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ON POSITIVE SOLUTIONS FOR FOURTH-ORDER FOUR-POINT BOUNDARY VALUE PROBLEMS WITH ALTERNATING COEFFICIENT ON TIME SCALES

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ABSTRACT. In this paper, by using four functionals fixed point theorem and five functionals fixed point theorem, we study the existence of at least one positive solution and three positive solutions respectively of a fourth-order four-point boundary value problem with alternating coefficient on a time scale. Examples are also included to illustrate our results.

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1. INTRODUCTION

The theory of time scales was introduced by Stefan Hilger [11] in his PhD thesis in 1988. Theoretically, this new theory has not only unify continuous and discrete equations, but has also exhibited much more complicated dynamics on time scales. Moreover, the study of dynamic equations on time scales has led to several important applications, for example, insect population models, biology, neural networks, heat transfer, and epidemic models, see [1, 2, 3, 10, 17]. Some preliminary definitions and theorems on time scales can be found in the books [7, 8] which are excellent references for the calculus of time scales. Due to the unification of the theory of differential and difference equations, there have been many investigations working on the existence of positive solutions to boundary value problems for dynamic equations on time scales [3, 4, 12, 15, 16, 19, 20].

There have been extensive studies on two-point and multi-point boundary value problems via many methods- for example [3, 4, 9, 10, 12, 13, 14] and references therein. In most of these studies, the coefficient function is assumed to be nonnegative. There is not much studies on multi-point boundary value problems with alternating coefficient, see [9, 13, 14, 18]. Especially the existence and positive solutions for fourth-order multi-point boundary value problems with alternating coefficient on time scales has never been discussed. So this paper fills the gap.

In this paper, we consider the following fourth-order four-point boundary value problem (BVP)

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$$\text{(1.1)}\qquad \begin{cases} y^{\Delta^4}(t) = h(t)f(t, y(t)), \ t \in [0,1] \subset \mathbb{T}, \\ y(0) = 0, \ y(1) = 0, \\ \alpha y^{\Delta^2}(\xi_1) - \beta y^{\Delta^3}(\xi_1) = 0, \quad \gamma y^{\Delta^2}(\xi_2) + \delta y^{\Delta^3}(\xi_2) = 0 \end{cases}$$

where $\mathbb{T}$ is a time scale, and $h(t)$ is the alternating coefficient on $[0,1]$.

Throughout this paper we assume that following conditions hold:

$(H1)$ $\alpha, \beta, \gamma, \delta \geq 0$, $0 < \xi_1 < \xi_2 < 1$, and $D = \alpha\gamma(\xi_2 - \xi_1) + \alpha\delta + \gamma\beta > 0$.

$(H2)$ $h : [0,1] \to \mathbb{R}$ is continuous and such that $h(t) \leq 0$, $t \in [0, \xi_1]$; $h(t) \geq 0$, $t \in [\xi_1, \xi_2]$; $h(t) \leq 0$, $t \in [\xi_2, 1]$. Moreover, it does not vanish identically on any subinterval of $[0,1]$.

$(H3)$ $f \in C([0,1] \times [0,\infty) \times [0,\infty))$.

By using four functionals fixed point theorem [5] and five functionals fixed point theorem [4], we get the existence of at least one positive solution and of at least three positive solutions respectively.

This work is organized as follows. After this section, we give some preliminary lemmas. In Section 3, we give our main results Theorems 3.3 and 3.4. Examples are also given to show our results.

2. PRELIMINARIES

In this section, we present auxiliary lemmas which will be used later.

Lemma 2.1. *Assume that the condition $(H1)$ is satisfied. If $g \in \mathcal{C}[0,1]$*

$$\begin{cases} y^{\Delta^2}(t) = g(t), \ t \in [0,1], \\ \alpha y(\xi_1) - \beta y^{\Delta}(\xi_1) = 0, \quad \gamma y(\xi_2) + \delta y^{\Delta}(\xi_2) = 0 \end{cases}$$

has a unique solution

$$y(t) = -\int_0^1 G(t,s) g(s) \Delta s, \quad t \in [0,1],$$

where

$$\text{(2.1)}\quad G(t,s) = \begin{cases} s \in [0,\xi_1], & \begin{cases} t - s, & t \leq s, \\ 0, & s \leq t, \end{cases} \\ \\ s \in [\xi_1, \xi_2], & \dfrac{1}{D} \begin{cases} (\alpha(t - \xi_1) + \beta)(\gamma(\xi_2 - s) + \delta), & t \leq s, \\ (\alpha(s - \xi_1) + \beta)(\gamma(\xi_2 - t) + \delta), & s \leq t, \end{cases} \\ \\ s \in [\xi_2, 1], & \begin{cases} 0, & t \leq s, \\ s - t, & s \leq t. \end{cases} \end{cases}$$

Proof. It is easy to see that $G(t,s)$ satisfies the boundary conditions

$$\begin{cases} \alpha y(\xi_1) - \beta y^{\Delta}(\xi_1) = 0, \\ \gamma y(\xi_2) + \delta y^{\Delta}(\xi_2) = 0. \end{cases}$$

For each $t \in [0,1]$, we consider three cases:

Case $1 : t \in [0, \xi_1]$.

$$y(t) = \int_t^{\xi_1} (s-t)h(s)f(s,y(s))\Delta s + \frac{1}{D}\int_{\xi_1}^{\xi_2} [\alpha(\xi_1 - t) - \beta][\gamma(\xi_2 - s) + \delta]g(s)\Delta s.$$

Case $2 : t \in [\xi_1, \xi_2]$.

$$\begin{aligned} y(t) &= \frac{1}{D}\int_{\xi_1}^{t} [\alpha(\xi_1 - s) - \beta][\gamma(\xi_2 - t) + \delta]h(s)f(s,y(s))\Delta s \\ &\quad + \frac{1}{D}\int_{t}^{\xi_2} [\alpha(\xi_1 - t) - \beta][\gamma(\xi_2 - s) + \delta]h(s)f(s,y(s))\Delta s. \end{aligned}$$

Case $3 : t \in [\xi_2, 1]$.

$$\begin{aligned} y(t) &= \frac{1}{D}\int_{\xi_1}^{\xi_2} [\alpha(\xi_1 - s) - \beta][\gamma(\xi_2 - t) + \delta]h(s)f(s,y(s))\Delta s \\ &\quad + \int_{\xi_2}^{t} (t-s)h(s)f(s,y(s))\Delta s. \end{aligned}$$

For each of three cases, we get $y^{\Delta^2}(t) = g(t)$. This completes the proof. $\square$

Lemma 2.2. *Assume that* $(H1)$ *holds. If* $\beta \geq \alpha\xi_1$ *and* $\delta \geq \gamma(1-\xi_2)$*, then* $G(t,s)$ *is nonpositive on* $[0,1] \times ([0,\xi_1] \cup [\xi_2, 1])$ *and nonnegative on* $[0,1] \times [\xi_1, \xi_2]$.

Proof. The lemma follows from (2.1) immediately. $\square$

Lemma 2.3. *Assume that the conditions* $(H1)$–$(H3)$ *are satisfied. Then the BVP* (1.1) *has a unique solution*

$$y(t) = \int_0^1 G_1(t,s)\int_0^1 G(s,\tau)h(\tau)f(\tau,y(\tau))\Delta\tau\Delta s, \quad t \in [0,1],$$

where G *is defined as in* (2.1) *and*

$$G_1(t,s) = \begin{cases} t(1-s), & t \leq s, \\ s(1-t), & s \leq t. \end{cases} \tag{2.2}$$

Proof. Consider the following boundary value problem

$$\begin{cases} y^{\Delta^2}(t) = -\displaystyle\int_0^1 G(t,s)h(s)f(s,y(s))\Delta s, \quad t \in [0,1], \\ y(0) = 0, \ y(1) = 0 \end{cases} \tag{2.3}$$

The Green's function associated with the BVP (2.3) is $G_1(t,s)$. This completes the proof. $\square$

Throughout this paper, we assume $\omega, \nu \in \mathbb{T}$ with $\xi_1 < \omega < \nu < \xi_2$.

Lemma 2.4. *Under the condition $(H1)$ the Green's function $G_1(t,s)$ satisfies*

$$\begin{cases} 0 \le G_1(t,s) \le G_1(s,s) \quad for \quad (t,s) \in [0,1] \times [0,1] \\ G_1(t,s) \ge kG_1(s,s) \quad for \quad (t,s) \in [\omega,\nu] \times [0,1] \end{cases} \tag{2.4}$$

where

$$k = \min\{\omega, 1-\nu\}. \tag{2.5}$$

Proof. From (2.2), one can easily see these inequalities (2.4). □

Lemma 2.5. *Let $(H1)$–$(H3)$ hold. Then the unique solution of BVP (1.1) satisfies*

$$y(t) \ge k\,\|y\| \quad for \ t \in [\omega,\nu],$$

where $\|y\| = \max_{t\in[0,1]} y(t)$ and k is as in (2.5).

Proof. We have from (2.4) that for all $t \in [0,1]$

$$y(t) \le \int_0^1 G_1(s,s) \int_0^1 G(s,\tau)h(\tau)f(\tau,y(\tau))\Delta\tau\Delta s,$$

which implies that

$$\|y\| \le \int_0^1 G_1(s,s) \int_0^1 G(s,\tau)h(\tau)f(\tau,y(\tau))\Delta\tau\Delta s.$$

Thus for $t \in [\omega,\nu]$,

$$\begin{aligned} y(t) &= \int_0^1 G_1(s,s) \int_0^1 G(s,\tau)h(\tau)f(\tau,y(\tau))\Delta\tau\Delta s \\ &= \int_0^1 \frac{G_1(t,s)}{G_1(s,s)} G_1(s,s) \int_0^1 G(s,\tau)h(\tau)f(\tau,y(\tau))\Delta\tau\Delta s \\ &\ge k \int_0^1 G_1(s,s) \int_0^1 G(s,\tau)h(\tau)f(\tau,y(\tau))\Delta\tau\Delta s \\ &\ge k\|y\|. \end{aligned}$$

□

Lemma 2.6. *Assume that $(H1)$–$(H3)$ hold. If $\beta \ge \alpha\xi_1$ and $\delta \ge \gamma(1-\xi_2)$, then the unique solution $y(t)$ of BVP (1.1) is positive on $[0,1]$ and concave on $[0,1]$.*

Proof. By Lemma 2.2, Lemma 2.3 and Lemma 2.4, $y(t)$ is positive on $[0,1]$.

For $t \in [0,\xi_1]$,

$$\begin{aligned} y^{\Delta^2}(t) \quad &= \quad \int_t^{\xi_1} (s-t)h(s)f(s,y(s))\Delta s \\ &\quad + \frac{1}{D}\int_{\xi_1}^{\xi_2} [\alpha(\xi_1 - t) - \beta][\gamma(\xi_2 - s)+\delta]h(s)f(s,y(s))\Delta s \le 0. \end{aligned}$$

Similarly, for $t \in [\xi_1,\xi_2]$ and $t \in [\xi_2,1]$, we get $y^{\Delta^2}(t) \le 0$. Hence $y(t)$ is concave on $[0,1]$. □

We work in the Banach space $\mathbb{B} = \{\mathcal{C}[0,1] : y(0) = y(1) = 0\}$ with the norm $\|y\| = \max_{t\in[0,1]} y(t)$. Then define a cone $\mathcal{P}$ in $\mathbb{B}$ by

$$\mathcal{P} = \{y \in \mathbb{B} : y(t) \geq 0,\ y \text{ is concave on } [0,1]\}.$$

Define an operator A by

$$Ay(t) := \int_0^1 G_1(t,s) \int_0^1 G(s,\tau)h(\tau)f(\tau, y(\tau))\Delta\tau\Delta s, \quad t \in [0,1].$$

Set

$$N = -\xi_1 \int_0^{\xi_1} h(\tau)\Delta\tau - \int_{\xi_2}^1 h(\tau)\Delta\tau + \tfrac{1}{D}[\beta + \alpha(\xi_2 - \xi_1)][\delta + \gamma(\xi_2 - \xi_1)] \int_{\xi_1}^{\xi_2} h(\tau)\Delta\tau. \tag{2.6}$$

Lemma 2.7. *Assume that (H1)–(H3) hold. If $\beta \geq \alpha\xi_1$ and $\delta \geq \gamma(1-\xi_2)$, then $Q(\alpha,\beta,r,R)$ is bounded and $A : Q(\alpha,\beta,r,R) \to \mathcal{P}$ is completely continuous.*

Proof. From the definition of A, it is clear that $A(P) \subset P$. It is obvious that A is continuous in view of continuity of f, G, and G_1. Let $\Omega \subset P$ be bounded. Then, there exist positive constant $C > 0$ such that $|f(t,y)| \leq C$, $\forall y \in \Omega$. Thus for all $y \in \Omega$, we have

$$\begin{aligned}
\|Ay\| &= \max_{t\in[0,1]} Ay(t)\\
&\leq C \max_{t\in[0,1]} \int_0^1 G_1(t,s)\Delta s \max\Bigg[\max_{s\in[0,\xi_1]} \int_0^1 G(s,\tau)h(\tau)\Delta\tau,\\
&\quad \max_{s\in[\xi_1,\xi_2]} \int_0^1 G(s,\tau)h(\tau)\Delta\tau,\ \max_{s\in[\xi_2,1]} \int_0^1 G(s,\tau)h(\tau)\Delta\tau\Bigg]\\
&\leq C \max_{t\in[0,1]} \int_0^1 G_1(t,s)\Delta s \max\Bigg[\max_{s\in[0,\xi_1]} \Bigg(\int_s^{\xi_1} (s-\tau)h(\tau)\Delta\tau\\
&\quad + \frac{1}{D}\int_{\xi_1}^{\xi_2} [\alpha(s-\xi_1)+\beta][\gamma(\xi_2-\tau)+\delta]h(\tau)\Delta\tau\Bigg),\\
&\quad \max_{s\in[\xi_1,\xi_2]} \frac{1}{D}\Bigg(\int_{\xi_1}^{s} [\alpha(\tau-\xi_1)+\beta][\gamma(\xi_2-s)+\delta]h(\tau)\Delta\tau\\
&\quad + \int_s^{\xi_2} [\alpha(s-\xi_1)+\beta][\gamma(\xi_2-\tau)+\delta]h(\tau)\Delta\tau\Bigg),\\
&\quad \max_{s\in[\xi_2,1]} \Bigg(\frac{1}{D}\int_{\xi_1}^{\xi_2} [\alpha(\tau-\xi_1)+\beta][\gamma(\xi_2-s)+\delta]h(\tau)\Delta\tau + \int_{\xi_2}^{s} (\tau-s)h(\tau)\Delta\tau\Bigg)\Bigg]\\
&\leq C \max_{t\in[0,1]} \int_0^1 G_1(t,s)\Delta s \max\Bigg[\int_0^{\xi_1} -\tau h(\tau)\Delta\tau + \frac{1}{D}\int_{\xi_1}^{\xi_2} \beta[\gamma(\xi_2-\tau)+\delta]h(\tau)\Delta\tau,\\
&\quad \frac{1}{D}\int_{\xi_1}^{s} [\alpha(\tau-\xi_1)+\beta][\gamma(\xi_2-s)+\delta]h(\tau)\Delta\tau\\
&\quad + \frac{1}{D}\int_s^{\xi_2} [\alpha(s-\xi_1)+\beta][\gamma(\xi_2-\tau)+\delta]h(\tau)\Delta\tau,
\end{aligned}$$

$$
\begin{aligned}
& \frac{1}{D}\int_{\xi_1}^{\xi_2}\delta[\alpha(\tau-\xi_1)+\beta]h(\tau)\Delta\tau+\int_{\xi_2}^{1}-(1-\tau)h(\tau)\Delta\tau\Big] \\
\le\ & C\max_{t\in[0,1]}\int_0^1 G_1(t,s)\max\Big[\int_0^{\xi_1}-\tau h(\tau)\Delta\tau+\frac{1}{D}\beta[\gamma(\xi_2-\xi_1)+\delta]\int_{\xi_1}^{\xi_2}h(\tau)\Delta\tau, \\
& \frac{1}{D}[\alpha(\xi_2-\xi_1)+\beta][\gamma(\xi_2-\xi_1)+\delta]\int_{\xi_1}^{\xi_2}h(\tau)\Delta\tau, \\
& \frac{1}{D}\delta[\alpha(\xi_2-\xi_1)+\beta]\int_{\xi_1}^{\xi_2}h(\tau)\Delta\tau-\int_{\xi_2}^{1}(1-\tau)h(\tau)\Delta\tau\Big] \\
\le\ & C\max_{t\in[0,1]}\int_0^1 G_1(t,s)\Delta s\Big[\int_0^{\xi_1}-\tau h(\tau)\Delta\tau-\int_{\xi_2}^{1}(1-\tau)h(\tau)\Delta\tau \\
& +\frac{1}{D}[\alpha(\xi_2-\xi_1)+\beta][\gamma(\xi_2-\xi_1)+\delta]\int_{\xi_1}^{\xi_2}h(\tau)\Delta\tau\Big] \\
\le\ & C\max_{t\in[0,1]}\int_0^1 G_1(t,s)\Delta s\Big[-\xi_1\int_0^{\xi_1}h(\tau)\Delta\tau-\int_{\xi_2}^{1}h(\tau)\Delta\tau \\
& +\frac{1}{D}[\alpha(\xi_2-\xi_1)+\beta][\gamma(\xi_2-\xi_1)+\delta]\int_{\xi_1}^{\xi_2}h(\tau)\Delta\tau\Big] \\
\le\ & CN\max_{t\in[0,1]}\int_0^1 G_1(t,s)\Delta s,
\end{aligned}
$$

which implies that the operator A is uniformly bounded.

In addition, for $t_1, t_2 \in [0,1]$, we have

$$
\begin{aligned}
|Ay(t_1)-Ay(t_2)| &= |\int_0^1 G_1(t_1,s)\int_0^1 G(s,\tau)h(\tau)f(\tau,y(\tau))\Delta\tau\Delta s \\
&\quad -\int_0^1 G_1(t_2,s)\int_0^1 G(s,\tau)h(\tau)f(\tau,y(\tau))\Delta\tau\Delta s| \\
&\le CN\int_0^1 |G_1(t_1,s)-G_1(t_2,s)|\Delta s
\end{aligned}
$$

which implies that A is equicontinuous on $[0,1]$. Thus, by the Arzela-Ascoli Theorem, $A : P \to P$ is completely continuous. □

3. MAIN RESULTS

In this section, we discuss the existence of at least one positive solution and three positive solutions for the BVP (1.1) by using Theorem 3.1 and Theorem 3.2 respectively.

Let α, ψ be nonnegative continuous concave functionals on P, and β and θ be nonnegative continuous convex functionals on P. Then for positive numbers r, j, l and R, we define the sets

$$Q(\alpha,\beta,r,R)=\{y\in\mathcal{P} : r\le\alpha(y),\ \beta(y)\le R\},$$

$$U(\psi, l) = \{y \in Q(\alpha, \beta, r, R) : l \leq \psi(y)\},$$
$$V(\theta, v) = \{y \in Q(\alpha, \beta, r, R) : \theta(y) \leq v\}.$$

Theorem 3.1 (Four Functionals Fixed Point Theorem [5]). *If P is a cone in a real Banach space $\mathbb{B}$, α and ψ are nonnegative continuous concave functionals on P, β and θ are nonnegative continuous convex functionals on P, and there exist nonnegative positive numbers r, l, v and R such that*

$$A : Q(\alpha, \beta, r, R) \to P$$

is a completely continuous operator, and $Q(\alpha, \beta, r, R)$ is a bounded set. If

(i) $\{y \in U(\psi, l) : \beta(x) < R\} \cap \{y \in V(\theta, v) : r < \alpha(y)\} \neq \emptyset$,
(ii) $\alpha(Ay) \geq r$, *for all* $y \in Q(\alpha, \beta, r, R)$, *with* $\alpha(y) = r$ *and* $v < \theta(Ay)$,
(iii) $\alpha(Ay) \geq r$, *for all* $y \in V(\theta, v)$, *with* $\alpha(y) = r$,
(iv) $\beta(Ay) \leq R$, *for all* $y \in Q(\alpha, \beta, r, R)$, *with* $\beta(y) = R$ *and* $\psi(Ay) < l$,
(v) $\beta(Ay) \leq R$, *for all* $y \in U(\psi, l)$, *with* $\beta(y) = R$,

then A has a fixed point in y in $Q(\alpha, \beta, r, R)$.

We are now in a position to present the five functionals fixed point theorem. Let γ, β, θ be nonnegative continuous convex functionals on P and α, φ nonnegative continuous concave functionals on P. For nonnegative numbers h, a, b, d, and c, define the following convex sets:

$$P(\gamma, c) = \{x \in P : \gamma(x) < c\},$$

$$P(\gamma, \alpha, a, c) = \{x \in P : a \leq \alpha(x),\ \gamma(x) \leq c\},$$

$$Q(\gamma, \beta, d, c) = \{x \in P : \beta(x) \leq d,\ \gamma(x) \leq c\},$$

$$P(\gamma, \theta, \alpha, a, b, c) = \{x \in P : a \leq \alpha(x),\ \theta(x) \leq b,\ \gamma(x) \leq c\}.$$

Theorem 3.2 (Five Functionals Fixed Point Theorem [4]). *Let P be a cone in a real Banach space $\mathbb{B}$. Suppose that there exist nonnegative numbers r and M, nonnegative continuous concave functionals α and φ on P, and nonnegative continuous convex functionals γ, β, and θ on P, with*

$$\alpha(x) \leq \beta(x), \quad \|x\| \leq M\gamma(x), \quad \forall x \in \overline{P(\gamma, c)}.$$

Suppose that $A : \overline{P(\gamma, r)} \to \overline{P(\gamma, r)}$ is completely continuous and there exist nonnegative numbers h, a, k, b with $0 < p < q$ such that

(i) $\{x \in P(\gamma, \theta, \alpha, q, v, r) : \alpha(x) > q\} \neq \emptyset$ *and* $\alpha(A(x)) > q$ *for* $x \in Q(\gamma, \theta, \alpha, q, v, r)$,
(ii) $\{x \in P(\gamma, \beta, \varphi, h, p, r) : \beta(x) < p\} \neq \emptyset$ *and* $\beta(A(x)) < p$ *for* $x \in Q(\gamma, \beta, \varphi, h, p, r)$,
(iii) $\alpha(Ax) > q$ *for* $x \in P(\gamma, \alpha, q, r)$ *with* $\theta(Ax) > v$,
(iv) $\beta(Ax) < p$ *for* $x \in Q(\gamma, \beta, p, r)$ *with* $\varphi(Ax) < h$,

then A has at least three fixed points $x_1,\ x_2,\ x_3 \in \overline{P(\gamma, r)}$ such that

$$\beta(x_1) < p,\ \ \alpha(x_2) > q,\ \ \beta(x_3) > p, \text{with } \alpha(x_3) < q.$$

Set

$$M = \max\{M_1, M_2\},$$

where

$$M_1 = \beta\delta \int_{\xi_1}^{\xi_2} G_1(\omega, s) \int_{\omega}^{\nu} h(\tau)\Delta\tau\Delta s,$$

$$M_2 = \beta\delta \int_{\xi_1}^{\xi_2} G_1(\nu, s) \int_{\omega}^{\nu} h(\tau)\Delta\tau\Delta s.$$

Let k and N be as in (2.5), (2.6), respectively.

Theorem 3.3. *Let $\beta \geq \alpha\xi_1$ and $\delta \geq \gamma(1-\xi_2)$. Assume that $(H1)-(H3)$ hold, if there exist constant $r,\ l,\ v,\ R$ with $v \geq \max\{2l, \frac{r}{k}\}$, $l > r$, $R > \max\{2l, \frac{l}{k}\}$, and suppose that $f(t, y)$ satisfies the following conditions:*

(i) $f(t,y) \leq \frac{R}{N}\left[\max_{t\in[0,1]} \int_0^1 G_1(t,s)\Delta s\right]^{-1}$, $(t,y) \in [0,1]\times[0,R]$,
(ii) $f(t,y) \geq \frac{r}{M}$, $(t,y) \in [\omega,\nu]\times[r,v]$,

then the boundary value problem (1.1) *has a fixed point $y \in P$ such that*

$$\min_{t\in[\omega,\nu]} y(t) \geq r, \quad \max_{t\in[\omega,\nu]} y(t) \leq R.$$

Proof. The BVP (1.1) has a solution $y = y(t)$ if and only if y solves the operator equation $y = Ay$. Thus we set out to verify that the operator A satisfies four functionals fixed point theorem which will prove the existence of a fixed point of A.

Define maps

$$\alpha(y) = \psi(y) = \min_{t\in[\omega,\nu]} y(t),$$

$$\beta(y) = \max_{t\in[0,1]} y(t), \quad \theta(y) = \max_{t\in[\omega,\nu]} y(t).$$

To check condition (i) of Theorem 3.1, we choose $y(t) = 2l$, $0 \leq t \leq 1$. It is easy to see that $\psi(y) \geq l$, $\beta(y) < R$, $\theta(y) \leq v$ and $\alpha(y) > r$. So, $y \in \{y \in U(\psi, l) : \beta(y) < R\} \cap \{y \in V(\theta, v) : r < \alpha(y)\} \neq \emptyset$ which means that (i) in Theorem 3.1 is satisfied.

For all $y \in Q(\alpha, \beta, r, R)$, with $\alpha(y) = r$ and $v < \theta(Ay)$, from Lemma 2.5, we have

$$\alpha(Ay) = \min_{t\in[\omega,\nu]} Ay(t) \geq k\|Ay\| \geq k\theta(Ay) > kv \geq r.$$

For all $y \in Q(\alpha, \beta, r, R)$ with $\beta(y) = R$ and $\psi(Ay) < l$,

$$\beta(Ay) \leq \frac{1}{k} \min_{t\in[\omega,\nu]} Ay(t) = \frac{1}{k}\psi(Ay) < \frac{l}{k} < R.$$

Hence (ii) and (iv) in Theorem 3.1 are hold.

For any $y \in V(\theta, v)$ with $\alpha(y) = r$, Case 1 :

$$\begin{aligned}\alpha(Ay) &= Ay(\omega) = \int_0^1 G_1(\omega,s)\int_0^1 G(s,\tau)f(\tau,y(\tau))h(\tau)\Delta\tau\Delta s\\ &> \beta\delta\int_{\xi_1}^{\xi_2} G_1(\omega,s)\int_\omega^\nu f(\tau,y(\tau))h(\tau)\Delta\tau\Delta s\\ &\geq \frac{r}{M}\beta\delta\int_{\xi_1}^{\xi_2} G_1(\omega,s)\int_\omega^\nu h(\tau)\Delta\tau\Delta s\\ &= \frac{r}{M}M_1 \geq r.\end{aligned}$$

Case 2 :

$$\begin{aligned}\alpha(Ay) &= Ay(\nu) = \int_0^1 G_1(\nu,s)\int_0^1 G(s,\tau)f(\tau,y(\tau))h(\tau)\Delta\tau\Delta s\\ &> \beta\delta\int_{\xi_1}^{\xi_2} G_1(\nu,s)\int_\omega^\nu f(\tau,y(\tau))h(\tau)\Delta\tau\Delta s\\ &\geq \frac{r}{M}\beta\delta\int_{\xi_1}^{\xi_2} G_1(\nu,s)\int_\omega^\nu h(\tau)\Delta\tau\Delta s\\ &= \frac{r}{M}M_2 \geq r.\end{aligned}$$

And for all $y \in U(\psi,l)$ with $\beta(y) = R$,

$$\begin{aligned}\beta(Ay) &= \max_{t\in[0,1]}\int_0^1 G_1(t,s)\int_0^1 G(s,\tau)f(\tau,y(\tau))h(\tau)\Delta\tau\Delta s\\ &\leq \frac{R}{N\max_{t\in[0,1]}\int_0^1 G_1(t,s)\Delta s}\max_{t\in[0,1]}\int_0^1 G_1(t,s)\int_0^1 G(s,\tau)h(\tau)\Delta\tau\Delta s\\ &\leq \frac{R}{N\max_{t\in[0,1]}\int_0^1 G_1(t,s)\Delta s}\max_{t\in[0,1]}\int_0^1 G_1(t,s)\Delta s[-\xi_1\int_0^{\xi_1} h(\tau)\Delta\tau\\ &\quad -\int_{\xi_2}^1 h(\tau)\Delta\tau + \frac{1}{D}[\beta+\alpha(\xi_2-\xi_1)][\delta+\gamma(\xi_2-\xi_1)]\int_{\xi_1}^{\xi_2} h(\tau)\Delta\tau]\\ &\leq N\frac{R}{N\max_{t\in[0,1]}\int_0^1 G_1(t,s)\Delta s}\max_{t\in[0,1]}\int_0^1 G_1(t,s)\Delta s = R.\end{aligned}$$

Hence (iii) and (v) in Theorem 3.1 hold. Thus, all the conditions of Theorem 3.1 are satisfied. A has a fixed point $y \in Q(\alpha,\beta,r,R)$. Therefore, the BVP (1.1) has at least one positive solution $y \in \mathcal{P}$ such that

$$\min_{t\in[\omega,\nu]} y(t) \geq r, \quad \max_{t\in[\omega,\nu]} y(t) \leq R.$$

□

Theorem 3.4. *Let $\beta \geq \alpha\xi_1$ and $\delta \geq \gamma(1-\xi_2)$. Assume that $(H1)$, $(H2)$, and $(H3)$ hold. If there exist constants p, q, r, v with $v \geq \max\{2q, \frac{q}{k}\}$, $h \leq \min\{kp, \frac{p}{2}\}$, $2q \leq r$, $p < q$ further suppose that $f(t,y)$ satisfies the following conditions:*

(*B*1) $f(t,y) < \frac{p}{N}\left[\max_{t\in[0,1]}\int_0^1 G_1(t,s)\Delta s\right]^{-1}$, $(t,y)\in[0,1]\times[0,p]$,
(*B*2) $f(t,y) > \frac{q}{M}$, $(t,y)\in[\omega,\nu]\times[q,v]$,
(*B*3) $f(t,y) \le \frac{r}{N}\left[\max_{t\in[0,1]}\int_0^1 G_1(t,s)\Delta s\right]^{-1}$, $(t,y)\in[0,1]\times[0,r]$,

then the BVP (1.1) *has at least three positive solutions* y_1 y_2 *and* y_3 *such that*

$$\max_{t\in[\omega,\nu]} y_1(t) < p < \max_{t\in[\omega,\nu]} y_3(t), \quad \min_{t\in[\omega,\nu]} y_3(t) < q < \min_{t\in[\omega,\nu]} y_2(t).$$

Proof. Define the maps

$$\alpha(y) = \varphi(y) = \min_{t\in[\omega,\nu]} y(t), \quad \beta(y) = \theta(y) = \max_{t\in[\omega,\nu]} y(t), \quad \gamma(y) = \max_{t\in[0,1]} y(t).$$

It is clear that

$$\alpha(y) \le \beta(y), \quad \|y\| \le \gamma(y), \quad \forall y \in \overline{P(\gamma,r)}.$$

From Lemma 2.7, we obtain that $A : \overline{P(\gamma,r)} \to P$ is completely continuous. Thus, we only need to show that $A : \overline{P(\gamma,r)} \to \overline{P(\gamma,r)}$. Let $y \in \overline{P(\gamma,r)}$, then from $(B3)$ we have

$$\begin{aligned}
\gamma(Ay) &= \max_{t\in[0,1]} Ay(t) \\
&\le \frac{r}{N\max_{t\in[0,1]}\int_0^1 G_1(t,s)\Delta s}\max_{t\in[0,1]}\int_0^1 G_1(t,s)\Delta s[-\xi_1\int_0^{\xi_1} h(\tau)\Delta\tau \\
&\quad -\int_{\xi_2}^1 h(\tau)\Delta\tau + \frac{1}{D}[\beta+\alpha(\xi_2-\xi_1)][\delta+\gamma(\xi_2-\xi_1)]\int_{\xi_1}^{\xi_2} h(\tau)\Delta\tau] \\
&\le N\frac{r}{N\max_{t\in[0,1]}\int_0^1 G_1(t,s)\Delta s}\max_{t\in[0,1]}\int_0^1 G_1(t,s)\Delta s = r
\end{aligned}$$

which implies that $A(\overline{P(\gamma,r)}) \subset \overline{P(\gamma,r)}$.

We take $y(t) = 2q$, $t \in [0,1]$. It is easy to see that $y(t) \in \mathcal{P}$, $\alpha(y) > q$, $\theta(y) \le v$ and $\gamma(y) \le r$. That is $\{y \in \mathcal{P}(\gamma,\theta,\alpha,q,v,r) : \alpha(y) > q\} \ne \emptyset$. For $y \in P(\gamma,\theta,\alpha,q,v,r)$, we have, by condition $(B2)$,

Case 1 :

$$\begin{aligned}
\alpha(Ay) &= (Ay)(\omega) > \beta\delta\int_{\xi_1}^{\xi_2} G_1(\omega,s)\int_\omega^\nu f(\tau,y(\tau))h(\tau)\Delta\tau\Delta s \\
&> \frac{q}{M}\beta\delta\int_{\xi_1}^{\xi_2} G_1(\omega,s)\int_\omega^\nu h(\tau)\Delta\tau\Delta s \\
&= \frac{q}{M}M_1 \ge q.
\end{aligned}$$

Case 2 :

$$\begin{aligned}
\alpha(Ay) &= (Ay)(\nu) > \beta\delta\int_{\xi_1}^{\xi_2} G_1(\nu,s)\int_\omega^\nu h(\tau)f(\tau,y(\tau))\Delta\tau\Delta s \\
&> \frac{q}{M}\beta\delta\int_{\xi_1}^{\xi_2} G_1(\nu,s)\int_\omega^\nu h(\tau)\Delta\tau\Delta s
\end{aligned}$$

$$= \frac{q}{M} M_2 \geq q.$$

So, $\alpha(Ay) > q$. Hence, condition (i) of Theorem 3.2 holds. We take $y(t) = \frac{p}{2}$. It is easy to see that $y(t) \in \mathcal{P}$, $\beta(y) < p$, $\varphi(y) \geq h$ and $\gamma(y) \leq r$. That is $\{y \in \mathcal{P}(\alpha, \beta, \varphi, h, p, r) : \beta(y) < p\} \neq \emptyset$.

By condition $(B1)$, we get for $y \in Q(\gamma, \beta, \varphi, h, p, r)$,

$$\begin{aligned}
\beta(Ay) &= \max_{t\in[\omega,\nu]} \int_0^1 G_1(t,s) \int_0^1 G(s,\tau)h(\tau)f(\tau, y(\tau))\Delta\tau\Delta s \\
&\leq \frac{p}{N \max_{t\in[0,1]} \int_0^1 G_1(t,s)\Delta s} \max_{t\in[0,1]} \int_0^1 G_1(t,s)\Delta s[-\xi_1 \int_0^{\xi_1} h(\tau)\Delta\tau \\
&\quad - \int_{\xi_2}^1 h(\tau)\Delta\tau + \frac{1}{D}[\beta + \alpha(\xi_2 - \xi_1)][\delta + \gamma(\xi_2 - \xi_1)] \int_{\xi_1}^{\xi_2} h(\tau)\Delta\tau] \\
&\leq N \frac{p}{N \max_{t\in[0,1]} \int_0^1 G_1(t,s)\Delta s} \max_{t\in[\omega,\nu]} \int_0^1 G_1(t,s)\Delta s < p.
\end{aligned}$$

Thus, condition (ii) of Theorem 3.2 is satisfied.

On the other hand, for $y \in P(\gamma, \alpha, q, r)$ with $\theta(Ay) > v$, we have

$$\alpha(Ay) = \min_{t\in[\omega,\nu]} Ay(t) \geq k\|Ay\| \geq k\theta(Ay) > kv \geq q.$$

For $y \in P(\gamma, \beta, p, r)$ with $\varphi(Ay) < h$, we can obtain

$$\beta(Ay) \leq \frac{1}{k} \min_{t\in[\omega,\nu]} Ay(t) = \frac{1}{k}\psi(Ay) < \frac{h}{k} \leq p.$$

Thus, (iii) and (iv) in Theorem 3.2 hold.

So, by Theorem 3.2, we obtain that the BVP (1.1) has at least three positive solutions $y_1, y_2, y_3 \in \overline{P(\gamma, r)}$ such that

$$\max_{t\in[\omega,\nu]} y_1(t) < p < \max_{t\in[\omega,\nu]} y_3(t), \quad \min_{t\in[\omega,\nu]} y_3(t) < q < \max_{t\in[\omega,\nu]} y_2(t).$$

□

4. EXAMPLES

We illustrate Theorem 3.3 with specific time scale $\mathbb{T} = \{2^{-n} : n \in \mathbb{N}^+\} \cup \{0\} \cup [1, 2]$. Consider the boundary value problem:

$$\begin{cases} y^{\Delta^4}(t) = h(t)f(t, y(t)),\ t \in [0, 1], \\ y(0) = 0,\ y(1) = 0, \\ 4y^{\Delta^2}(\frac{1}{16}) - 2y^{\Delta^3}(\frac{1}{16}) = 0, \quad 6y^{\Delta^2}(\frac{1}{2}) + 5y^{\Delta^3}(\frac{1}{2}) = 0, \end{cases} \tag{4.1}$$

where $f(t,y)=\frac{2}{19}(68y-65)$, $\xi_1=\frac{1}{16}$, $\xi_2=\frac{1}{2}$, $\omega=\frac{1}{8}$, $\nu=\frac{1}{4}$, $\alpha=4$, $\beta=2$, $\gamma=6$, $\delta=5$,

$$h(t)=\begin{cases}-1, & t\in[0,\dfrac{1}{16}],\\ 1, & t\in[\dfrac{1}{16},\dfrac{1}{2}],\\ -1, & t\in[\dfrac{1}{2},1]\end{cases}$$

After some calculation, we have

$$M=M_2=\frac{315}{4096},\ M_1=\frac{415}{8192},\ N=\frac{3474}{4352},\ k=\frac{1}{8}.$$

Choose $r=1$, $l=2$, $v=10$, $R=20$.

Consequently f satisfies

$$f(t,y)\le 150<150.3281=\frac{6R}{N},\quad (t,y)\in[0,1]\times[0,20],$$

$$f(t,y)\ge 14>13.0031=\frac{r}{M},\quad (t,y)\in[\frac{1}{8},\frac{1}{4}]\times[1,10].$$

Then all conditions of Theorem 3.3 hold. Thus, with Theorem 3.3, problem (4.1) has a fixed point $y\in P$ such that

$$\min_{t\in[\frac{1}{8},\frac{1}{4}]} y(t)\ge 1,\quad \max_{t\in[\frac{1}{8},\frac{1}{4}]} y(t)\le 20.$$

Example 4.2 We illustrate Theorem 3.4 with specific time scale $\mathbb{T}=\{\frac{n+1}{10}:n\in\mathbb{N}\}$. Consider the boundary value problem:

$$(4.2)\qquad \begin{cases} y^{\Delta^4}(t)=h(t)f(t,y(t)),\ t\in[0,1],\\ y(0)=0,\ y(1)=0,\\ \dfrac{1}{4}y^{\Delta^2}(\dfrac{1}{5})-10y^{\Delta^3}(\dfrac{1}{5})=0,\quad \dfrac{1}{2}y^{\Delta^2}(\dfrac{1}{2})+20y^{\Delta^3}(\dfrac{1}{2})=0,\end{cases}$$

where $f(t,y)=5y$, $\xi_1=\frac{1}{5}$, $\xi_2=\frac{1}{2}$, $\omega=\frac{1}{4}$, $\nu=\frac{1}{3}$, $\alpha=\frac{1}{4}$, $\beta=10$, $\gamma=\frac{1}{2}$, $\delta=20$,

$$h(t)=\begin{cases}-2, & t\in[0,\dfrac{1}{5}],\\ t, & t\in[\dfrac{1}{5},\dfrac{1}{2}],\\ -1, & t\in[\dfrac{1}{2},1],\end{cases}$$

It follows from a direct calculation that

$$M_1=\frac{377}{1920},\ M=M_2=\frac{21315}{97200},\ N=\frac{2393161}{1606000},\ k=\frac{1}{4}.$$

Choose $p=\frac{1}{2}$, $q=1$, $r=4$, $v=5$.

Consequently f satisfies

$$f(t,y) \le \frac{5}{2} < 2.68431 = \frac{8p}{N}, \quad (t,y) \in [0,1] \times [0, \frac{1}{2}],$$

$$f(t,y) \ge 5 > 4.5601 = \frac{q}{M}, \quad (t,y) \in [\frac{1}{4}, \frac{1}{3}] \times [1,5],$$

$$f(t,y) \le 20 < 21.4745 = \frac{8r}{N}, \quad (t,y) \in [0,1] \times [0,4].$$

Then all conditions of Theorem 3.4 hold. Thus, with Theorem 3.4, problem (4.2) has a fixed point $y \in P$ such that

$$\max_{t \in [\frac{1}{4}, \frac{1}{3}]} y_1(t) < \frac{1}{2} < \max_{t \in [\frac{1}{4}, \frac{1}{3}]} y_3(t), \quad \min_{t \in [\frac{1}{4}, \frac{1}{3}]} y_3(t) < 1 < \min_{t \in [\frac{3}{2}, \frac{5}{2}]} y_2(t).$$

REFERENCES

[1] R. P. Agarwal and M. Bohner, Basic calculus on time scales and some of its applications, *Result Math.*, 35:3–22, 1999.

[2] R. P. Agarwal and D. O'Regan, Nonlinear boundary value problems on time scales, *Nonlinear Anal.*, 44:527–535, 2001.

[3] D. R. Anderson and I. Y. Karaca, Higher-order three-point boundary value problem on time scales, *Comput. Math. Appl.*, 56:2429–2443, 2008.

[4] D. R. Anderson and C. Zhai, Positive solutions to semi-positone second-order three-point problems on time scales, *Appl. Math. Comput.*, 215:3713–3720, 2010. 3713-3720.

[5] R. I. Avery, A generalization of the Leggett-Williams fixed point theorem, *Math. Sci. Res. Hot-Line*, 2:9–14, 1998.

[6] R. Avery, J. Henderson and D. O'Regan, Four functionals fixed point theorem, *Math. Comput. Modelling*, 48:1081–1089, 2008.

[7] M. Bohner and A. Peterson, Dynamic Equations on time scales, *A*n Introduction with Applications, Birkhauser, 2001.

[8] M. Bohner and A. Peterson, *A*dvances in Dynamic Equations on Time Scales, Birkhauser, Boston, 2003.

[9] Z. Fang, C. Li and C. Bai, Multiple positive solutions of fourth-order four-point boundary value problems with changing sign coefficient, *Electron. J. Differential Equations*, 2008:1–10, 2008.

[10] W. Han and S. Kang, Multiple positive solutions of nonlinear third-order bvp for a class of p-laplacian dynamic equations on time scales, *Math. Comput. Modelling*, 49:527–535, 2009.

[11] S. Hilger, Ein masskettenkalkl mit anwendug auf zentrumsmanningfaltigkeiten. PhD thesis, Universitat Würzburg, 1988.

[12] L. G. Hu and X.-F. Zhou, Positive solutions of the singular eigenvalue problem for a higher-order differential equation on time scales, *Math. Comput. Modelling*, 53:667–677, 2011.

[13] I. Y. Karaca, Nonlinear triple-point problems with change of sign, *Comput. Math. Appl.*, 55:691–703, 2008.

[14] I. Y. Karaca, On the existence of positive solutions for three-point boundary value problems with alternating coefficients, *Math. Comput. Modelling*, 47:1019–1034, 2008.

[15] I. Y. Karaca, Multiple positive solutions for dynamic m-point boundary value problems, *Dynam. Systems Appl.*, 17:25–42, 2008.

[16] I. Y. Karaca, O. Yilmaz, Fourth-order m-point boundary value problems on time scales, *Dynam. Systems Appl.*, 19:249–269, 2010.

[17] V. Lakshmikantham, S. Sivasundaram and B. Kaymakcalan, *Dynamic Systems on Measure Chains*, Kluwer, Dordrecht, 1996.

[18] B. Liu, Positive solutions of second-order three-point boundary-value problems with change of sign, *Comput. Math. Appl.*, 47:1351–1361, 2004.

[19] H. Luo and C. Gao, Positive solutions of a nonlinear fourth-order dynamic eigenvalue problem on time scales, *Abstr. Appl. Anal.*, 2012 Art. ID 798796 17 pp., 2012.

[20] Z. Zhang, L. Yin, Three positive solutions to BVP for p-Laplacian impulsive functional dynamic equations on a time scale, *Ann. Differential Equations*, 30:236–243, 2014.