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IMPROVEMENTS OF DYNAMIC OPIAL-TYPE INEQUALITIES AND APPLICATIONS

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ABSTRACT. In this paper, we present some new improvements of dynamic Opial-type inequalities of first and higher order on time scales. We employ the new inequalities to prove several results related to the spacing between consecutive zeros of a solution and/or a zero of its derivative of a second-order dynamic equation with a damping term. The main results are proved by making use of a recently introduced new technique for Opial dynamic inequalities, the time scales integration by parts formula, the time scales chain rule, the time scales Taylor formula, and classical as well as time scales versions of Hölder's inequality.

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1. INTRODUCTION

The study of dynamic inequalities on time scales has received a lot of attention in the literature and has become a major field in pure and applied mathematics. The subject of time scales has been created in [8] in order to unify the study of differential and difference equations, and it also extends these classical cases to cases "in between", e.g., to so-called q-difference equations. The general idea is to prove a result for a dynamic equation or inequality, where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$. The books [5, 6] by Bohner and Peterson summarize and organize much of time scales calculus. The three most popular examples of calculus on time scales are differential calculus, difference calculus and quantum calculus (see [9]), i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$. It is worth to mention here that many results concerning differential inequalities carry over quite easily to corresponding results for difference inequalities, while other results seem to be completely different from their continuous counterparts.

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The Polish mathematician Z. Opial in 1960 [12] published an inequality, involving integrals of a function and its derivative, of the form

$$\int_0^h |f(t)f'(t)|\,\mathrm{d}t \le \frac{h}{4}\int_0^h |f'(t)|^2\,\mathrm{d}t, \tag{1.1}$$

where $f \in \mathrm{C}^1[0,h]$, $f(0) = f(h) = 0$, and $f > 0$ on $(0,h)$, and the constant $h/4$ is the best possible. Olech [11] extended the inequality (1.1) and proved that if f is absolutely continuous on $[0,h]$ and $f(0) = 0$, then

$$\int_0^h |f(t)f'(t)|\,\mathrm{d}t \le \frac{h}{2}\int_0^h |f'(t)|^2\,\mathrm{d}t. \tag{1.2}$$

Since the discovery of Opial's inequality, much work has been done, and many papers which deal with new proofs, various generalizations, extensions, and their discrete analogues have been also proved in the literature. A discrete analogue of (1.1) has been proved in [10], and a discrete analogue of (1.2) has been proved in [2, Theorem 5.2.1]. In [3] (see also [5, Theorem 6.23]), the authors extended (1.1) to an arbitrary time scale $\mathbb{T}$ and proved that if $f \in \mathrm{C}^1_{\mathrm{rd}}([0,h]_{\mathbb{T}},\mathbb{R})$ satisfies $f(0) = 0$, then

$$\int_0^h \left|\left(f^2\right)^{\Delta}(t)\right|\Delta t \le h\int_0^h \left(f^{\Delta}\right)^2(t)\Delta t. \tag{1.3}$$

In [14], the following result was proved.

Theorem 1.1 (See [14, Theorem 1])**.** *Assume that* $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,

$$r, s \in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}},(0,\infty)), \quad \text{and} \quad f \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R}).$$

If $f(a) = 0$, *then*

$$\int_a^b s(t)\left|(f^2)^{\Delta}(t)\right|\Delta t \le K\int_a^b r(t)\left|f^{\Delta}(t)\right|^2\Delta t,$$

where

$$K = \sqrt{2\int_a^b \frac{s^2(t)}{r^2(t)}\left(\int_a^t \frac{\Delta s}{r(s)}\right)\Delta t} + \sup_{a\le t\le b}\frac{\mu(t)s(t)}{r(t)},$$

and μ *is the graininess of the time scale* $\mathbb{T}$.

Using a novel technique in [4], Theorem 1.1 was improved in the following way.

Theorem 1.2 (See [4, Theorem 5.2])**.** *Assume that* $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,

$$r, s \in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}},(0,\infty)), \quad \text{and} \quad f \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R}).$$

If $f(a) = 0$, *then*

$$\int_a^b s(t)\left|(f^2)^{\Delta}(t)\right|\Delta t \le K\int_a^b r(t)\left|f^{\Delta}(t)\right|^2\Delta t,$$

where

$$K = \sqrt{\int_a^b s^2(t)(R^2)^{\Delta}(t)\Delta t} \quad \text{with} \quad R(t) = \int_a^t \frac{\Delta s}{r(s)}.$$

The purpose of this paper is to apply the new technique that was developed in [4] in order to improve the main results of the papers [13, 15, 17]. We now present the results obtained in [13, 15, 17]. These results are improvements of Theorem 1.1.

Theorem 1.3 (See [15, Theorem 2.1])**.** *Assume that $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,*

$$r, s \in \mathrm{C_{rd}}([a,b]_{\mathbb{T}}, (0,\infty)), \quad \text{and} \quad f \in \mathrm{C^1_{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

Let $\lambda \geq 1$ and $\delta \geq 0$. If $f(a) = 0$ and f^{Δ} does not change sign in $(a,b)_{\mathbb{T}}$, then

$$\int_a^b s(t)\,|f(t)+f^{\sigma}(t)|^{\lambda}\left|f^{\Delta}(t)\right|^{\delta}\Delta t \leq K\int_a^b r(t)\left|f^{\Delta}(t)\right|^{\lambda+\delta}\Delta t,$$

where

$$K = 2^{2\lambda-1}\left(\frac{\delta}{\lambda+\delta}\right)^{\frac{\delta}{\lambda+\delta}}\left(\int_a^b \frac{(s(t))^{\frac{\lambda+\delta}{\lambda}}}{(r(t))^{\frac{\delta}{\lambda}}}\left(\int_a^t \frac{\Delta\tau}{(r(\tau))^{\frac{1}{\lambda+\delta-1}}}\right)^{\lambda+\delta-1}\Delta t\right)^{\frac{\lambda}{\lambda+\delta}} + 2^{\lambda-1}\sup_{a\leq t\leq b}\left(\frac{\mu^{\lambda}(t)s(t)}{r(t)}\right),$$

and μ is the graininess of the time scale $\mathbb{T}$.

Theorem 1.4 (See [13, Theorem 2.1])**.** *Assume that $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,*

$$r, s \in \mathrm{C_{rd}}([a,b]_{\mathbb{T}}, (0,\infty)), \quad \text{and} \quad f \in \mathrm{C^1_{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

Let $0 < \lambda \leq 1$, $\delta > 0$, and $\lambda + \delta > 1$. If $f(a) = 0$ and $f \geq 0$ on $[a,b]_{\mathbb{T}}$, then

$$\int_a^b s(t)\,|f(t)+f^{\sigma}(t)|^{\lambda}\left|f^{\Delta}(t)\right|^{\delta}\Delta t \leq K\int_a^b r(t)\left|f^{\Delta}(t)\right|^{\lambda+\delta}\Delta t,$$

where

$$K = 2^{\lambda}\left(\frac{\delta}{\lambda+\delta}\right)^{\frac{\delta}{\lambda+\delta}}\left(\int_a^b \frac{(s(t))^{\frac{\lambda+\delta}{\lambda}}}{(r(t))^{\frac{\delta}{\lambda}}}\left(\int_a^t \frac{\Delta\tau}{(r(\tau))^{\frac{1}{\lambda+\delta-1}}}\right)^{\lambda+\delta-1}\Delta t\right)^{\frac{\lambda}{\lambda+\delta}} + \sup_{a\leq t\leq b}\left(\frac{\mu^{\lambda}(t)s(t)}{r(t)}\right),$$

and μ is the graininess of the time scale $\mathbb{T}$.

Theorem 1.5 (See [17, Theorem 2.1])**.** *Let $n \in \mathbb{N}$. Assume that $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,*

$$r, s \in \mathrm{C_{rd}}([a,b]_{\mathbb{T}}, (0,\infty)), \quad \text{and} \quad f \in \mathrm{C^n_{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

Let $\lambda > 1$ and $\delta = \lambda/(\lambda-1)$. If $f^{\Delta^k}(a) = 0$ for all $0 \leq k \leq n-1$, then

$$\int_a^b s(t)\,|f(t)+f^{\sigma}(t)|^{\lambda}\left|f^{\Delta^n}(t)\right|^{\delta}\Delta t \leq K\int_a^b r(t)\left|f^{\Delta^n}(t)\right|^{\lambda+\delta}\Delta t,$$

where

$$K = 2^{2\lambda-1}\left(\frac{\delta}{\lambda+\delta}\right)^{\frac{\delta}{\lambda+\delta}}\left(\int_a^b \frac{(s(t))^{\frac{\lambda+\delta}{\lambda}}}{(r(t))^{\frac{\delta}{\lambda}}}\left(\int_a^t \frac{(h_{n-1}(t,\sigma(\tau)))^{\frac{\lambda+\delta}{\lambda+\delta-1}}}{(r(\tau))^{\frac{1}{\lambda+\delta-1}}}\Delta\tau\right)^{\lambda+\delta-1}\Delta t\right)^{\frac{\lambda}{\lambda+\delta}}$$

$$+ 2^{\lambda-1}\left(\frac{\delta}{\lambda+\delta}\right)^{\frac{\delta}{\lambda+\delta}}\times$$

$$\times\left(\int_a^b \frac{(\mu(t))^{\lambda+\delta}(s(t))^{\frac{\lambda+\delta}{\lambda}}}{(r(t))^{\frac{\delta}{\lambda}}}\left(\int_a^t \frac{(h_{n-2}(t,\sigma(\tau)))^{\frac{\lambda+\delta}{\lambda+\delta-1}}}{(r(\tau))^{\frac{1}{\lambda+\delta-1}}}\Delta\tau\right)^{\lambda+\delta-1}\Delta t\right)^{\frac{\lambda}{\lambda+\delta}},$$

and μ is the graininess of the time scale $\mathbb{T}$.

One can see that in Theorems 1.1, 1.3, 1.4, and 1.5, there is an additional term in the constants of the inequalities that depends on the graininess function μ. However, in Theorem 1.2, our new technique was able to remove this additional term. In this paper, we now want to present improvements of Theorems 1.3, 1.4, and 1.5, which will not have the additional term depending on the graininess function. The technique used is the same one as introduced in [4].

The paper is organized as follows: In Section 2, we present the basic definitions of time scales calculus that will be used in the sequel. In Section 3, we address the improvements of Theorems 1.3 and 1.4 and prove some new dynamic inequalities of Opial type involving first-order delta derivatives. In Section 4, we address the improvement of Theorem 1.5 and prove some new dynamic inequalities of Opial type involving higher-order delta derivatives. Finally, in Section 5, we employ the new inequalities to prove several results related to a second-order dynamic equation with a damping term, generalizing the previous published work [16] on the same problem. We prove our main results by using the time scales chain rule, the time scales integration by parts formula, the time scales Taylor formula, and classical as well as time scales versions of Hölder's inequality.

2. **TIME SCALES PRELIMINARIES**

In this section, we briefly present some basic definitions and results concerning the delta calculus on time scales that we will use in this article. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ for $t \in \mathbb{T}$. The mapping $\mu : \mathbb{T} \to [0, \infty)$ defined by $\mu(t) := \sigma(t) - t$ is called the graininess of $\mathbb{T}$. A function $f : [a, b] \to \mathbb{R}$ is called rd-continuous, denoted by $f \in \mathrm{C}_{\mathrm{rd}}$, if it is continuous at each right-dense point (i.e., $\sigma(t) = t$) and there exists a finite left-sided limit at all left-dense points (i.e., $\rho(t) = t$, where the backward jump ρ is defined in a similar way as the forward jump σ). For the definition of the delta derivative f^Δ, we refer to [5, 6]. A simple useful formula is $f^\sigma = f + \mu f^\Delta$, where $f^\sigma := f \circ \sigma$. The product rule for the derivative of the product fg of two differentiable functions f and g reads

$$fg^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma.$$

For the definition of the integral of $f \in \mathrm{C_{rd}}$, we refer again to [5,6]. What will be used in this paper is the *time scales integration by parts formula*

$$\int_a^b f(t)g^\Delta(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(t)g^\sigma(t)\Delta t.$$

If $f \in \mathrm{C}^1(\mathbb{R},\mathbb{R})$ and $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable, then the *time scales chain rule*, see [5, Theorem 1.90], states that

$$(f \circ g)^\Delta = g^\Delta \int_0^1 f'(hg^\sigma + (1-h)g^\Delta)\mathrm{d}h,$$

and a special case, which we will use in this paper, is given by

$$(f^\gamma)^\Delta = \gamma f^\Delta \int_0^1 (hf^\sigma + (1-h)f)^{\gamma-1}\,\mathrm{d}h \quad \text{for} \quad \gamma > 0.$$

The *time scales Hölder inequality*, see [5, Theorem 6.13], says

$$\int_a^b |f(t)g(t)|\,\Delta t \le \left\{\int_a^b |f(t)|^\gamma\,\Delta t\right\}^{\frac{1}{\gamma}} \left\{\int_a^b |g(t)|^\nu\,\Delta t\right\}^{\frac{1}{\nu}},$$

where $a, b \in \mathbb{T}$, $f, g \in \mathrm{C_{rd}}([a,b]_\mathbb{T},\mathbb{R})$, $\gamma > 1$, and $\nu = \gamma/(\gamma-1)$. For the definition and properties of the Taylor monomials h_k, we refer to [1,5]. In this paper, we will use the *time scales Taylor formula*, see [5, Theorem 1.113],

$$f(t) = \sum_{k=0}^{n-1} h_k(t,\alpha)f^{\Delta^k}(a) + \int_a^t h_{n-1}(t,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau$$

for a function $f \in \mathrm{C}^n_{\mathrm{rd}}$ with $n \in \mathbb{N}$, more precisely, its special case

$$f(t) = \int_a^t h_{n-1}(t,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau$$

that holds provided $f^{\Delta^k}(a) = 0$ for all $0 \le k \le n-1$.

3. INEQUALITIES WITH FIRST-ORDER DERIVATIVES

Theorem 3.1. *Assume that $a \in \mathbb{T}$, $b \in (a,\infty)_\mathbb{T}$,*

$$r, s \in \mathrm{C_{rd}}([a,b]_\mathbb{T},(0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_\mathbb{T},\mathbb{R}).$$

Let $\alpha > 1$ and $\beta \ge 0$. If $f(a) = 0$, then

$$\int_a^b s(t)\left|(f^\alpha)^\Delta(t)(f^\Delta(t))^\beta\right|\Delta t \le K\int_a^b r(t)\left|f^\Delta(t)\right|^{\alpha+\beta}\Delta t,$$

where

$$K = \frac{\alpha(\beta+1)^{\frac{\beta+1}{\alpha+\beta}}}{\alpha+\beta}\left\{\int_a^b (s(t))^{\frac{\alpha+\beta}{\alpha-1}}\frac{(R^{\alpha+\beta})^\Delta(t)}{(r(t))^{\frac{\beta(\alpha+\beta)}{(\alpha-1)(\alpha+\beta-1)}}}\Delta t\right\}^{\frac{\alpha-1}{\alpha+\beta}}$$

with

$$R(t) = \int_a^t \frac{\Delta\tau}{(r(\tau))^{\frac{1}{\alpha+\beta-1}}}.$$

Proof. Define

$$g(t) := \int_a^t r(\tau)\left|f^\Delta(\tau)\right|^{\alpha+\beta}\Delta\tau.$$

Then $g(a) = 0$,

$$g^\Delta = r\left|f^\Delta\right|^{\alpha+\beta} \quad \text{so that} \quad \left|f^\Delta\right| = \left(\frac{g^\Delta}{r}\right)^{\frac{1}{\alpha+\beta}},$$

and

$$\begin{aligned}
|f(t)| &= \left|\int_a^t \frac{1}{(r(\tau))^{\frac{1}{\alpha+\beta}}}(r(\tau))^{\frac{1}{\alpha+\beta}} f^\Delta(\tau)\Delta\tau\right| \\
&\le \int_a^t \frac{1}{(r(\tau))^{\frac{1}{\alpha+\beta}}}(r(\tau))^{\frac{1}{\alpha+\beta}}\left|f^\Delta(\tau)\right|\Delta\tau \\
&\le \left\{\int_a^t \frac{\Delta\tau}{(r(\tau))^{\frac{1}{\alpha+\beta-1}}}\right\}^{\frac{\alpha+\beta-1}{\alpha+\beta}} \left\{\int_a^t r(\tau)\left|f^\Delta(\tau)\right|^{\alpha+\beta}\Delta\tau\right\}^{\frac{1}{\alpha+\beta}} \\
&= (R(t))^{\frac{\alpha+\beta-1}{\alpha+\beta}}(g(t))^{\frac{1}{\alpha+\beta}},
\end{aligned}$$

where we have used the time scales Hölder inequality with conjugate exponents $\frac{\alpha+\beta}{\alpha+\beta-1}$ and $\alpha+\beta > 1$. Thus, for $h \in [0,1]$, we obtain

$$\begin{aligned}
|hf^\sigma + (1-h)f| &\le h|f^\sigma| + (1-h)|f| \\
&\le h(R^\sigma)^{\frac{\alpha+\beta-1}{\alpha+\beta}}(g^\sigma)^{\frac{1}{\alpha+\beta}} + (1-h)R^{\frac{\alpha+\beta-1}{\alpha+\beta}}g^{\frac{1}{\alpha+\beta}} \\
&= (hR^\sigma)^{\frac{\alpha+\beta-1}{\alpha+\beta}}(hg^\sigma)^{\frac{1}{\alpha+\beta}} + ((1-h)R)^{\frac{\alpha+\beta-1}{\alpha+\beta}}((1-h)g)^{\frac{1}{\alpha+\beta}} \\
&\le (hR^\sigma + (1-h)R)^{\frac{\alpha+\beta-1}{\alpha+\beta}}(hg^\sigma + (1-h)g)^{\frac{1}{\alpha+\beta}},
\end{aligned}$$

where we have used the classical Hölder inequality for sums with conjugate exponents $\frac{\alpha+\beta}{\alpha+\beta-1}$ and $\alpha+\beta > 1$. Hence

$$\begin{aligned}
&\left|\int_0^1 (hf^\sigma + (1-h)f)^{\alpha-1}\mathrm{d}h\right| \le \int_0^1 |hf^\sigma + (1-h)f|^{\alpha-1}\mathrm{d}h \\
&\le \int_0^1 (hR^\sigma + (1-h)R)^{\frac{(\alpha+\beta-1)(\alpha-1)}{\alpha+\beta}}(hg^\sigma + (1-h)g)^{\frac{\alpha-1}{\alpha+\beta}}\mathrm{d}h \\
&\le \left\{\int_0^1 (hR^\sigma + (1-h)R)^{\alpha+\beta-1}\mathrm{d}h\right\}^{\frac{\alpha-1}{\alpha+\beta}} \left\{\int_0^1 (hg^\sigma + (1-h)g)^{\frac{\alpha-1}{\beta+1}}\mathrm{d}h\right\}^{\frac{\beta+1}{\alpha+\beta}},
\end{aligned}$$

where we have used the classical Hölder inequality for integrals with conjugate exponents $\frac{\alpha+\beta}{\alpha-1}$ and $\frac{\alpha+\beta}{\beta+1} > 1$. Therefore, using the time scales chain rule three times, we get

$$\left|(f^\alpha)^\Delta (f^\Delta)^\beta\right| = \alpha\left|f^\Delta\right|^{\beta+1}\left|\int_0^1 (hf^\sigma + (1-h)f)^{\alpha-1}\mathrm{d}h\right|$$

$$
\begin{aligned}
&= \frac{\alpha (g^\Delta)^{\frac{\beta+1}{\alpha+\beta}}}{r^{\frac{\beta+1}{\alpha+\beta}}} \left| \int_0^1 (hf^\sigma + (1-h)f)^{\alpha-1}\, \mathrm{d}h \right| \\
&\le \frac{\alpha (g^\Delta)^{\frac{\beta+1}{\alpha+\beta}}}{r^{\frac{\beta+1}{\alpha+\beta}}} \left\{ \int_0^1 (hR^\sigma + (1-h)R)^{\alpha+\beta-1}\, \mathrm{d}h \right\}^{\frac{\alpha-1}{\alpha+\beta}} \\
&\quad \times \left\{ \int_0^1 (hg^\sigma + (1-h)g)^{\frac{\alpha-1}{\beta+1}}\, \mathrm{d}h \right\}^{\frac{\beta+1}{\alpha+\beta}} \\
&= \frac{\alpha(\beta+1)^{\frac{\beta+1}{\alpha+\beta}}}{(\alpha+\beta) r^{\frac{\beta}{\alpha+\beta-1}}} \left\{ (\alpha+\beta) R^\Delta \int_0^1 (hR^\sigma + (1-h)R)^{\alpha+\beta-1}\, \mathrm{d}h \right\}^{\frac{\alpha-1}{\alpha+\beta}} \\
&\quad \times \left\{ \frac{\alpha+\beta}{\beta+1} g^\Delta \int_0^1 (hg^\sigma + (1-h)g)^{\frac{\alpha+\beta}{\beta+1}-1}\, \mathrm{d}h \right\}^{\frac{\beta+1}{\alpha+\beta}} \\
&= \frac{\alpha(\beta+1)^{\frac{\beta+1}{\alpha+\beta}}}{(\alpha+\beta) r^{\frac{\beta}{\alpha+\beta-1}}} \left\{ \left(R^{\alpha+\beta}\right)^\Delta \right\}^{\frac{\alpha-1}{\alpha+\beta}} \left\{ \left(g^{\frac{\alpha+\beta}{\beta+1}} \right)^\Delta \right\}^{\frac{\beta+1}{\alpha+\beta}},
\end{aligned}
$$

and thus finally

$$
\begin{aligned}
&\int_a^b s(t) \left| \left(f^\alpha\right)^\Delta (t) \left(f^\Delta(t)\right)^\beta \right| \Delta t \\
&\le \frac{\alpha(\beta+1)^{\frac{\beta+1}{\alpha+\beta}}}{\alpha+\beta} \int_a^b s(t) \frac{\left\{ \left(R^{\alpha+\beta}\right)^\Delta (t) \right\}^{\frac{\alpha-1}{\alpha+\beta}}}{(r(t))^{\frac{\beta}{\alpha+\beta-1}}} \left\{ \left(g^{\frac{\alpha+\beta}{\beta+1}} \right)^\Delta (t) \right\}^{\frac{\beta+1}{\alpha+\beta}} \Delta t \\
&\le \frac{\alpha(\beta+1)^{\frac{\beta+1}{\alpha+\beta}}}{\alpha+\beta} \left\{ \int_a^b (s(t))^{\frac{\alpha+\beta}{\alpha-1}} \frac{\left(R^{\alpha+\beta}\right)^\Delta (t)}{(r(t))^{\frac{\beta(\alpha+\beta)}{(\alpha-1)(\alpha+\beta-1)}}} \Delta t \right\}^{\frac{\alpha-1}{\alpha+\beta}} \left\{ \int_a^b \left(g^{\frac{\alpha+\beta}{\beta+1}} \right)^\Delta (t) \Delta t \right\}^{\frac{\beta+1}{\alpha+\beta}} \\
&= K \left\{ g^{\frac{\alpha+\beta}{\beta+1}}(b) \right\}^{\frac{\beta+1}{\alpha+\beta}} \\
&= K g(b),
\end{aligned}
$$

where we have used one last time the time scales Hölder inequality with conjugate exponents $\frac{\alpha+\beta}{\alpha-1}$ and $\frac{\alpha+\beta}{\beta+1} > 1$. $\square$

If we let $\beta = 0$ in Theorem 3.1, then we get the following result.

Corollary 3.2. *Assume that* $a \in \mathbb{T}$, $b \in (a, \infty)_{\mathbb{T}}$,

$$r, s \in \mathrm{C_{rd}}([a,b]_{\mathbb{T}}, (0,\infty)), \quad \textit{and} \quad f \in \mathrm{C^1_{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

Let $\alpha > 1$*. If* $f(a) = 0$*, then*

$$\int_a^b s(t) \left| (f^\alpha)^\Delta (t) \right| \Delta t \le K \int_a^b r(t) \left| f^\Delta(t) \right|^\alpha \Delta t,$$

where

$$K = \left\{ \int_a^b (s(t))^{\frac{\alpha}{\alpha-1}} \left(R^\alpha\right)^\Delta (t) \Delta t \right\}^{\frac{\alpha-1}{\alpha}}$$

with

$$R(t) = \int_a^t \frac{\Delta\tau}{(r(\tau))^{\frac{1}{\alpha-1}}}.$$

If we let $\alpha = 2$ in Corollary 3.2, then we obtain Theorem 1.2.

Remark 3.3. The results in this section are given when $f(a) = 0$. Similarly as in [13–15], results can also be given when $f(b) = 0$ or when $f(a) = f(b) = 0$, but since no new ideas are needed, we refrain from stating these other results.

4. INEQUALITIES WITH HIGHER-ORDER DERIVATIVES

Theorem 4.1. *Let $n \in \mathbb{N}$. Assume that $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,*

$$r, s \in \mathrm{C_{rd}}([a,b]_{\mathbb{T}}, (0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^n_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

Let $\alpha > 1$ and $\beta \geq 0$. If $f^{\Delta^k}(a) = 0$ for all $0 \leq k \leq n-1$, then

$$\int_a^b s(t) \left| (f^\alpha)^\Delta(t) (f^{\Delta^n}(t))^\beta \frac{f^{\Delta^n}(t)}{f^\Delta(t)} \right| \Delta t \leq K \int_a^b r(t) \left| f^{\Delta^n}(t) \right|^{\alpha+\beta} \Delta t,$$

where

$$K = \frac{\alpha(\beta+1)^{\frac{\beta+1}{\alpha+\beta}}}{\alpha+\beta} \left\{ \int_a^b (s(t))^{\frac{\alpha+\beta}{\alpha-1}} \frac{(R^{\alpha+\beta})^\Delta(t)}{(r(t))^{\frac{\beta+1}{\alpha-1}} R^\Delta(t)} \Delta t \right\}^{\frac{\alpha-1}{\alpha+\beta}}$$

with

$$R(t) = \int_a^t \frac{(h_{n-1}(t,\sigma(\tau)))^{\frac{\alpha+\beta}{\alpha+\beta-1}}}{(r(\tau))^{\frac{1}{\alpha+\beta-1}}} \Delta\tau.$$

Proof. Define

$$g(t) := \int_a^t r(\tau) \left| f^{\Delta^n}(\tau) \right|^{\alpha+\beta} \Delta\tau.$$

Then $g(a) = 0$,

$$g^\Delta = r \left| f^{\Delta^n} \right|^{\alpha+\beta} \quad \text{so that} \quad \left| f^{\Delta^n} \right| = \left(\frac{g^\Delta}{r} \right)^{\frac{1}{\alpha+\beta}},$$

and

$$\begin{aligned}
|f(t)| &= \left| \int_a^t h_{n-1}(t,\sigma(\tau)) f^{\Delta^n}(\tau) \Delta\tau \right| \\
&= \left| \int_a^t \frac{h_{n-1}(t,\sigma(\tau))}{(r(\tau))^{\frac{1}{\alpha+\beta}}} (r(\tau))^{\frac{1}{\alpha+\beta}} f^{\Delta^n}(\tau) \Delta\tau \right| \\
&\leq \int_a^t \frac{h_{n-1}(t,\sigma(\tau))}{(r(\tau))^{\frac{1}{\alpha+\beta}}} (r(\tau))^{\frac{1}{\alpha+\beta}} \left| f^{\Delta^n}(\tau) \right| \Delta\tau \\
&\leq \left\{ \int_a^t \frac{(h_{n-1}(t,\sigma(\tau)))^{\frac{\alpha+\beta}{\alpha+\beta-1}}}{(r(\tau))^{\frac{1}{\alpha+\beta-1}}} \Delta\tau \right\}^{\frac{\alpha+\beta-1}{\alpha+\beta}} \left\{ \int_a^t r(\tau) \left| f^{\Delta^n}(\tau) \right|^{\alpha+\beta} \Delta\tau \right\}^{\frac{1}{\alpha+\beta}} \\
&= (R(t))^{\frac{\alpha+\beta-1}{\alpha+\beta}} (g(t))^{\frac{1}{\alpha+\beta}},
\end{aligned}$$

where we have used the time scales Hölder inequality with conjugate exponents $\frac{\alpha+\beta}{\alpha+\beta-1}$ and $\alpha+\beta>1$. Thus, exactly as in the proof of Theorem 3.1, we obtain

$$\begin{aligned}
&\left|\int_0^1 (hf^\sigma+(1-h)f)^{\alpha-1}\,\mathrm{d}h\right| \\
&\qquad \le \left\{\int_0^1 (hR^\sigma+(1-h)R)^{\alpha+\beta-1}\,\mathrm{d}h\right\}^{\frac{\alpha-1}{\alpha+\beta}} \left\{\int_0^1 (hg^\sigma+(1-h)g)^{\frac{\alpha-1}{\beta+1}}\,\mathrm{d}h\right\}^{\frac{\beta+1}{\alpha+\beta}}.
\end{aligned}$$

Therefore, using the time scales chain rule three times, we obtain

$$\begin{aligned}
\left|(f^\alpha)^\Delta (f^{\Delta^n})^\beta \frac{f^{\Delta^n}}{f^\Delta}\right| &= \alpha\left|f^{\Delta^n}\right|^{\beta+1}\left|\int_0^1 (hf^\sigma+(1-h)f)^{\alpha-1}\,\mathrm{d}h\right| \\
&= \frac{\alpha(g^\Delta)^{\frac{\beta+1}{\alpha+\beta}}}{r^{\frac{\beta+1}{\alpha+\beta}}}\left|\int_0^1 (hf^\sigma+(1-h)f)^{\alpha-1}\,\mathrm{d}h\right| \\
&\le \frac{\alpha(g^\Delta)^{\frac{\beta+1}{\alpha+\beta}}}{r^{\frac{\beta+1}{\alpha+\beta}}}\left\{\int_0^1 (hR^\sigma+(1-h)R)^{\alpha+\beta-1}\,\mathrm{d}h\right\}^{\frac{\alpha-1}{\alpha+\beta}} \\
&\quad \times\left\{\int_0^1 (hg^\sigma+(1-h)g)^{\frac{\alpha-1}{\beta+1}}\,\mathrm{d}h\right\}^{\frac{\beta+1}{\alpha+\beta}} \\
&= \frac{\alpha(\beta+1)^{\frac{\beta+1}{\alpha+\beta}}}{(\alpha+\beta)r^{\frac{\beta+1}{\alpha+\beta}}(R^\Delta)^{\frac{\alpha-1}{\alpha+\beta}}}\left\{(\alpha+\beta)R^\Delta\int_0^1 (hR^\sigma+(1-h)R)^{\alpha+\beta-1}\,\mathrm{d}h\right\}^{\frac{\alpha-1}{\alpha+\beta}} \\
&\quad \times\left\{\frac{\alpha+\beta}{\beta+1}g^\Delta\int_0^1 (hg^\sigma+(1-h)g)^{\frac{\alpha+\beta}{\beta+1}-1}\,\mathrm{d}h\right\}^{\frac{\beta+1}{\alpha+\beta}} \\
&= \frac{\alpha(\beta+1)^{\frac{\beta+1}{\alpha+\beta}}}{(\alpha+\beta)r^{\frac{\beta+1}{\alpha+\beta}}(R^\Delta)^{\frac{\alpha-1}{\alpha+\beta}}}\left\{\left(R^{\alpha+\beta}\right)^\Delta\right\}^{\frac{\alpha-1}{\alpha+\beta}}\left\{\left(g^{\frac{\alpha+\beta}{\beta+1}}\right)^\Delta\right\}^{\frac{\beta+1}{\alpha+\beta}},
\end{aligned}$$

and thus finally

$$\begin{aligned}
&\int_a^b s(t)\left|(f^\alpha)^\Delta(t)\left(f^\Delta(t)\right)^\beta\frac{f^{\Delta^n}(t)}{f^\Delta(t)}\right|\Delta t \\
&\le \frac{\alpha(\beta+1)^{\frac{\beta+1}{\alpha+\beta}}}{\alpha+\beta}\int_a^b s(t)\frac{\left\{\left(R^{\alpha+\beta}\right)^\Delta(t)\right\}^{\frac{\alpha-1}{\alpha+\beta}}}{(r(t))^{\frac{\beta+1}{\alpha+\beta}}(R^\Delta)^{\frac{\alpha-1}{\alpha+\beta}}}\left\{\left(g^{\frac{\alpha+\beta}{\beta+1}}\right)^\Delta(t)\right\}^{\frac{\beta+1}{\alpha+\beta}}\Delta t \\
&\le \frac{\alpha(\beta+1)^{\frac{\beta+1}{\alpha+\beta}}}{\alpha+\beta}\left\{\int_a^b (s(t))^{\frac{\alpha+\beta}{\alpha-1}}\frac{\left(R^{\alpha+\beta}\right)^\Delta(t)}{(r(t))^{\frac{\beta+1}{\alpha-1}}R^\Delta(t)}\Delta t\right\}^{\frac{\alpha-1}{\alpha+\beta}}\left\{\int_a^b\left(g^{\frac{\alpha+\beta}{\beta+1}}\right)^\Delta(t)\Delta t\right\}^{\frac{\beta+1}{\alpha+\beta}} \\
&= K\left\{g^{\frac{\alpha+\beta}{\beta+1}}(b)\right\}^{\frac{\beta+1}{\alpha+\beta}} \\
&= Kg(b),
\end{aligned}$$

where we have used one last time the time scales Hölder inequality with conjugate exponents $\frac{\alpha+\beta}{\alpha-1}$ and $\frac{\alpha+\beta}{\beta+1}>1$. ◻

If we let $n=1$ in Theorem 4.1, then we get Theorem 3.1.

Remark 4.2. The results in this section are given when $f^{\Delta^k}(a) = 0$ for all $0 \le k \le n-1$. Similarly as in [17], results can also be given when $f^{\Delta^k}(b) = 0$ for all $0 \le k \le n-1$ or when $f^{\Delta^k}(a) = f^{\Delta^k}(b) = 0$ for all $0 \le k \le n-1$, but since no new ideas are needed, we refrain from stating these other results.

5. APPLICATIONS

In this section, we consider the second-order dynamic equation with a damping term

$$\left(r\left(f^{\Delta}\right)^{\gamma}\right)^{\Delta}(t) + p(t)\left(f^{\Delta}(t)\right)^{\gamma} + q(t)\left(f^{\sigma}(t)\right)^{\gamma} = 0, \quad t \in [a,b]_{\mathbb{T}} \tag{5.1}$$

on an arbitrary time scale $\mathbb{T}$. The results given in this section improve the results from [16].

Theorem 5.1. *Assume that $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$, $\gamma \ge 1$,*

$$p, q \in \mathrm{C_{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}),\ r \in \mathrm{C_{rd}}([a,b]_{\mathbb{T}}, (0,\infty)), \quad \textit{and} \quad f \in \mathrm{C^1_{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

If f is a nontrivial and nonnegative solution of (5.1) such that $f(a) = 0$, $f^{\Delta}(t) \ge 0$ for all $t \in [a,b)_{\mathbb{T}}$, and $(f^{\Delta}(b))^{\gamma} \le 0$, then

$$K_1 + K_2 \ge 1,$$

where

$$K_1 = \frac{2\gamma^{\frac{\gamma}{\gamma+1}}}{\gamma+1}\left\{\int_a^b |p(t)|^{\gamma+1}\frac{(R^{\gamma+1})^{\Delta}(t)}{(r(t))^{\frac{\gamma^2-1}{\gamma}}}\Delta t\right\}^{\frac{1}{\gamma+1}}$$

and

$$K_2 = \left\{\int_a^b |Q(t)|^{\frac{\gamma+1}{\gamma}}(R^{\gamma+1})^{\Delta}(t)\Delta t\right\}^{\frac{\gamma}{\gamma+1}}$$

with

$$R(t) = \int_a^t \frac{\Delta\tau}{(r(\tau))^{\frac{1}{\gamma}}}, \quad Q(t) = \int_t^b q(\tau)\Delta\tau.$$

Proof. First note that Theorem 3.1 yields (with $\alpha = 2$ and $\beta = \gamma - 1$)

$$\int_a^b |p(t)|\left(f^2\right)^{\Delta}(t)\left(f^{\Delta}(t)\right)^{\gamma-1}\Delta t \le K_1\int_a^b r(t)\left(f^{\Delta}(t)\right)^{\gamma+1}\Delta t,$$

and Corollary 3.2 yields (with $\alpha = \gamma + 1$)

$$\int_a^b |Q(t)|\left(f^{\gamma+1}\right)^{\Delta}(t)\Delta t \le K_2\int_a^b r(t)\left(f^{\Delta}(t)\right)^{\gamma+1}\Delta t.$$

Using this, together with two applications of the time scales integration by parts rule, we obtain

$$\int_a^b r(t)\left(f^{\Delta}(t)\right)^{\gamma+1}\Delta t = \int_a^b r(t)\left(f^{\Delta}(t)\right)^{\gamma} f^{\Delta}(t)\Delta t$$

$$
\begin{aligned}
&= r(b)\left(f^{\Delta}(b)\right)^{\gamma} f(b) - r(a)\left(f^{\Delta}(a)\right)^{\gamma} f(a) - \int_a^b \left(r\left(f^{\Delta}\right)^{\gamma}\right)^{\Delta}(t) f^{\sigma}(t)\Delta t \\
&\le -\int_a^b \left(r\left(f^{\Delta}\right)^{\gamma}\right)^{\Delta}(t) f^{\sigma}(t)\Delta t \\
&= \int_a^b \left\{p(t)\left(f^{\Delta}(t)\right)^{\gamma} + q(t)\left(f^{\sigma}(t)\right)^{\gamma}\right\} f^{\sigma}(t)\Delta t \\
&= \int_a^b p(t)\left(f^{\Delta}(t)\right)^{\gamma} f^{\sigma}(t)\Delta t - \int_a^b Q^{\Delta}(t)\left(f^{\sigma}(t)\right)^{\gamma+1}\Delta t \\
&= \int_a^b p(t)\left(f^{\Delta}(t)\right)^{\gamma-1} f^{\Delta}(t) f^{\sigma}(t)\Delta t - Q(b)(f(b))^{\gamma+1} + Q(a)(f(a))^{\gamma+1} \\
&\quad + \int_a^b Q(t)\left(f^{\gamma+1}\right)^{\Delta}(t)\Delta t \\
&= \int_a^b p(t)\left(f^{\Delta}(t)\right)^{\gamma-1} f^{\Delta}(t) f^{\sigma}(t)\Delta t + \int_a^b Q(t)\left(f^{\gamma+1}\right)^{\Delta}(t)\Delta t \\
&\le \int_a^b |p(t)|\left(f^{\Delta}(t)\right)^{\gamma-1} f^{\Delta}(t)\left(f^{\sigma}(t)+f(t)\right)\Delta t + \int_a^b |Q(t)|\left(f^{\gamma+1}\right)^{\Delta}(t)\Delta t \\
&= \int_a^b |p(t)|\left(f^{\Delta}(t)\right)^{\gamma-1}\left(f^{2}\right)^{\Delta}(t)\Delta t + \int_a^b |Q(t)|\left(f^{\gamma+1}\right)^{\Delta}(t)\Delta t \\
&\le K_1\int_a^b r(t)\left(f^{\Delta}(t)\right)^{\gamma+1}\Delta t + K_2\int_a^b r(t)\left(f^{\Delta}(t)\right)^{\gamma+1}\Delta t \\
&= (K_1+K_2)\int_a^b r(t)\left(f^{\Delta}(t)\right)^{\gamma+1}\Delta t,
\end{aligned}
$$

and so the claim follows after cancelling the integral. □

Remark 5.2. If $\gamma \ge 1$ is a quotient of odd positive integers, then the assumption $(f^{\Delta}(b))^{\gamma} \le 0$ in Theorem 5.1 may be replaced by the assumption $f^{\Delta}(b) \le 0$.

If we let $\gamma = 1$ in Theorem 5.1, then we get the following result.

Corollary 5.3. *Assume that $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,*

$$p, q \in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}}, (0,\infty)),\ r \in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}), \quad \textit{and} \quad f \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

If f is a nontrivial and nonnegative solution of (5.1) *with $\gamma = 1$ such that $f(a) = 0$, $f^{\Delta}(t) \ge 0$ for all $t \in [a,b)_{\mathbb{T}}$, and $f^{\Delta}(b) \le 0$, then*

$$\sqrt{\int_a^b (p(t))^2\left(R^2\right)^{\Delta}(t)\Delta t} + \sqrt{\int_a^b (Q(t))^2\left(R^2\right)^{\Delta}(t)\Delta t} \ge 1,$$

where

$$R(t) = \int_a^t \frac{\Delta\tau}{r(\tau)}, \quad Q(t) = \int_t^b q(\tau)\Delta\tau.$$

If we let $r(t) \equiv 1$ in Corollary 5.3, then we get the following result.

Corollary 5.4. *Assume that* $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,

$$p, q \in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}}, (0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

If f *is a nontrivial and nonnegative solution of* (5.1) *with* $\gamma = 1$ *and* $r(t) \equiv 1$ *such that* $f(a) = 0$, $f^{\Delta}(t) \geq 0$ *for all* $t \in [a,b)_{\mathbb{T}}$, *and* $f^{\Delta}(b) \leq 0$, *then*

$$\sqrt{\int_a^b (p(t))^2 (t + \sigma(t) - 2a)\Delta t} + \sqrt{\int_a^b (Q(t))^2 (t + \sigma(t) - 2a)\Delta t} \geq 1,$$

where

$$Q(t) = \int_t^b q(\tau)\Delta\tau.$$

If we let $p(t) \equiv 0$ in Corollary 5.4, then we get the following result.

Corollary 5.5. *Assume that* $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,

$$q \in \mathrm{C}_{\mathrm{rd}}([a,b]_{\mathbb{T}}, (0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

If f *is a nontrivial and nonnegative solution of*

$$f^{\Delta\Delta}(t) + q(t) f^{\sigma}(t) = 0, \quad t \in [a,b]_{\mathbb{T}}$$

such that $f(a) = 0$, $f^{\Delta}(t) \geq 0$ *for all* $t \in [a,b)_{\mathbb{T}}$, *and* $f^{\Delta}(b) \leq 0$, *then*

$$\int_a^b (Q(t))^2 (t + \sigma(t) - 2a)\Delta t \geq 1, \quad \textit{where} \quad Q(t) = \int_t^b q(\tau)\Delta\tau.$$

If we let $\mathbb{T} = \mathbb{R}$ in Corollary 5.5, then we get the following classical result of Brown and Hinton [7, Theorem 3.1].

Example 5.6. Assume that $a \in \mathbb{R}$, $b \in (a,\infty)$,

$$q \in \mathrm{C}([a,b], (0,\infty)), \quad \text{and} \quad f \in \mathrm{C}^1([a,b]_{\mathbb{T}}, \mathbb{R}).$$

If f is a nontrivial and nonnegative solution of

$$f''(t) + q(t) f(t) = 0, \quad t \in [a,b]$$

such that $f(a) = 0$, $f'(t) \geq 0$ for all $t \in [a,b)$, and $f'(b) = 0$, then

$$2\int_a^b (Q(t))^2 (t - a)\mathrm{d}t \geq 1, \quad \text{where} \quad Q(t) = \int_t^b q(\tau)\mathrm{d}\tau.$$

If we let $\mathbb{T} = \mathbb{Z}$ in Corollary 5.5, then we get the following result.

Example 5.7. Assume that $a \in \mathbb{Z}$, $b \in (a,\infty)_{\mathbb{Z}}$,

$$q : [a,b]_{\mathbb{Z}} \to (0,\infty), \quad \text{and} \quad f : [a,b]_{\mathbb{Z}} \to \mathbb{R}.$$

If f is a nontrivial and nonnegative solution of

$$\Delta^2 f(t) + q(t) f(t+1) = 0, \quad t \in [a,b]_{\mathbb{Z}}$$

such that $f(a) = 0$, $\Delta f(t) \geq 0$ for all $t \in [a, b)_{\mathbb{Z}}$, and $\Delta f(b) \leq 0$, then

$$\sum_{t=a}^{b-1} (Q(t))^2(2t - 2a + 1) \geq 1, \quad \text{where} \quad Q(t) = \sum_{\tau=t}^{b-1} q(\tau).$$

Remark 5.8. The results in this section are given when $f(a) = 0$ and f^{Δ} has a generalized zero at b. Similarly as in [16], results can also be given when $f(b) = 0$ and f^{Δ} has a generalized zero at a or when f has two consecutive zeros at a and at b, but since no new ideas are needed, we refrain from stating these other results. These other results give criteria for disfocality and disconjugacy of (5.1), see [16].

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