $u \in C^1(J, \mathbb{R})$ is a lower solution of (1.1), (1.3) if

$$u'(t) \le Fu(t),\ t \in J, \quad u(T) \ge \lambda[u] + d,$$

and it is an upper solution of (1.1), (1.3) if the above inequalities are reversed.

Theorem 4.4. *Assume that Assumption H_1 holds. Let $y_0, z_0 \in C^1(J, \mathbb{R})$ be lower and upper solutions of* (1.1), (1.3), *respectively and $z_0(t) \le y_0(t)$, $t \in J$. In addition, we assume that the following assumption H_8 holds with:*

H_8 : *there exists a function $M \in C(J, \mathbb{R})$ such that condition* (2.4) *holds and*

$$f(t, u_1) - f(t, v_1) \ge -M(t)[v_1 - u_1]$$

if $z_0(t) \leq u_1 \leq v_1 \leq y_0(t)$.

Then problem (1.1), (1.3) *has, in the sector* $[z_0, y_0]$, *extremal solutions.*

Proof. For each $h \in C(J, \mathbb{R})$, problem (2.2) has a unique solution x given in Lemma 2.2. Indeed, x is a unique fixed point of operator D_h, so $x = D_h x$. Choose $h_1, h_2 \in C(J, \mathbb{R})$ such that $h_1(t) \leq h_2(t)$ on J. Let x_1, x_2 denote the solutions of problem (2.2) with h_1, h_2 instead of h, respectively. Put $p = x_1 - x_2$. Then,

$$\begin{cases} p'(t) &= M(t)p(t) - h_1(t) + h_2(t) \geq M(t)p(t), \quad t \in J, \\ p(T) &= \lambda[p]. \end{cases}$$

In view of Lemma 2.4, we see that $x_1(t) \leq x_2(t)$ on J; so the operator D_h is nondecreasing. It is also continuous.

For $u \in [z_0, y_0]$, we put

$$\mathcal{F}u(t) = Fu(t) - M(t)u(t),$$

where the operator F is defined as in problem (1.1). We define the operator $D = D_{\mathcal{F}}$. Let $x_1 = Dz_0$, $x_2 = Dy_0$, so

$$\begin{cases} x_1'(t) &= M(t)x_1(t) + \mathcal{F}z_0(t), \\ x_1(T) &= \lambda[x_1] + d, \end{cases}$$

and

$$\begin{cases} x_2'(t) &= M(t)x_2(t) + \mathcal{F}y_0(t), \\ x_2(T) &= \lambda[x_2] + d. \end{cases}$$

Now, apply Lemma 2.4 with $p = x_2 - y_0$; so it is easy to show, using the definition of the lower solution y_0, that $y_0 \geq x_2 = Dy_0$. Similarly, we can show $Dz_0 = x_1 \geq z_0$. Put $x = x_1 - x_2$. Then

$$\begin{cases} x'(t) &= M(t)x(t) + Fz_0(t) - Fy_0(t) + M(t)[y_0(t) - z_0(t)] \geq M(t)x(t), \\ x(T) &= \lambda[x]. \end{cases}$$

Using again Lemma 2.4, we see that $x_1 \leq x_2$; so the operator D is nondecreasing. It means that $z_0 \leq Du \leq y_0$ for $u \in [z_0, y_0]$. Hence $D : [z_0, y_0] \to [z_0, y_0]$ and operator D is bounded because $\|Du\| \leq \max(\|y_0\|, \|z_0\|) = B$. Similarly as in the proof of Theorem 4.2, we can show that problem (1.1), (1.3) has minimal and maximal solutions in $[z_0, y_0]$. This ends the proof. □

Theorem 4.5. *Assume that all assumptions of Theorem* 4.4 *hold. In addition, we assume that the following assumption* H_9 *holds with:*

H_9 : *there exists a function* $L \in C(J, \mathbb{R})$ *such that* $M(t) + L(t) \geq 0$, $t \in J$, *condition* (2.4) *holds with* $-L$ *instead of* M *and*

$$f(t, v_1) - f(t, u_1) \geq -L(t)[v_1 - u_1]$$

if $z_0(t) \leq u_1 \leq v_1 \leq y_0(t)$.

Then problem (1.1), (1.3) *has, in the sector* $[z_0, y_0]$, *a unique solution.*

Proof. In view of Theorem 4.4, $z_0 \le z \le y \le y_0$, where z, y are corresponding minimal and maximal solutions of problem (1.1), (1.3) in $[z_0, y_0]$. Put $p = y - z$. Hence,

$$\begin{cases} p'(t) &= Fy(t) - Fz(t) \ge -L(t)p(t), \\ p(T) &= \lambda[p]. \end{cases}$$

By Lemma 2.4, $y \le z$, so the assertion holds. □

5. EXISTENCE OF SOLUTIONS TO PROBLEMS (1.1), (1.4) and (1.1), (1.5)

Note that BCs (1.4) and (1.5) have minus before the functional λ. It means that in our considerations will appear the notations of coupled lower and upper solutions. As the consequence of it, we have to discuss systems of equations or inequalities giving corresponding lemmas.

First, we consider the linear system of the form:

$$\begin{cases} y'(t) &= -M(t)y(t) + h_1(t), \quad t \in J, \quad y(0) = -\lambda[z] + d, \\ z'(t) &= -M(t)z(t) + h_2(t), \quad t \in J, \quad z(0) = -\lambda[y] + d. \end{cases} \tag{5.1}$$

Lemma 5.1. *Suppose that Assumption H_2 holds with h_1, h_2 instead of h.*

Then problem (5.1) *has a unique solution given by*

$$\begin{aligned} y(t) &= P(t)\left\{\frac{1}{\Delta}\left[d - \int_0^T P(s)H_2(s)dA(s)\right.\right. \\ &\quad \left.\left. - \left(d - \int_0^T P(s)H_1(s)dA(s)\right)\int_0^T P(s)dA(s)\right] + H_1(t)\right\}, \\ z(t) &= P(t)\left\{\frac{1}{\Delta}\left[d - \int_0^T P(s)H_1(s)dA(s)\right.\right. \\ &\quad \left.\left. - \left(d - \int_0^T P(s)H_2(s)dA(s)\right)\int_0^T P(s)dA(s)\right] + H_2(t)\right\} \end{aligned}$$

with

$$\begin{aligned} P(t) &= \exp\left(-\int_0^t M(\xi)d\xi\right), \quad \Delta = 1 - \left(\int_0^T P(s)dA(s)\right)^2, \\ H_1(t) &= \int_0^t P^{-1}(s)h_1(s)ds, \quad H_2(t) = \int_0^t P^{-1}(s)h_2(s)ds. \end{aligned}$$

Proof. Note that

$$\begin{aligned} y(t) &= P(t)[y(0) + H_1(t)], \\ z(t) &= P(t)[z(0) + H_2(t)]. \end{aligned}$$

Now, using the boundary conditions to eliminate $y(0), z(0)$ and Assumption H_2, we have the assertion of this lemma. □

Lemma 5.2. *Suppose that Assumption H_3 holds with h_1, h_2 instead of h.*

Then problem

$$\left\{\begin{array}{rcll} y'(t) & = & M(t)y(t) - h_1(t), \ \ t \in J, & y(T) = -\lambda[z] + d, \\ z'(t) & = & M(t)z(t) - h_2(t), \ \ t \in J, & z(T) = -\lambda[y] + d \end{array}\right. \tag{5.2}$$

has a unique solution (y, z) given as in Lemma 5.1 with

$$\begin{array}{rcl} P(t) & = & \exp\left(-\int_t^T M(\xi)d\xi\right), \quad \Delta = 1 - \left(\int_0^T P(s)dA(s)\right)^2, \\ H_1(t) & = & \int_t^T P^{-1}(s)h_1(s)ds, \quad H_2(t) = \int_t^T P^{-1}(s)h_2(s)ds. \end{array}$$

Lemma 5.3. *Suppose that Assumption H_4 holds.*

Let $p, q \in C^1(J, \mathbb{R})$ and

$$\left\{\begin{array}{rcll} p'(t) & \leq & M(t)p(t), \ \ t \in J, & p(0) \leq \lambda[q], \\ q'(t) & \leq & M(t)q(t), \ \ t \in J, & q(0) \leq \lambda[p]. \end{array}\right.$$

Then $p(t) \leq 0$, $q(t) \leq 0$ on J.

Proof. Indeed,

$$\begin{array}{rcl} p(t) & \leq & \exp\left(-\int_0^t M(s)ds\right) p(0), \\ q(t) & \leq & \exp\left(-\int_0^t M(s)ds\right) q(0). \end{array}$$

Now, using the conditions for $p(0)$, $q(0)$ and (2.3), we have the assertion. $\square$

Lemma 5.4. *Suppose that Assumption H_5 holds.*

Let $p, q \in C^1(J, \mathbb{R})$ and

$$\left\{\begin{array}{rcll} p'(t) & \geq & M(t)p(t), \ \ t \in J, & p(T) \leq \lambda[q], \\ q'(t) & \geq & M(t)q(t), \ \ t \in J, & q(T) \leq \lambda[p]. \end{array}\right.$$

Then $p(t) \leq 0$, $q(t) \leq 0$ on J.

Proof. Indeed,

$$\begin{array}{rcl} p(t) & \leq & \exp\left(-\int_t^T M(s)ds\right) p(T), \\ q(t) & \leq & \exp\left(-\int_t^T M(s)ds\right) q(T). \end{array}$$

Now, using the conditions for $p(0)$, $q(0)$ and (2.4), we have the assertion. $\square$

We say that $u, v \in C^1(J, \mathbb{R})$ are coupled lower and upper solutions of problem (1.1), (1.4) if

$$\left\{\begin{array}{rcll} u'(t) & \leq & Fu(t), \ t \in J, & u(0) \leq -\lambda[v] + d, \\ v'(t) & \geq & Fv(t), \ t \in J, & v(0) \geq -\lambda[u] + d. \end{array}\right.$$

We say that $U, V \in C^1(J,\mathbb{R})$ are coupled quasi-solutions of problem (1.1), (1.4) if

$$\begin{cases} U'(t) &= FU(t),\ t \in J, \quad U(0) = -\lambda[V] + d, \\ V'(t) &= FV(t),\ t \in J, \quad V(0) = -\lambda[U] + d. \end{cases}$$

Theorem 5.5. *Assume that Assumptions H_1, H_6 hold. Let $y_0, z_0 \in C^1(J,\mathbb{R})$ be coupled lower and upper solutions of* (1.1), (1.4) *and $y_0(t) \le z_0(t)$, $t \in J$.*

Then problem (1.1), (1.4) *has, in the sector $[y_0, z_0]$, coupled quasi-solutions y, z and $y \le z$.*

Proof. Let $\eta, \xi \in [y_0, z_0]$. Put $\varphi(t) = \sup[\eta(t), \xi(t)]$, $\Phi(t) = \inf[\eta(t), \xi(t)]$. Consider the following system:

$$\begin{cases} v'(t) &= F\Phi(t) - M(t)[v(t) - \Phi(t)],\ \ t \in J, \quad v(0) = -\lambda[w] + d, \\ w'(t) &= F\varphi(t) - M(t)[w(t) - \varphi(t)],\ \ t \in J, \quad w(0) = -\lambda[v] + d. \end{cases} \tag{5.3}$$

By Lemma 5.1, system (5.3) has a unique solution. Therefore, we can define the operator

$$B : \bar{\Omega} \to C^1(J) \times C^1(J), \quad B(\eta, \xi) = (v, w), \tag{5.4}$$

where (v, w) is the solution of (5.3), $\bar{\Omega} = [y_0, z_0] \times [y_0, z_0]$.

Now, we want to show that

$$y_0(t) \le v(t) \le w(t) \le z_0(t),\ t \in J. \tag{5.5}$$

Put $p = y_0 - v$, $q = w - z_0$. Then

$$\begin{aligned} p'(t) &\le Fy_0(t) - F\Phi(t) + M(t)[v(t) - \Phi(t)] \le M(t)[\Phi(t) - y_0(t)] = -M(t)p(t), \\ q'(t) &\le F\varphi(t) - M(t)[w(t) - \varphi(t)] - Fz_0(t) \le -M(t)q(t). \end{aligned}$$

Moreover

$$\begin{aligned} p(0) &\le -\lambda[z_0] + \lambda[w] = \lambda[q], \\ q(0) &\le -\lambda[v] + \lambda[y_0] = \lambda[p]. \end{aligned}$$

This and Lemma 5.3 show that $y_0(t) \le v(t)$, $w(t) \le z_0(t)$, $t \in J$. To show that $v(t) \le w(t)$, $t \in J$, we put $p = v - w$. Then

$$\begin{aligned} p'(t) &= F\Phi(t) - F\varphi(t) - M(t)[v(t) - \Phi(t) - w(t) + \varphi(t)] \le -M(t)p(t), \\ p(0) &= -\lambda[w] + \lambda[v] = \lambda[p]. \end{aligned}$$

This and Lemma 2.3 show that $v(t) \le w(t)$, $t \in J$ so (5.5) holds.

Hence $B : \bar{\Omega} \to \bar{\Omega}$. Using (5.3), we can define two sequences $\{y_n, z_n\}$ by relations

$$\begin{cases} y'_{n+1}(t) &= Fy_n(t) - M(t)[y_{n+1}(t) - y_n(t)],\ \ t \in J, \quad y_{n+1}(0) = -\lambda[z_{n+1}] + d, \\ z'_{n+1}(t) &= Fz_n(t) - M(t)[z_{n+1}(t) - z_n(t)],\ \ t \in J, \quad z_{n+1}(0) = -\lambda[y_{n+1}] + d \end{cases}$$

for $n = 0, 1, \ldots$. In view of (5.5), we have

$$y_0(t) \le y_1(t) \le \cdots \le y_n(t) \le y_{n+1}(t) \le z_{n+1}(t) \le z_n(t) \le \cdots \le z_1(t) \le z_0(t).$$

The sequence $\{y_n\}$ is nondecreasing while $\{z_n\}$ is nonincreasing. Note that the sequences $\{y_n, z_n\}$ are uniformly bounded. Indeed, y_n, z_n are equicontinuous too.

The Arzela-Ascoli theorem guarantees the existence of subsequences $\{y_{n_k}, z_{n_k}\}$ of $\{y_n, z_n\}$, respectively, and continuous functions y, z with y_{n_k}, z_{n_k} converging uniformly on J to y and z, respectively. Note that y_{n_k}, z_{n_k} satisfy the integral equations

$$\begin{cases} y_{n_k+1}(t) &= y_{n_k+1}(0) + \int_0^t [Fy_{n_k}(s) - M(t)(y_{n_k+1}(s) - y_{n_k}(s))]\, ds, \quad t \in J, \\ z_{n_k+1}(t) &= z_{n_k+1}(0) + \int_0^t [Fz_{n_k}(s) - M(t)(z_{n_k+1}(s) - z_{n_k}(s))]\, ds, \quad t \in J, \end{cases}$$

and

$$\begin{cases} y_{n_k+1}(0) &= -\lambda[z_{n_k+1}] + d, \\ z_{n_k+1}(0) &= -\lambda[y_{n_k+1}] + d. \end{cases}$$

If $n_k \to \infty$, then from the above relations, we have

$$\begin{cases} y(t) &= y(0) + \int_0^t Fy(s)ds, \quad t \in J, \quad y(0) = -\lambda[z] + d, \\ z(t) &= z(0) + \int_0^t Fz(s)ds, \quad t \in J, \quad z(0) = -\lambda[y] + d \end{cases}$$

because f is continuous. Thus $y, z \in C^1(J)$ and

$$y'(t) = Fy(t), \quad z'(t) = Fz(t), \quad t \in J.$$

It proves that y, z are coupled quasi-solutions of problem (1.1), (1.4) and $y \leq z$. This ends the proof. $\square$

Our next theorem concerns the case when problem (1.1), (1.4) has a unique solution.

Theorem 5.6. *Assume that all assumptions of Theorem* 5.5 *are satisfied. In addition assume that Assumption H_7 holds.*

Then problem (1.1), (1.4) *has, in the sector* $[y_0, z_0]$, *a unique solution.*

Proof. Theorem 5.5 guarantees that functions y, z are coupled quasi-solutions of problem (1.1), (1.4) and $y_0(t) \leq y(t) \leq z(t) \leq z_0(t)$, $t \in J$. We first show that $y(t) = z(t)$, $t \in J$. Put $p = z - y$. Then

$$\begin{aligned} p'(t) &= Fz(t) - Fy(t) \leq L(t)p(t), \quad t \in J, \\ p(0) &= \lambda[p]. \end{aligned}$$

In view of Lemma 2.3, $y(t) \geq z(t)$, $t \in J$. It proves that $y = z$, so problem (1.1), (1.4) has a solution.

It remains to show that $y = z$ is a unique solution of (1.1), (1.4) in the sector $[y_0, z_0]$. Let $w \in [y_0, z_0]$ be any solution of (1.1), (1.4). We assume that $y_m(t) \leq$

$w(t) \le z_m(t)$, $t \in J$ for some m. Let $p = y_{m+1} - w$, $q = w - z_{m+1}$, where y_m, z_m are defined as in Theorem 5.5. Then,

$$\begin{aligned} p'(t) &= Fy_m(t) - M(t)[y_{m+1}(t) - y_m(t)] - Fw(t) \le -M(t)p(t), \\ q'(t) &= Fw(t) - Fz_m(t) + M(t)[z_{m+1}(t) - z_m(t)] \le -M(t)q(t), \end{aligned}$$

and

$$p(0) = \lambda[q], \quad q(0) = \lambda[p].$$

It gives $y_{m+1}(t) \le w(t) \le z_{m+1}(t)$, $t \in J$. By induction, $y_n(t) \le w(t) \le z_n(t)$, $t \in J$, $n = 0, 1, \dots$. If $n \to \infty$, then $y = z = w$ which proves the assertion of our theorem. □

Now, we will discuss problem (1.1), (1.5). We say that $u, v \in C^1(J, \mathbb{R})$ are coupled lower and upper solutions of problem (1.1), (1.5) if

$$\begin{cases} u'(t) \le Fu(t), \ t \in J, & u(T) \ge -\lambda[v] + d, \\ v'(t) \ge Fv(t), \ t \in J, & v(T) \le -\lambda[u] + d. \end{cases}$$

We say that $U, V \in C^1(J, \mathbb{R})$ are coupled quasi-solutions of problem (1.1), (1.4) if

$$\begin{cases} U'(t) = FU(t), \ t \in J, & U(T) = -\lambda[V] + d, \\ V'(t) = FV(t), \ t \in J, & V(T) = -\lambda[U] + d. \end{cases}$$

Theorem 5.7. *Assume that Assumptions H_1, H_8 hold. Let $y_0, z_0 \in C^1(J, \mathbb{R})$ be coupled lower and upper solutions of* (1.1), (1.5) *and $z_0(t) \le y_0(t)$, $t \in J$.*

Then problem (1.1), (1.5) *has, in the sector $[z_0, y_0]$, coupled quasi-solutions y, z and $z \le y$.*

Proof. We introduce only the definitions of sequences:

$$\begin{aligned} y'_{n+1}(t) &= Fy_n(t) + M(t)[y_{n+1}(t) - y_n(t)], \quad y_{n+1}(T) = -\lambda[z_{n+1}] + d, \\ z'_{n+1}(t) &= Fz_n(t) + M(t)[z_{n+1}(t) - z_n(t)], \quad z_{n+1}(T) = -\lambda[y_{n+1}] + d. \end{aligned}$$

Similarly as before, we can prove the assertion of this theorem. □

Our next theorem concerns the case when problem (1.1), (1.5) has a unique solution.

Theorem 5.8. *Assume that all assumptions of Theorem* 5.7 *are satisfied. In addition assume that Assumption H_7 holds.*

Then problem (1.1), (1.5) *has, in the sector $[z_0, y_0]$, a unique solution.*

6. EXAMPLES

Example 6.1. Consider the following differential equation:

$$x'(t) = 2\,[\exp(x(t)) - \exp(-1)] \equiv Fx(t), \quad t \in J = [0, T]. \tag{6.1}$$

Note that $M(t) = 2\exp(t)$, see the condition for f in Assumption H_8. Now we consider equation (6.1) with the boundary condition defined in points **1**. or **2**.

1. Let $\lambda[x] = \int_0^T x(s)g(s)ds, \quad g \in C(J, \mathbb{R}_+)$, so

$$x(T) = \int_0^T x(s)g(s)ds, \quad g \in C(J, \mathbb{R}_+). \tag{6.2}$$

Let us assume that

$$\int_0^T \exp[2\exp(s)]g(s)ds < \exp[2\exp(T)], \tag{6.3}$$

$$\int_0^T g(s)ds \le 1, \quad \int_0^T sg(s)ds \le T. \tag{6.4}$$

Take $y_0(t) = t$, $z_0(t) = -1$, $t \in J$. Then

$$\begin{aligned} Fy_0(t) &= 2[\exp(t) - \exp(-1)] > 1 = y_0'(t), \\ Fz_0(t) &= 0 = z_0'(t) \end{aligned}$$

and

$$\begin{aligned} \int_0^T y_0(s)g(s)ds &= \int_0^T sg(s)ds \le T = y_0(T), \\ \int_0^T z_0(s)g(s)ds &= -\int_0^T g(s)ds \ge -1 = z_0(T), \end{aligned}$$

by (6.4). It proves that y_0, z_0 are lower and upper solutions of problem (6.1), (6.2). By Theorem 4.4, this problem has extremal solutions in the sector $[-1, t]$.

For example, if we take $g(t) = t$, $T < \sqrt{2}$, then conditions (6.3) and (6.4) are satisfied.

2. Let $\lambda[x] = -\int_0^T x(s)g(s)ds - 1, \quad g \in C(J, \mathbb{R}_+)$, so

$$x(T) = -\int_0^T x(s)g(s)ds - 1, \quad g \in C(J, \mathbb{R}_+). \tag{6.5}$$

Let us assume that conditions (6.3) and (6.6) hold with

$$\int_0^T g(s)ds \le 1. \tag{6.6}$$

Put $y_0(t) = 0$, $z_0(t) = -1$, $t \in J$. Then

$$Fy_0(t) > 0 = y_0'(t), \quad Fz_0(t) = 0 = z_0'(t)$$

and

$$\begin{aligned} -\int_0^T z_0(s)g(s)ds - 1 &= \int_0^T g(s)ds - 1 \le 0 = y_0(T), \\ -\int_0^T y_0(s)g(s)ds - 1 &= -1 = z_0(T), \end{aligned}$$

by (6.6). It proves that y_0, z_0 are coupled lower and upper solutions of problem (6.1), (6.5). By Theorem 5.7, this problem has quasi-solutions in the sector $[-1, 0]$.

Example 6.2 (see [5])**.** Consider the following differential equation:

$$x'(t) = \exp\big(t \sin^2 x(t)\big) \equiv Fx(t), \quad t \in J = [0, T] \text{ with } T = \ln 2. \tag{6.7}$$

We consider equation (6.7) with the condition defined in points **1** or **2** (the results are taken from paper [5],

1. Let $\lambda[x] = \int_0^T x(s)ds$, so

$$x(0) = \int_0^T x(s)ds. \tag{6.8}$$

Note that $y_0(t) = 0$, $z_0(t) = \exp(t)$, $t \in J$ are lower and upper solutions of (6.7), respectively. Problem (6.7), (6.8) has extremal solutions in the sector $[y_0, z_0]$, by Theorem 4.2.

2. Let $\lambda[x] = -\int_0^T x(s)ds$, so

$$x(0) = -\int_0^T x(s)ds. \tag{6.9}$$

Indeed, $y_0(t) = -1$, $z_0(t) = \exp(t)$, $t \in J$ are coupled lower and upper solutions of (6.7). Problem (6.7), (6.9) has quasi-solutions in the sector $[y_0, z_0]$, by Theorem 5.5.

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