Dynamic Systems and Applications 24 (2015) 349-360

THE DIFFERENTIAL GEOMETRY OF REGULAR CURVES ON A REGULAR TIME-LIKE SURFACE

EMIN OZYILMAZ AND YUSUF YAYLI

Department of Mathematics, Ege University, Bornova, Izmir, 35100, Turkey
Department of Mathematics, Ankara University, Dogol cad.
Ankara, 06100, Turkey

ABSTRACT. In this study, we consider time-like regular surface in Minkowski space as $y = y(u, v)$ and investigate Darboux vectors of the time-like curves on time-like surface as (c), (c_1) and (c_2) which are not intersect perpendicularly. Moreover, we give a relation between the Darboux vectors of these Darboux frames. By this relation we obtain general Liouville formula and general form Euler and O. Bonnet.

AMS (MOS) Subject Classification. 53A04,53B30.

1. INTRODUCTION

It is well-known that, if a curve differentiable on an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called Frenet frame or moving frame vectors. The set, whose elements are frame vectors and curvatures of a curve, is called Frenet apparatus of the curves. In recent years, the theory of degenerate submanifolds has been treated by researchers and some classical differential geometry topics have been extended to Lorentz manifolds. For instance, in [6], the authors extended and studied spacelike involute-evolute curves in Minkowski space. Classical differential geometry of the curves may be surrounded by the topics which are general helices, involute-evolute curve couples, spherical curves and Bertrand curves. Such special curves are investigated and used in some of real world problems like mechanical design or robotics by well-known Frenet-Serret equations. Because, we think of curves as the path of a moving particle in the Euclidean space.

At the beginning of the twentieth century Einstein's theory opened a door to new geometries such as Minkowski space-time, which is simultaneously the geometry of special relativity and the geometry induced on each fixed tangent space of an arbitrary Lorentzian manifold. Some authors have aimed to determine Frenet–Serret invariants in higher dimensions. There exists a vast literature on this subject, for instance [1–4, 6, 7, 8]. In the light of the available literature, in [4] the author extended spherical

Received April 2, 2015

images of curves to a four-dimensional Lorentzian space and studied such curves in the case where the base curve is a space-like curve according to the signature $(+++-)$. By using the Darboux vector, various well-known formulas of differential geometry had been produced by [5]. Then, in [1], authors had been given these formulas in Minkowski 3-space .

In this work, we study to investigate the formulae between the Darboux vectors of the curve (c), the parameter curves (c_1) and (c_2) which are not intersecting perpendicularly. Thus, we will find an opportunity to investigate regular time-like surface by taking the parameter curves which are intersect under the angle θ.

2. PRELIMINARIES

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space R_1^3 are briefly presented. (A more complete elementary treatment can be found in [1].) The Minkowski 3-space provided with the standard flat metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 - dx_3^2 \tag{2.1}$$

where (x_1, x_2, x_3) is rectangular coordinate system in R_1^3. Recall that, the norm of an arbitrary vector $a \in R_1^3$ is given by $\|\vec{a}\| = |\langle \vec{a}, \vec{a} \rangle|$. Let $\Phi = \Phi(s)$ be a regular curve in R_1^3. Φ is called an unit speed curve if the velocity vector $\vec{v}$ of Φ satisfies $\|\vec{v}\| = 1$. For the vectors $\vec{u}, \vec{w} \in R_1^3$ it is said to be orthogonal if and only if $\langle \vec{u}, \vec{w} \rangle = 0$.

On the other hand, the vector $\vec{w}$ is called angular velocity vector of motion. If we consider any orthogonal trihedron as $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$, we can write their derivative formulas as follows:

$$\frac{d\vec{e_i}}{ds} = \vec{w} \wedge \vec{e_i}, i = 1, 2, 3 \tag{2.2}$$

where $\wedge$ is Lorentzian vectorial product, [1].

Let us take a time-like surface as $\vec{y} = \vec{y}(u, v)$. Denote by $\left\{\vec{t}, \vec{N}, \vec{B}\right\}$ the moving Frenet-Serret frame along the time-like curve (c) on $\vec{y} = \vec{y}(u, v)$. Another orthogonal frame on $\vec{y} = \vec{y}(u, v)$ is the Darboux trihedron as $\left\{\vec{t}, \vec{g}, \vec{N}\right\}$. For an arbitrary time-like curve (c) on time-like surface, the orientation of the Darboux trihedron is written as

$$\vec{N} \wedge \vec{t} = -\vec{g}, \quad \vec{t} \wedge \vec{g} = -\vec{N}, \quad \vec{g} \wedge \vec{N} = \vec{t} \tag{2.3}$$

and the Darboux vector of this trihedron is written as

$$\vec{w} = \frac{\vec{t}}{T_g} + \frac{\vec{g}}{R_n} - \frac{\vec{N}}{R_g} \tag{2.4}$$

where $\vec{t}.\vec{t} = -1$ $\vec{g}.\vec{g} = 1$ $\vec{N}.\vec{N} = 1$ and $\frac{1}{T_g}, \frac{1}{R_n}$ and $\frac{1}{R_g}$ are geodesic torsion, normal curvature and geodesic curvature, respectively. Also,the Darboux derivative formulae can be written as follows:

$$\frac{d\vec{t}}{ds} = \vec{w} \wedge \vec{t}, \quad \frac{d\vec{g}}{ds} = \vec{w} \wedge \vec{g}, \quad \frac{d\vec{N}}{ds} = \vec{w} \wedge \vec{N} \tag{2.5}$$

[1].

3. THE DARBOUX VECTOR FOR THE DARBOUX TRIHEDRON OF A TIME-LIKE CURVE

Let us express the parameter curves $u =$ const. as (c_1) and $v =$ const. as (c_2) which are constant on a time-like surface $y = y(u, v)$. But, these curves are intersect under the angle θ (not perpendicular). Let any time-like curve that is passing through a point P on the surface be (c). Let us take time-like curves which are passing through the same point P as (c_1) and (c_2). Let the unit tangent vectors of curves (c), (c_1) and (c_2) and at the point P be $\vec{t}$,$\vec{t}_1$and $\vec{t}_2$, respectively. From [1], the edges of the Darboux trihedrons of parameter curves are

$$\vec{N} \wedge \vec{t}_1 = -\vec{g}_1, \quad \vec{t}_1 \wedge \vec{g}_1 = -\vec{N}, \quad \vec{g}_1 \wedge \vec{N} = \vec{t}_1. \tag{3.1}$$

Here, three Darboux trihedrons are written as below:

$$\left\{\vec{t}, \vec{g}, \vec{N}\right\}, \qquad \left\{\vec{t}_1, \vec{g}_1, \vec{N}\right\}, \qquad \left\{\vec{t}_2, \vec{g}_2, \vec{N}\right\}.$$

Let s, s_1 and s_2 be the arc-elements of the curves (c), (c_1) and (c_2), respectively. Thus, we can write

$$\vec{t}_1 = \frac{\vec{r}_u}{\|\vec{r}_u\|} = \frac{\vec{r}_u}{\sqrt{E}}, \quad \vec{t}_2 = \frac{\vec{r}_v}{\|\vec{r}_v\|} = \frac{\vec{r}_v}{\sqrt{G}}, \quad \vec{t} = \vec{r}_u\frac{du}{ds} + \vec{r}_v\frac{dv}{ds} \tag{3.2}$$

Moreover, because of the parameter curves are intersect under the angle θ we have

$$\vec{t}_1.\vec{t}_2 = -ch\theta \tag{3.3}$$

Then, the normal vector of time-like surface is

$$\vec{N} = \frac{\vec{t}_1 \wedge \vec{t}_2}{\left\|\vec{t}_1 \wedge \vec{t}_2\right\|} = \frac{\vec{t}_1 \wedge \vec{t}_2}{sh\theta} \tag{3.4}$$

Other than, considering the first two formulae of (3.2) in the third term,

$$\vec{t} = \vec{r}_u\frac{du}{ds} + \vec{r}_v\frac{dv}{ds} = \vec{t}_1\sqrt{E}\frac{du}{ds} + \vec{t}_2\sqrt{G}\frac{dv}{ds} \tag{3.5}$$

is written, [1].

On the other hand, let us consider the hyperbolic angle between $\vec{t}$ and $\vec{t}_1$ as α, and if we take inner product both sides of (3.5) with $\vec{t}_1$ and $\vec{t}_2$ then

$$\vec{t}.\vec{t}_1 = -ch\alpha = -\sqrt{E}\frac{du}{ds} - ch\theta\sqrt{G}\frac{dv}{ds} \tag{3.6}$$

$$\vec{t}.\vec{t}_2 = -ch(\theta - \alpha) = -ch\theta\sqrt{E}\frac{du}{ds} - \sqrt{G}\frac{dv}{ds} \tag{3.7}$$

are obtained. Thus, from (3.6) and (3.7)

$$\frac{sh(\theta-\alpha)}{sh\theta}=\sqrt{E}\frac{du}{ds} \tag{3.8}$$

$$\frac{sh\alpha}{sh\theta}=\sqrt{G}\frac{dv}{ds}$$

are written. Finally, if we put (3.8) into (3.5), we have the following equation between the tangent vectors of the curves (c), (c_1) and (c_2) as

$$\vec{t}=\frac{sh(\theta-\alpha)}{sh\theta}\vec{t}_1+\frac{sh\alpha}{sh\theta}\vec{t}_2 \tag{3.9}$$

Here, we shall denote the arc elements ds, ds_1 and ds_2 of the parameter curves which are belongs to time-like surface $y=y(u,v)$, and then we express as follows:

$$ds^2=Edu^2+2Fdudv+Gdv^2 \tag{3.10}$$

$$ds_1^2=Edu^2$$

$$ds_2^2=Gdv^2$$

Thus, considering (3.8) and (3.10), we have

$$\frac{sh(\theta-\alpha)}{sh\theta}=\sqrt{E}\frac{du}{ds}=\frac{ds_1}{ds} \tag{3.11}$$

$$\frac{sh\alpha}{sh\theta}=\sqrt{G}\frac{dv}{ds}=\frac{ds_2}{ds}$$

Corollary 3.1. *The third elements $\vec{g}$, $\vec{g}_1$ and $\vec{g}_2$ of the Darboux trihedrons $\left\{\vec{t},\vec{g},\vec{N}\right\}$, $\left\{\vec{t}_1,\vec{g}_1,\vec{N}\right\}$ and $\left\{\vec{t}_2,\vec{g}_2,\vec{N}\right\}$are linear dependent.*

Proof. If we substitute the equation (3.9) in the first equality of (2.3) and consider the Darboux trihedrons of (c_1) and (c_2) we have

$$\vec{g}=\frac{sh(\theta-\alpha)}{sh\theta}\vec{g}_1+\frac{sh\alpha}{sh\theta}\vec{g}_2 \tag{3.12}$$

Thus, we get the expression. □

Theorem 3.1. *The Darboux trihedrons $\left\{\vec{t}_1,\vec{g}_1,\vec{N}\right\}$ and $\left\{\vec{t}_2,\vec{g}_2,\vec{N}\right\}$ of the parameters curves (c_1) and (c_2) of the time-like surface are written by Darboux instantaneous vectors as follows:*

$$\frac{\partial\vec{t}_i}{\partial s_j}=\vec{w}_i\wedge\vec{t}_i,\quad \frac{\partial\vec{g}_i}{\partial s_j}=\vec{w}_i\wedge\vec{t}_i\quad \frac{\partial\vec{N}}{\partial s_j}=\vec{w}_i\wedge\vec{N},\qquad (i,j=1,2) \tag{3.13}$$

Proof. If we consider the Darboux trihedrons $\left\{\vec{t}_1,\vec{g}_1,\vec{N}\right\}$ and $\left\{\vec{t}_2,\vec{g}_2,\vec{N}\right\}$, we see that the normal vector $\vec{N}$ is coincide. Then, considering (3.4)

$$\vec{g}_1=\vec{t}_1\wedge\vec{N}=\vec{t}_1\wedge\left(\frac{\vec{t}_1\wedge\vec{t}_2}{sh\theta}\right)=\frac{\vec{t}_2(\vec{t}_1^2)-\vec{t}_1(\vec{t}_1.\vec{t}_2)}{sh\theta}$$

$$\vec{g}_1 = \frac{\vec{t}_1 ch\theta - \vec{t}_2}{sh\theta} \tag{3.14}$$

$$\vec{g}_2 = \vec{t}_2 \wedge \vec{N} = \vec{t}_2 \wedge \left(\frac{\vec{t}_1 \wedge \vec{t}_2}{sh\theta}\right) = \frac{\vec{t}_2(\vec{t}_1.\vec{t}_2) - \vec{t}_1(\vec{t}_2^2)}{sh\theta}$$

$$\vec{g}_2 = \frac{-\vec{t}_2 ch\theta + \vec{t}_1}{sh\theta} \tag{3.15}$$

are obtained. From (2.2), we write

$$\frac{\partial \vec{t}_1}{\partial s_1} = \vec{w}_1 \wedge \vec{t}_1, \qquad \frac{\partial \vec{g}_1}{\partial s_1} = \vec{w}_1 \wedge \vec{g}_1, \qquad \frac{\partial \vec{N}}{\partial s_1} = \vec{w}_1 \wedge \vec{N} \tag{3.16}$$

$$\frac{\partial \vec{t}_2}{\partial s_2} = \vec{w}_2 \wedge \vec{t}_2, \quad \frac{\partial \vec{g}_2}{\partial s_2} = \vec{w}_2 \wedge \vec{g}_2, \quad \frac{\partial \vec{N}}{\partial s_2} = \vec{w}_2 \wedge \vec{N} \tag{3.17}$$

If (3.14) is substituted in the third equality (3.16), we get

$$\begin{aligned} \frac{\partial \vec{g}_1}{\partial s_1} &= \frac{\partial \left[\frac{\vec{t}_1 ch\theta - \vec{t}_2}{sh\theta}\right]}{\partial s_1} = \frac{1}{sh\theta}\left(\frac{\partial \vec{t}_1}{\partial s_1} ch\theta - \frac{\partial \vec{t}_2}{\partial s_1}\right) \\ &= \frac{1}{sh\theta}\left[(\vec{w}_1 \wedge \vec{t}_1) ch\theta - \frac{\partial \vec{t}_2}{\partial s_1}\right] \end{aligned} \tag{3.18}$$

$$\frac{\partial \vec{g}_1}{\partial s_1} = \vec{w}_1 \wedge \vec{g}_1 = \vec{w}_1 \wedge \left[\frac{\vec{t}_1 ch\theta - \vec{t}_2}{sh\theta}\right] = \frac{1}{sh\theta}[(\vec{w}_1 \wedge \vec{t}_1) ch\theta - (\vec{w}_1 \wedge \vec{t}_2). \tag{3.19}$$

Then, from (3.18) and (3.19), we have

$$\frac{\partial \vec{t}_2}{\partial s_1} = \vec{w}_1 \wedge \vec{t}_2. \tag{3.20}$$

Thus, the derivative of $\vec{t}_2$ with respect to s_1 is written by the Lorentzian vectorial product of $\vec{w}_1$ and $\vec{t}_2$. Similarly, it is easy to see that the other vectors can be written by the same method. □

Corollary 3.2. *By using the vectors $\vec{t}_1, \vec{t}_2$ and $\vec{N}$, we can express $\vec{w}, \vec{w}_1$ and $\vec{w}_2$ as follows:*

$$\begin{aligned} \vec{w} &= \frac{\vec{t}}{T_g} + \frac{\vec{g}}{R_n} - \frac{\vec{N}}{R_g} \\ &= \frac{\vec{t}_1}{sh\theta}\left[\frac{sh(\theta-\alpha)}{T_g} + \frac{ch(\theta-\alpha)}{R_n}\right] + \frac{\vec{t}_2}{sh\theta}\left[\frac{sh\alpha}{T_g} - \frac{ch\alpha)}{R_n}\right] - \frac{\vec{N}}{R_g} \end{aligned} \tag{3.21}$$

$$\vec{w}_1 = \vec{t}_1\left[\frac{1}{T_{g_1}} + \frac{ch\theta}{sh\theta R_{n_1}}\right] - \frac{\vec{t}_2}{sh\theta R_{n_1}} - \frac{\vec{N}}{R_{g_1}} \tag{3.22}$$

$$\vec{w}_2 = \frac{1}{sh\theta R_{n_2}}\vec{t}_1 + \left(\frac{1}{T_{g_2}} - \frac{ch(\theta)}{sh\theta R_{n_2}}\right)\vec{t}_2 + \frac{\vec{N}}{R_{g_2}} \tag{3.23}$$

where $\frac{1}{T_g}$, $\frac{1}{R_n}$ *and* $\frac{1}{R_g}$ *are geodesic torsion, normal curvature and geodesic curvature, respectively.*

From (2.4), we can write the Darboux vectors of the $\left\{\vec{t},\vec{g},\vec{N}\right\}$, $\left\{\vec{t_1},\vec{g_1},\vec{N}\right\}$ and $\left\{\vec{t_2},\vec{g_2},\vec{N}\right\}$ as

$$\vec{w}=\frac{\vec{t}}{T_g}+\frac{\vec{g}}{R_n}-\frac{\vec{N}}{R_g}, \tag{3.24}$$

$$\vec{w_1}=\frac{\vec{t_1}}{T_{g_1}}+\frac{\vec{g_1}}{R_{n_1}}-\frac{\vec{N}}{R_{g_1}},$$

$$\vec{w_2}=\frac{\vec{t_2}}{T_{g_2}}+\frac{\vec{g}}{R_{n_2}}-\frac{\vec{N}}{R_{g_2}}.$$

Then, if we consider the equations (3.9), (3.14) and (3.15) according to the vectors $\vec{t_1}$ and $\vec{t_2}$, and substitute in (3.24), we get (3.21), (3.22) and (3.23).

Theorem 3.2. *If we consider the tangent vectors* $\vec{t_1}$ *and* $\vec{t_2}$ *of the parameter curves* (c_1) *and* (c_2) *on the time-like surface, then we obtain the following relations:*

$$i)\quad \vec{t_1}\frac{\partial\vec{t_2}}{\partial s_1}=-\vec{t_2}\,\frac{\partial\vec{t_1}}{\partial s_1}=\frac{\left(\sqrt{E}\right)_v+ch\theta\left(\sqrt{G}\right)_u}{\sqrt{EG}}, \tag{3.25}$$

$$ii)\quad \vec{t_2}\frac{\partial\vec{t_1}}{\partial s_2}=-\vec{t_1}\,\frac{\partial\vec{t_2}}{\partial s_2}=\frac{\left(\sqrt{G}\right)_u+ch\theta\left(\sqrt{E}\right)_v}{\sqrt{EG}}. \tag{3.26}$$

Proof. i) From (3.2), $\vec{t_1}=\frac{\vec{r}_u}{\sqrt{E}}$ and $\vec{t_2}=\frac{\vec{r}_v}{\sqrt{G}}$ are written. And also, we know that

$$\vec{t_1}\vec{t_2}=-ch\theta\Rightarrow\vec{r}_u\vec{r}_v=-ch\theta\sqrt{E}\sqrt{G}, \tag{3.27}$$

$$E=\left(\sqrt{E}\right)^2=\left(\vec{r}_u\right)^2\Rightarrow\sqrt{E}\left(\sqrt{E}\right)_v=\vec{r}_{uv}\vec{r}_u, \tag{3.28}$$

$$G=\left(\sqrt{G}\right)^2=\left(\vec{r}_v\right)^2\Rightarrow\sqrt{G}\left(\sqrt{G}\right)_u=\vec{r}_{vu}\vec{r}_v. \tag{3.29}$$

By taking differential from $\vec{t_1}=\frac{\vec{r}_u}{\sqrt{E}}$ and $\vec{t_2}=\frac{\vec{r}_v}{\sqrt{G}}$, we obtain

$$\frac{\partial\vec{t_1}}{\partial v}=\frac{\vec{r}_{uv}\left(\sqrt{E}\right)-\left(\sqrt{E}\right)_v\vec{r}_u}{E},$$

$$\frac{\partial\vec{t_2}}{\partial u}=\frac{\vec{r}_{uv}\left(\sqrt{G}\right)-\left(\sqrt{G}\right)_u\vec{r}_v}{G}.$$

Thus, we write

$$\vec{t_2}\frac{\partial\vec{t_1}}{\partial v}=\frac{\vec{r}_v}{\sqrt{G}}\left(\frac{\vec{r}_{uv}\left(\sqrt{E}\right)-\left(\sqrt{E}\right)_v\vec{r}_u}{E}\right)=\frac{\left(\sqrt{G}\right)_u+ch\theta\left(\sqrt{E}\right)_v}{\sqrt{E}}, \tag{3.30}$$

$$\vec{t_1}\frac{\partial \vec{t_2}}{\partial u} = \frac{\vec{r}_u}{\sqrt{E}}\left(\frac{\vec{r}_{uv}\left(\sqrt{G}\right) - \left(\sqrt{G}\right)_u \vec{r}_v}{G}\right) = \frac{\left(\sqrt{E}\right)_v + ch\theta\left(\sqrt{G}\right)_u}{\sqrt{G}}. \tag{3.31}$$

On the other hand, we have

$$\frac{\partial \vec{t_1}}{\partial s_1} = \frac{\partial \vec{t_1}}{\partial v}\frac{\partial v}{\partial s_1} = \frac{1}{\sqrt{G}}\frac{\partial \vec{t_1}}{\partial v}(ds_2 = \sqrt{G}du) \Rightarrow \frac{\partial v}{\partial s_2} = \frac{1}{\sqrt{G}}, \tag{3.32}$$

$$\frac{\partial \vec{t_2}}{\partial s_1} = \frac{\partial \vec{t_1}}{\partial u}\frac{\partial u}{\partial s_1} = \frac{1}{\sqrt{E}}\frac{\partial \vec{t_2}}{\partial u}(ds_1 = \sqrt{E}du) \Rightarrow \frac{\partial u}{\partial s_1} = \frac{1}{\sqrt{E}}. \tag{3.33}$$

Thus, taking inner product of (3.32) and (3.33) by the vector $\vec{t_2}$ and the vector $\vec{t_1}$, and considering (3.30) and (3.31), we have (3.26) and (3.25). The other cases can be seen easily. □

Result 3.1. If we take differential from $\vec{t_1}\vec{t_2} = -ch\theta$ with respect to u and v, we get

$$\vec{t_1}\frac{\partial \vec{t_2}}{\partial v} = -\vec{t_2}\frac{\partial \vec{t_1}}{\partial v} = -\frac{\left(\sqrt{G}\right)_u + ch\theta\left(\sqrt{E}\right)_v}{\sqrt{E}}, \tag{3.34}$$

$$\vec{t_1}\frac{\partial \vec{t_2}}{\partial u} = \frac{\left(\sqrt{E}\right)_v + ch\theta\left(\sqrt{G}\right)_u}{\sqrt{G}}. \tag{3.35}$$

Thus, we have

$$\begin{aligned}\frac{\partial}{\partial v}\left(\vec{t_1}\frac{\partial \vec{t_2}}{\partial u}\right) - \frac{\partial}{\partial u}\left(\vec{t_1}\frac{\partial \vec{t_2}}{\partial v}\right) &= \frac{\partial}{\partial v}\left(\frac{\left(\sqrt{E}\right)_v + ch\theta\left(\sqrt{G}\right)_u}{\sqrt{G}}\right)\\ &+ \frac{\partial}{\partial u}\left(\frac{\left(\sqrt{G}\right)_u + ch\theta\left(\sqrt{E}\right)_v}{\sqrt{E}}\right).\end{aligned} \tag{3.36}$$

Theorem 3.3. *The following geodesic curvature equalities are satisfied for the parameter curves* (c_1) *and* (c_2)

$$\frac{1}{R_{g_1}} = -\frac{1}{sh\theta\sqrt{EG}}\left(ch\theta\left(\sqrt{G}\right)_u + \left(\sqrt{E}\right)_v\right) \tag{3.37}$$

$$\frac{1}{R_{g_2}} = -\frac{1}{sh\theta\sqrt{EG}}\left(\left(\sqrt{G}\right)_u + sh\theta\left(\sqrt{E}\right)_v\right). \tag{3.38}$$

Proof. i) From (3.25) and (3.20), we write

$$\vec{t}_1\frac{\partial \vec{t}_2}{\partial s_1}=\left(\frac{\left(\sqrt{E}\right)_v+ch\theta\left(\sqrt{G}\right)_u}{\sqrt{EG}}\right)$$

$$\frac{\partial \vec{t}_2}{\partial s_1}=\vec{w}_1\wedge\vec{t}_2.$$

From here, we have

$$\frac{\left(\sqrt{E}\right)_v+ch\theta\left(\sqrt{G}\right)_u}{\sqrt{EG}}=\vec{t}_1(\vec{w}_1\wedge\vec{t}_2)=\vec{w}_1(\vec{t}_2\wedge\vec{t}_1)=-sh\theta\vec{w}_1.$$

Then, from (3.24), if we take inner product both of side $\vec{w}_1$ with $-sh\theta\vec{N}$ we obtain

$$-sh\theta\vec{N}\vec{w}_1=\frac{sh\theta\vec{N}^2}{(R_g)_1}\Rightarrow\vec{N}\vec{w}_1=-\frac{1}{(R_g)_1}.$$

Thus,

$$\frac{1}{(R_g)_1}=\frac{-1}{sh\theta\sqrt{EG}}\left(ch\theta\left(\sqrt{G}\right)_u+\left(\sqrt{E}\right)_v\right)$$

is obtained. Similarly, (ii) can be proofed. □

Theorem 3.4. *Let us consider any curve (c) on the time-like surface and the arc elements of curves (c), (c_1) and (c_2) as s, s_1 and s_2, respectively. Let the Darboux instantaneous rotation vectors of (c_1) and (c_2) be $\vec{w}_1$ and $\vec{w}_2$, and if the hyperbolic angle between the tangent $\vec{t}$ of curve (c) and $\vec{t}_1$ is α, then*

$$\left(\frac{sh(\theta-\alpha)}{sh\theta}\vec{w}_1+\frac{sh\alpha}{sh\theta}\vec{w}_2\right)\wedge\vec{t}_1=\vec{A}\wedge\vec{t}_1 \tag{3.39}$$

$$\frac{d\vec{t}_1}{ds}=\vec{A}\wedge\vec{t}_1,\qquad\frac{d\vec{t}_2}{ds}=\vec{A}\wedge\vec{t}_2,\qquad\frac{d\vec{N}}{ds}=\vec{A}\wedge\vec{N} \tag{3.40}$$

are satisfied.

Proof. If we consider (3.11) and (3.13), then

$$\begin{aligned}\frac{d\vec{t}_1}{ds}&=\frac{\partial\vec{t}_1}{\partial s_1}\frac{ds_1}{\partial s}+\frac{\partial\vec{t}_1}{\partial s_2}\ \frac{ds_2}{\partial s}=\frac{sh(\theta-\alpha)}{sh\theta}\left(\vec{w}_1\wedge\vec{t}_1\right)+\frac{sh\alpha}{sh\theta}\left(\vec{w}_2\wedge\vec{t}_1\right)\\&=\left(\frac{sh(\theta-\alpha)}{sh\theta}\vec{w}_1+\frac{sh\alpha}{sh\theta}\vec{w}_2\right)\wedge\vec{t}_1=\vec{A}\wedge\vec{t}_1.\end{aligned}$$

is obtained. Similarly, the others are satisfied. □

Result 3.2. The following equality

$$\vec{t}_2\frac{d\vec{t}_1}{ds}=-\vec{t}_1\frac{d\vec{t}_2}{ds}=sh\theta\vec{A}\vec{N} \tag{3.41}$$

is valid.

Theorem 3.5. *Let us consider the curves (c), (c_1) and (c_2) which are intersect a point P on time-like surface. Let the Darboux instantaneous rotation vectors of these curves at the point P be $\vec{w}$, $\vec{w}_1$ and $\vec{w}_2$, respectively. Then the following equality is satisfied*

$$\vec{w} = \frac{sh(\theta-\alpha)}{sh\theta}\vec{w}_1 + \frac{sh\alpha}{sh\theta}\vec{w}_2 + \vec{N}\frac{d\alpha}{ds}. \tag{3.42}$$

Proof. From (3.9)

$$\vec{t} = \frac{sh(\theta-\alpha)}{sh\theta}\vec{t}_1 + \frac{sh\alpha}{sh\theta}\vec{t}_2, \tag{3.43}$$

can be written. Then, by taking derivatives with respect to s from equation (3.43), we obtain

$$\frac{d\vec{t}}{ds} = \frac{sh(\theta-\alpha)}{sh\theta}\frac{d\vec{t}_1}{ds} + \frac{sh\alpha}{sh\theta}\frac{d\vec{t}_2}{ds} - \left(\frac{ch(\theta-\alpha)}{sh\theta}\vec{t}_1 - \frac{ch\alpha}{sh\theta}\vec{t}_2\right)\frac{d\alpha}{ds}. \tag{3.44}$$

On the other hand, considering the Darboux trihedrons $\left\{\vec{t}_1, \vec{g}_1, \vec{N}\right\}$ and $\left\{\vec{t}_2, \vec{g}_2, \vec{N}\right\}$, we write

$$\vec{t}_1 = \vec{g}_1 \wedge \vec{N}, \qquad \vec{t}_2 = \vec{g}_2 \wedge \vec{N}. \tag{3.45}$$

From (3.14) and (3.15), if $\vec{g}_1$ and $\vec{g}_2$ are substituted in (3.45) we obtain

$$\vec{t}_1 = \frac{1}{sh\theta}\left(ch\theta\vec{t}_1 - \vec{t}_2\right) \wedge \vec{N}, \tag{3.46}$$

$$\vec{t}_2 = \frac{1}{sh\theta}\left(-ch\theta\vec{t}_2 + \vec{t}_1\right) \wedge \vec{N}.$$

And then, substituting the equations (3.46) in (3.44), we have

$$\begin{aligned}\frac{d\vec{t}}{ds} = {} & \frac{sh(\theta-\alpha)}{sh\theta}\frac{d\vec{t}_1}{ds} + \frac{sh\alpha}{sh\theta}\frac{d\vec{t}_2}{ds} - \Big(\frac{ch(\theta-\alpha)}{sh^2\theta}\left(ch\theta\vec{t}_1 - \vec{t}_2\right) \wedge \vec{N} \\ & - \frac{ch\alpha}{sh^2\theta}\left(\vec{t}_1 - \vec{t}_2 ch\theta\right) \wedge \vec{N}\Big)\frac{d\alpha}{ds}\end{aligned}$$

According to the Theorem 3.4,

$$\frac{d\vec{t}_1}{ds} = \vec{A} \wedge \vec{t}_1, \qquad \frac{d\vec{t}_2}{ds} = \vec{A} \wedge \vec{t}_2, \qquad \vec{A} = \frac{sh(\theta-\alpha)}{sh\theta}\vec{w}_1 + \frac{sh\alpha}{sh\theta}\vec{w}_2$$

are known. And, by using the trigonometric expression, we find

$$\begin{aligned}\frac{d\vec{t}}{ds} &= \frac{sh(\theta-\alpha)}{sh\theta}\vec{A} \wedge \vec{t}_1 + \frac{sh\alpha}{sh\theta}\vec{A} \wedge \vec{t}_2 - \frac{d\alpha}{ds}\left(\frac{sh(\theta-\alpha)}{sh\theta}\vec{t}_1 + \frac{sh\alpha}{sh\theta}\vec{t}_2\right) \wedge \vec{N} \\ &= \vec{A} \wedge \left(\frac{sh(\theta-\alpha)}{sh\theta}\vec{t}_1 + \frac{sh\alpha}{sh\theta}\vec{t}_2\right) - \frac{d\alpha}{ds}\left(\frac{sh(\theta-\alpha)}{sh\theta}\vec{t}_1 + \frac{sh\alpha}{sh\theta}\vec{t}_2\right) \wedge \vec{N} \\ &= \vec{A} \wedge \vec{t} + \left(\vec{N} \wedge \vec{t}\right)\frac{d\alpha}{ds} \\ &= \left(\vec{A} + \vec{N}\frac{d\alpha}{ds}\right) \wedge \vec{t}.\end{aligned}$$

Thus,

$$\frac{d\vec{t}}{ds} = \vec{b} \wedge \vec{t} \tag{3.47}$$

$$\vec{b} = \vec{A} + \vec{N}\frac{d\alpha}{ds} \tag{3.48}$$

are written. After that,

$$\vec{A} = \vec{b} - \vec{N}\frac{d\alpha}{ds} \tag{3.49}$$

is obtained. By writing (3.49) in the third expression of (3.40) we obtain

$$\frac{d\vec{N}}{ds} = \vec{A} \wedge \vec{N} = \left(\vec{b} - \vec{N}\frac{d\alpha}{ds}\right) \wedge \vec{N} = \vec{b} \wedge \vec{N}. \tag{3.50}$$

Since $\vec{w}$ is Darboux vector, we have

$$\frac{d\vec{t}}{ds} = \vec{w} \wedge \vec{t}, \qquad \frac{d\vec{N}}{ds} = \vec{w} \wedge \vec{N}. \tag{3.51}$$

Then, considering (2.5), (3.47), (3.50) and (3.51)

$$\frac{d\vec{t}}{ds} = \vec{w} \wedge \vec{t} = \vec{b} \wedge \vec{t} \Rightarrow \vec{b} \wedge \vec{t} - \vec{w} \wedge \vec{t} = \left(\vec{b} - \vec{w}\right) \wedge \vec{t}$$

$$\Rightarrow \left(\vec{b} - \vec{w}\right) = \lambda\vec{t}, \tag{3.52}$$

$$\frac{d\vec{N}}{ds} = \vec{w} \wedge \vec{N} = \vec{b} \wedge \vec{N} \Rightarrow \vec{b} \wedge \vec{N} - \vec{w} \wedge \vec{N} = \left(\vec{b} - \vec{w}\right) \wedge \vec{N}, \tag{3.53}$$

are written. At the end, if we make equal (3.52) to (3.53), we have

$$\left(\vec{b} - \vec{w}\right) = \lambda\vec{t} = \mu\vec{N} \Rightarrow \lambda = \mu = 0.$$

Finally, $\vec{b} - \vec{w} = \vec{0}$ can be written. Thus, we get the theorem. □

Corollary 3.3 (General Form of Euler Formula)**.** *Taking dot product both of the* (3.42) *with* $\vec{g}$*, we have following equation among the timelike curves* (c)*,* (c_1) *and* (c_2)*:*

$$\frac{1}{R_n} = \frac{sh\alpha sh(\theta - \alpha)}{sh\theta}\left[\frac{1}{T_{g_1}} - \frac{1}{T_{g_2}}\right] + \frac{1}{sh\theta}\left[\frac{ch\alpha sh(\theta - \alpha)}{R_{n_1}} - \frac{sh\alpha ch(\theta - \alpha)}{R_{n_2}}\right], \tag{3.54}$$

where $\frac{1}{T_{g_i}}$*,* $\frac{1}{R_{n_i}}$ $(i = 1, 2)$ *and* $\frac{1}{R_n}$ *are geodesic torsion, normal curvatures of the parameter curves and the normal curve, respectively.*

Corollary 3.4 (General Form of O. Bonnet)**.** *Taking dot product both of the* (3.42) *with* $\vec{t}$*, we have following equation among the timelike curves* (c)*,* (c_1) *and* (c_2)*:*

$$\frac{1}{T_g} = \frac{sh(\theta - \alpha)}{sh\theta}\left[\frac{ch\alpha}{T_{g_1}} + \frac{sh\alpha}{R_{n_2}}\right] + \frac{sh\alpha}{sh\theta}\left[\frac{ch(\theta - \alpha)}{T_{g_2}} - \frac{sh(\theta - \alpha)}{R_{n_2}}\right], \tag{3.55}$$

where $\frac{1}{T_{g_i}}, \frac{1}{R_{n_i}}$ $(i = 1, 2)$ *and* $\frac{1}{T_g}$ *are geodesic torsions, normal curvatures of the parameter curves and the normal torsion of normal curve, respectively.*

Corollary 3.5 (Liouville Formula)**.** *Taking dot product both of the* (3.42) *with* $\vec{N}$, *we have following equation among the timelike curves* (c), (c_1) *and* (c_2)*:*

$$\frac{1}{R_g} = \frac{1}{R_{g_1}} \frac{sh(\theta-\alpha)}{sh\theta} + \frac{1}{R_{g_2}} \frac{sh\alpha}{sh\theta} + \frac{d\alpha}{ds}, \tag{3.56}$$

where $\frac{1}{R_{g_i}}$ *and* $\frac{1}{R_g}$ $(i = 1, 2)$ *are geodesic curvatures of the parameter curves and the normal curve, respectively. Now, we give some special cases of the formulae* (3.54) *and* (3.55)*.*

Corollary 3.6. *If we take* $\frac{1}{T_{g_1}} = \frac{1}{T_{g_2}} = 0$ *(i.e. the parameter curves are curvature lines) in* (3.55) *we have*

$$\frac{1}{T_g} = \frac{sh\alpha sh(\theta-\alpha)}{sh\theta} \left[\frac{1}{R_{n_1}} - \frac{1}{R_{n_2}}\right], \tag{3.57}$$

where $\frac{1}{T_g}$ *and* $\frac{1}{R_{n_i}}$ $(i = 1, 2)$ *are geodesic torsion of normal curve and normal curvature of parameter curves, respectively.*

Result 3.3: If we take $\frac{1}{T_{g_1}} = \frac{1}{T_{g_2}} = 0$ (i.e. the parameter curves are curvature lines) in (3.54) we have

$$\frac{1}{R_n} = \frac{1}{sh\theta} \left[\frac{ch\alpha sh(\theta-\alpha)}{R_{n_1}} - \frac{sh\alpha ch(\theta-\alpha)}{R_{n_2}}\right], \tag{3.58}$$

where $\frac{1}{R_{n_i}}$ $(i = 1, 2)$ and $\frac{1}{R_n}$ are normal curvatures of parameter curves and normal curve, respectively.

Result 3.4: If we take $\frac{1}{R_{n_1}} = \frac{1}{R_{n_2}} = 0$ (i.e. the parameter curves are asymptotic) in (3.54) we have

$$\frac{1}{R_n} = \frac{sh\alpha sh(\theta-\alpha)}{sh\theta} \left[\frac{1}{T_{g_1}} - \frac{1}{T_{g_2}}\right], \tag{3.59}$$

where $\frac{1}{T_{g_i}}$ $(i = 1, 2)$ and $\frac{1}{R_n}$ are geodesic torsions of parameter curves and normal curvature of normal curve, respectively.

REFERENCES

[1] Ugurlu, H. and Çalışkan, A. "The Space-like and Time-like Surface Geometry by the Darboux Vector" The press of the Celal Bayar Univ., 2012.

[2] Yılmaz, S.; Ozyilmaz, E. and Turgut, M. "On The Differential Geometry of the Curves in Minkowski Space-Time II," Int. Jour. of Comput Math Sci 3:2, 2009.

[3] Yaylı, Y. and Hacısalihoğlu, H.H. "Closed Curves in the Minkowski 3-space," Hadronic J. 23 (2000), No: 3

[4] Yılmaz, S. "Spherical Indictors of Curves and Characterizations of Some Special Curves in Four Dimensional Lorentzian Space," Ph.D. Dissertation, Dept. Math. Dokuz Eylül Univ., İzmir, Turkey, 2001.

[5] Akbulut, F. “Bir Yüzey Üzerindeki Eğrilerin Darboux Vektörleri”, E. Ü. Faculty of Science, İzmir, 1983.

[6] Kızıltuğ, S. and Yaylı, Y. “Space-Like Curve on Space-like Parallel Surface in Minkowski 3-space, ”Int. J. Math. Comput., 19 (2013), No. 2, 94–105

[7] Ali, A. T.; Lopez, R. and Turgut, M. “k-type Partially Null and Nseudo Null Slant Helices in Minkowski 4-space”, Math. Commun., 17, (2012), No. 1, 93–103.

[8] Turgut, M. and Yilmaz, S. “On The Frenet Frame and A Characterization of Space-Like Involute-Evolute Curve Couple in Minkowski Space-Time”, Int. Math. Forum, Vol.3, No. 16 (2008), 793–801.